CHAPTER 4

Internal solitary waves

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Abstract

In this chapter we review various approximate models (from weakly to strongly non-
linear) describing internal solitary waves in the density-stratified fluids. The waves
may have different shapes and polarities depending on the fluid stratification. These
properties as well as the processes of generation, interaction and transformation are
discussed.

1 Introduction

The physical nature of waves propagating on the boundary between two layers of
fluids of different densities is well known and explained by particle motion under
the action of gravity and inertial forces; the familiar example of such waves is the
surface waves (e.g. wind waves, swells, tides, tsunamis and so on) seen in seas
and oceans. However, the density and shear flow in the ocean and atmosphere are
stratified in the vertical direction, and hence internal waves can propagate at vari-
ous depths in the ocean and at various heights in the atmosphere. Due to the rather
weak density variation, the restoring force is also weak, and very large vertical
displacements of fluid particles may be achieved by the impact of external forces.
The amplitudes of natural internal waves can reach 100 m, significantly exceeding
the limiting amplitudes of the waves on the sea–air boundary (such as wind waves
and swells). Figure 1 gives an example of an observation of large-amplitude (up
to 40 m) long internal waves in the North Atlantic to the north of Ireland (Small
et al. 1999). Many intense waves of this kind have solitary-like shapes that propa-
gate for a long time without significant change of energy. They can be interpreted
and well described as internal solitary waves (solitons). Strictly speaking, the term
‘soliton’ is reserved for solitary waves in integrable systems, but we will nevertheless follow the widely used custom and call these waves internal solitons. As we will describe, such a terminology is partially justified here because these waves can be successfully modelled by integrable equations of the Korteweg–de Vries (KdV) type. Observations of large-amplitude long internal waves in the environment are compiled in several reviews: in the oceans (Ostrovsky and Stepanyants 1989; Jeans 1995; Holloway et al. 2002; Global Ocean Associates 2004; Sabinin and Serebranny 2005; Helfrich and Melville 2006), in the earth’s atmosphere (Cheung and Little 1990; Rottman and Grimshaw 2002) and in stratified laboratory tanks (Ostrovsky and Stepanyants 2005). Various appropriate mathematical models have been developed to describe nonlinear internal waves (Grimshaw 2002). The first study of internal solitons was based on the nonlinear boundary problem for the stream function (Dubriel-Jacotin 1932), and this approach remains popular in numerical simulations of large-amplitude solitons. The unsteady dynamics of small-amplitude soliton-like internal waves was first analysed in the framework of the KdV equation (Benney 1966), then more complicated models based on the extended versions of the KdV and Boussinesq equations have been developed for waves of moderate and large amplitudes. Now, the direct numerical simulation of the basic 2D Euler equations is being actively carried out (Grue et al. 1999; Lamb 2002; Vlasenko et al. 2005) to study the generation, propagation and breaking of internal solitons.

This chapter reviews the modern approaches used to describe internal solitons. Different mathematical models (from weakly to strongly nonlinear) are summarized in Section 2. The steady-state solitary wave solutions are discussed in Section 3. It is shown that their shape critically depends on the sign of the cubic nonlinearity. Another type of localized nonlinear waves (breathers) is briefly discussed. The process of internal soliton generation from the long-scale disturbances is described in Section 4 within the framework of the extended KdV (Gardner) equation. The transformation of a soliton passing a transition zone, where the variable stratification leads to the change of signs of the model coefficients (which often happens on ocean shelves) is discussed in Section 5.
2 Approximate evolution equations

Weakly nonlinear models for long internal waves are based on the KdV equation and its generalizations (Benney 1966; Lee and Beardsley 1974; Kakutani and Yamasaki 1978; Lamb and Yan 1996; Grimshaw 2002; Holloway et al. 2002 and Grimshaw et al. 2002b). Such equations are derived by using a multiscale asymptotic procedure on the governing Euler equations for inviscid incompressible stratified fluids. For simplicity, we consider a two-dimensional flow; then the basic equations are:

\[
\begin{align*}
\rho \frac{du}{dt} + \frac{\partial P}{\partial x} &= 0, \\
\rho \frac{dw}{dt} + \frac{\partial P}{\partial z} + \rho g &= 0, \\
\frac{d\rho}{dt} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0,
\end{align*}
\]

(1)

(2)

where \( \vec{V} = \{u, w\} \) is the fluid velocity, \( \rho(x, z, t) \) is the fluid density, \( P \) is the pressure, \( g \) is gravitational acceleration, \( \{x, z\} \) are spatial coordinates (horizontal and vertical – see Fig. 2) and \( d/dt \) is the convective time derivative. The fluid is assumed to be confined between two rigid boundaries: \( z = -h \) (bottom) and \( z = 0 \) (free surface). In fact, the approximation of the rigid boundary on the free surface ‘works’ very well in natural water basins due to the weak variation of the water density compared with the density jump on the surface. Somewhat more laborious computations are necessary to take into account the free moving surface (Grimshaw et al. 2002b).

Thus, the boundary conditions for eqns (1) and (2) have the form:

\[
w = 0 \text{ at } z = -h \quad \text{and} \quad z = 0.
\]

(3)

Since the density of the fluid particles does not change, it is convenient to introduce the isopycnal (Lagrangian) coordinate

\[
y = z - \zeta(x, z, t),
\]

(4)

where \( \zeta(x, z, t) \) is the vertical displacement of a fluid particle from its rest position. Thus, density \( \rho(x, z, t) = \rho_0(y) \) is ‘frozen’ in this representation. We also assume

![Figure 2: Coordinate system and the background flow (\( \xi = 0 \) in the rigid boundary approximation).](image-url)
that the horizontal velocity field $u(x,y,t)$ can be decomposed into the unperturbed horizontal shear flow velocity $U(y)$ and its perturbation $u'(x,y,t)$: $u = U + u'$.

Let us consider the weakly nonlinear limit, when the wave amplitude is small with respect to depth (with the corresponding small parameter $\mu \ll 1$) but finite. Another small parameter $\varepsilon = h/L$ ($L$ is the characteristic wavelength scale and $h$ is the total depth) characterizes the weak dispersion (long waves) and defines slow variables: $X = \varepsilon x, T = \varepsilon t$. We anticipate the KdV-scaling $\mu = \varepsilon^2$ to implement the contribution of the first nonlinear and dispersive corrections to the wave equation in the same order of smallness. To enable building a higher-order evolution equation, we introduce also the set of new temporal variables $\tau_i = \mu^i T$:

$$\frac{\partial}{\partial T} = -c \frac{\partial}{\partial s} + \mu \frac{\partial}{\partial \tau_1} + \mu^2 \frac{\partial}{\partial \tau_2} + \cdots,$$

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial s}, \quad s = X - cT.$$  

(5)

The co-moving coordinate $s$ contains the linear long-wave speed $c$, which is yet to be determined.

After tedious, but straightforward, calculations the systems (1) and (2) lead to one equation containing $\zeta$ only,

$$\frac{\partial}{\partial y} \left\{ \rho_0 (c - U)^2 \frac{\partial^2 \zeta}{\partial s \partial y} \right\} + \rho_0 N^2 \frac{\partial \zeta}{\partial s} = M, \quad (6)$$

where $N(y)$ is the Brunt–Väisälä (buoyancy) frequency profile, $N^2(y) = -(g/\rho_0(y))(d\rho_0/dy)$, and $M$ is a complicated expression containing nonlinear and dispersive terms of different orders (Grimshaw et al. 2002b). The boundary conditions for eqn (6) are $\zeta = 0$, at the bottom $y = -h$ and on the surface $y = 0$.

Next, we assume that the isopycnal displacement $\zeta$ is represented by the asymptotic series

$$\zeta(s,y,\tau) = \mu A(s,\tau) \Phi(y) + \mu^2 \zeta_2(s,y,\tau) + \mu^3 \zeta_3(s,y,\tau) + \cdots,$$  

(7)

where all the functions should be determined after substituting the series (7) into eqn (6) and collecting the terms of the same order of $\mu$. In the lowest order, the linear eigenvalue problem for the modal function $\Phi(y)$ is obtained:

$$L \Phi \equiv \frac{d}{dy} \left[ \rho_0 (c - U)^2 \frac{d \Phi}{dy} \right] + \rho_0 N^2(y) \Phi = 0,$$  

(8)

with zero boundary conditions on the bottom and the surface of the fluid. The solution of (8) may be simplified by neglecting the weak density variation in natural water and taking $\rho_0(y) \equiv \rho_0$ a constant; this corresponds to the Boussinesq approximation, which will be used hereafter. Real solutions (stable internal waves) of eqn (8) can be obtained only for large values of the Richardson number ($\text{Re} = N^2/U^2_z > 1/4$) (see, for instance, Miles 1961). It is well known that the eigenvalue problem (8) has, in general, an infinite sequence of modes $\Phi_n^\pm$ with
corresponding speeds (eigenvalues) $c_{n}^{\pm}$ \((n = 0, 1, 2, \ldots)\); here the modes with $c_{n}^{\pm}$ propagate faster/slower than the background current. A formal theory can be developed for any of these modes, but the amplitude of an internal soliton belonging to a higher mode gradually decreases due to the energy transfer to radiated shorter lower-mode dispersive waves (Akylas and Grimshaw 1992). This is why theories are developed mainly for solitary waves of the lowest \((n = 0)\) mode only, which has the greatest speed $c$. The corresponding modal function $\Phi(y)$ is defined by (8).

We choose the normalization of the modal function in a way that its extreme value is unity, i.e. $\Phi_{\text{max}} = 1$ at $y_{\text{max}}$.

The compatibility conditions to solve each order of the inhomogeneous problem give the sequence of evolution equations defining the amplitude dynamics up to different orders of accuracy. At $O(\mu^{2})$, we obtain the famous KdV equation derived first in this context by Benney (1966):

$$\frac{\partial A}{\partial \tau} + \alpha A \frac{\partial A}{\partial s} + \beta \frac{\partial^{3} A}{\partial s^{3}} = 0, \quad (9)$$

where

$$\alpha = \frac{3}{2I} \int (c - U)^{2}(d\Phi/dy)^{3}dy, \quad \beta = \frac{1}{2I} \int (c - U)^{2}\Phi^{2}dy,$$

$$I = \int (c - U)(d\Phi/dy)^{2}dy,$$  \(10\)

and all integrals are calculated over the fluid depth. The KdV equation is a very popular model to demonstrate the soliton properties of the internal wave field in the ocean and atmosphere.

After solving the compatibility condition for eqn (6) in order $O(\mu^{2})$ with respect to $\zeta$, one obtains

$$\zeta_{2} = A^{2}T_{n}(y) + \frac{\partial^{2} A}{\partial s^{2}}T_{d}(y), \quad (11)$$

where $T_{n}(y)$ is the first nonlinear and $T_{d}(y)$ is the first dispersion correction to the modal structure $\Phi(y)$ of the internal wave; these are solutions of corresponding inhomogeneous boundary problems:

$$LT_{n} = -\alpha \frac{d}{dy} \left\{ \rho_{0}(c - U) \frac{d\Phi}{dy} \right\} + \frac{3}{2} \frac{d}{dy} \left\{ \rho_{0}(c - U)^{2} \left( \frac{d\Phi}{dy} \right)^{2} \right\}, \quad (12)$$

and

$$LT_{d} = -2\beta \frac{d}{dy} \left\{ \rho_{0}(c - U) \frac{d\Phi}{dy} \right\} - \rho_{0}(c - U)^{2}\Phi, \quad (13)$$

with zero boundary conditions on the bottom and the surface of the fluid. It is important to note that solutions of the boundary-value problems (12) and (13) are unique only up to additive multiples of $\Phi$. It is convenient to let $A(s, \tau)$ represent the
isopycnal displacement at the level \( y_{\text{max}} \). Hence, we choose the auxiliary conditions: 
\[ T_n(y_{\text{max}}) = T_d(y_{\text{max}}) = 0. \]
In this case, series \( \xi \), at the level \( y = y_{\text{max}} \) gives 
\[ \xi(s, y_{\text{max}}, \tau) = \mu A(s, \tau) + O(\mu^3); \]  
(14)
of course, other normalizations can be used if convenient.

The compatibility condition for \( \xi \) at \( O(\mu^3) \), along with (5) and \( \xi \), leads to the second-order KdV equation:
\[
\frac{\partial A}{\partial \tau} + \alpha A \frac{\partial A}{\partial s} + \beta \frac{\partial^3 A}{\partial s^3} + \mu \left( \alpha_1 A^2 \frac{\partial A}{\partial s} + \beta_1 \frac{\partial^5 A}{\partial s^5} + \gamma_1 \frac{\partial^3 A}{\partial s^3} + \gamma_2 \frac{\partial A \partial^2 A}{\partial s \partial s^2} \right) = 0,
\]  
(15)
where the new coefficients are given by
\[
\alpha_1 = \frac{1}{2T} \int dy \left[ 3(c - U)^2 \left\{ 3(dT_n/dy) - 2(d\Phi/dy)^2 \right\} (d\Phi/dy)^2 \right.
+ \alpha(c - U) \left[ 5(d\Phi/dy)^2 - 4(dT_n/dy) \right] (d\Phi/dy) \left. - \alpha^2 (d\Phi/dy)^2 \right],
\]  
(16a)
\[
\beta_1 = \frac{1}{2T} \int dy \left[ 2\beta(c - U) \left\{ \Phi^2 - (d\Phi/dy)(dT_d/dy) \right\} 
- \beta^2 (d\Phi/dy)^2 + (c - U)^2 \Phi T_d \right],
\]  
(16b)
\[
\gamma_1 = -\frac{1}{2T} \int dy \left\{ 2(c - U) \left[ \alpha(dT_d/dy) + 2\beta(dT_n/dy) \right] (d\Phi/dy) 
+ 2\alpha\beta (d\Phi/dy)^2 - 2\alpha(c - U) \Phi^2 + (c - U)^2 \Phi^2 (d\Phi/dy) 
- 4\beta(c - U)(d\Phi/dy)^3 - (c - U)^2 \left\{ 3(dT_d/dy)(d\Phi/dy)^2 + 2T_n \Phi \right\} \right],
\]  
(16c)
\[
\gamma_2 = \frac{1}{2T} \int dy \left\{ (c - U) \left[ 2\beta(d\Phi/dy)^3 + 6\alpha \Phi^2 \right] - 2\alpha\beta(d\Phi/dy)^2 
- 2(c - U)^2 \left\{ \Phi^2 (d\Phi/dy) - 3T_n \Phi \right\} 
- 6\alpha(c - U)(dT_d/dy)(d\Phi/dy) + 3(c - U)^2 dT_d/dy)(d\Phi/dy)^2 \right\}.
\]  
(16d)

This asymptotic procedure allows us to obtain extensions of the KdV equation to any order. The asymptotically obtained evolution equations may acquire some specific features that are absent in the exact solutions of the Euler equations. For instance, if the linear wave solutions of the extended KdV eqn \( \xi \) are considered, their wave frequency \( \omega \) and wavenumber \( k \) satisfy the dispersion relation
\[
\omega = ck - \mu \beta k^3 + \mu^2 \beta_1 k^5.
\]  
(17)
Since \( \beta > 0 \) (see (10)), a positive coefficient \( \beta_1 \) (at least for the case of a two-layer fluid) implies that the group velocity of short waves may exceed the speed \( c \) of
long waves, in contradiction with the exact linear dispersion relation of the original physical problem. Thus, asymptotically derived equations may lead to different (wrong) wave dynamics. This takes place when the formal ranges of applicability of asymptotic expansions are not satisfied and happens due to the truncation of the asymptotic series. An asymptotically derived evolution equation may be written in many asymptotically close forms. All of them are equally justified, but may show different properties when the applicability conditions are broken. This freedom may be used to obtain improved models that are more preferable from this or that point of view. One regular way of obtaining the models with a more accurate dispersion law is based on the Padé approximation, which requires a modification of the asymptotic procedure. As a result, the recently derived Boussinesq-like models can be applied to fully nonlinear and almost fully dispersive surface waves (Madsen et al. 2003). The same approach may be applied in the case of nonlinear internal waves as well.

The extended KdV eqn (15) may be reduced to the KdV eqn (9) using the smallness of nonlinear and dispersive parameter $\mu$. The following near-identity asymptotical transformation of the wave field,

$$B = A + \mu \left\{ \frac{1}{2} \lambda_1 A^2 + \lambda_2 A_x + \lambda_3 A_x \int_{s_0}^{s} A ds + \lambda_4 A_x \right\}, \quad (18)$$

(where $\lambda_1, \ldots, \lambda_4$ are constants) can reduce the second-order evolution eqn (15) to its integrable version, or to the first-order KdV eqn (9) for the field $B$ with accuracy up to $O(\mu^2)$ (see, for example, Kodama 1985; Fokas et al. 1996), where the transformation (18) is introduced for the particular case of surface waves. In the general case, the coefficients $\lambda_1, \ldots, \lambda_4$ are given by

$$\lambda_1 = -\frac{18 \beta^2 \alpha_1 + 2 \alpha^2 \beta_1 + 3 \alpha \beta \gamma_1}{9 \alpha \beta^2}, \quad \lambda_2 = -\frac{6 \beta^2 \alpha_1 - \alpha^2 \beta_1 + \alpha \beta \gamma_2}{2 \alpha^2 \beta},$$

$$\lambda_3 = \frac{4 \alpha \beta \gamma_1 - 3 \beta \gamma_1}{9 \beta^2}, \quad \lambda_4 = -\frac{\beta_1}{3 \beta^2}, \quad (19)$$

assuming that nonlinear and dispersive coefficients $\alpha$ and $\beta$ in the KdV eqn (9) differ from zero. Thus, in general, eqn (15) is asymptotically reducible to the integrable KdV eqn (9).

The coefficients $\alpha$ and $\beta$ of the KdV equation are determined by integral expressions (10) and can vary over a rather wide range. The dispersion coefficient $\beta$ is always positive since the integrands in (19) are positive expressions. At the same time, the nonlinear coefficient $\alpha$ can have either sign and may even be zero. A two-layer fluid is a popular example of such a case; then $\alpha = 3c(h_1 - h_2)/(2h_1 h_2)$, where $h_1$ and $h_2$ are the thicknesses of the upper and lower layers (Kakutani and Yamasaki 1978). If $\alpha$ approaches zero, the classical hierarchy of small parameters gets broken and the asymptotic procedure must be modified. This anomalous smallness of the nonlinear term in the KdV equation requires taking into account the next nonlinear term with the coefficient $\alpha_1$ in the second-order eqn (15) for the description of nonlinear processes. The other second-order terms in (15) turn out to be of the next order of smallness in this degenerate case. Such an extension of the
KdV equation is often called the *Gardner* equation. To derive the Gardner equation using the asymptotic expansions, one should explicitly take into consideration the smallness of $\alpha \sim \delta$, where $\delta \ll 1$, assuming $\delta \sim \mu$, modifying the KdV-scaling as $\mu = \varepsilon$ and revising all the series. Then the following equation results:

$$\frac{\partial A}{\partial \tau} + \alpha A \frac{\partial A}{\partial s} + \beta \frac{\partial^3 A}{\partial s^3} + \alpha_1 A^2 \frac{\partial A}{\partial s} = 0. \quad (20)$$

It is necessary to note that this equation implies a less strict condition for the smallness of wave amplitude: classical KdV assumption $\mu \ll 1$ transforms to $\mu^2 \ll 1$ for the Gardner equation (note that now $\tau \sim \mu^2 T$), and therefore, the wave amplitude can be moderate, not just weak.

Equation (20) demonstrates quite different strongly nonlinear wave dynamics determined by the sign of the cubic nonlinear term $\alpha_1$, which can be negative as well as positive depending on the stratification (Grimshaw 2002). In the particular case of a two-layer fluid, it is always negative (Kakutani and Yamasaki 1978). The Gardner equation has been the subject of many investigations (Miles 1979, 1981; Grimshaw *et al.* 1999; Slyunyaev and Pelinovskii 1999; Slyunyaev 2001; Grimshaw *et al.* 2002a; Nakoulima *et al.* 2004). This equation is now widely used for modelling of internal solitons in the ocean (Liu *et al.* 1998; Small 1999; Holloway *et al.* 2002).

The extension of the Gardner eqn (20) to the next order in asymptotic theory can also be derived:

$$\frac{\partial A}{\partial \tau} + \alpha A \frac{\partial A}{\partial s} + \beta \frac{\partial^3 A}{\partial s^3} + \alpha_1 A^2 \frac{\partial A}{\partial s} + \mu^2 \left( \alpha_2 A^4 \frac{\partial A}{\partial s} + \alpha_3 A^4 \frac{\partial A}{\partial s} + \gamma_1 A^2 \frac{\partial A}{\partial s^2} + \gamma_2 A \frac{\partial^2 A}{\partial s^2} + \gamma_3 A \frac{\partial^3 A}{\partial s^3} + \beta_1 A \frac{\partial^5 A}{\partial s^5} \right) = 0, \quad (21)$$

and the asymptotic transformation similar to (18) can be suggested:

$$B = A + \mu^2 \left[ \lambda_1 A^2 + \lambda_2 A^3 + \lambda_3 A^4 + \lambda_4 A_1 \int_{s_0}^{s} A^2 ds + \lambda_5 A_1 \int_{s_0}^{s} A^3 ds + \lambda_6 X A \tau \right],$$

$$\lambda_1 = \frac{3\alpha_1 \beta_1 + \alpha_1 \beta (\gamma_1 - \gamma_2) + \alpha \beta (3\gamma_{31} - \gamma_{32} - \gamma_{33})}{6\alpha_1 \beta^2}, \quad \lambda_2 = \frac{2\alpha_1 \beta_1 + \beta (2\gamma_{31} - \gamma_{32})}{6\beta^2}, \quad \lambda_3 = \frac{4\alpha_1 \beta_1 + 3\beta (3\gamma_{31} - \gamma_{32} - \gamma_{33})}{6\alpha_1 \beta},$$

$$\lambda_4 = \frac{4\alpha_1 \beta_1 - 3\beta \gamma_1}{9\beta^2}, \quad \lambda_5 = \frac{4\alpha_1 \beta_1 - 3\beta \gamma_{33}}{9\beta^2}, \quad \lambda_6 = -\frac{\beta_1}{3\beta^2}. \quad (22)$$
which gives us the equation for $B$ in a more convenient form than (21) but still different from the Gardner eqn (18) (see Slunyaev et al. 2003):

$$\frac{\partial B}{\partial \tau} + \left( aB + \alpha_1 B^2 + \mu \alpha_2 B^3 + \mu \alpha_3 B^4 \right) \frac{\partial B}{\partial s} + \beta \frac{\partial^3 B}{\partial s^3} = 0,$$

$$\alpha'_1 = \alpha_1 + \mu \rho, \quad \rho = \frac{\alpha_1^2}{6 \alpha_1 \beta} \left( 3 \gamma_{31} - \gamma_{32} - \gamma_{33} \right) - \frac{\alpha \gamma_2}{6 \beta} + \frac{7 \alpha_1^2 \rho_1}{18 \beta^2},$$

$$\alpha'_2 = \alpha_2 + \frac{2 \alpha_1 (\gamma_1 - 3 \gamma_2) + \alpha (24 \gamma_{31} - 9 \gamma_{32} - 8 \gamma_{33})}{18 \beta} + \frac{10 \alpha \alpha_1 \rho_1}{9 \beta^2},$$

$$\alpha'_3 = \alpha_3 + \frac{\alpha_1 (4 \gamma_{31} - 2 \gamma_{32} - \gamma_{33})}{6 \beta} + \frac{5 \alpha^2 \rho_1}{9 \beta^2}.$$

Equation (23) contains two additional nonlinear terms and conserves at least two first integrals, but has the same order of asymptotic accuracy, and is therefore preferable for numerical modelling rather than (21).

To derive fully nonlinear and dispersive generalizations of the KdV equation, the heuristic or variational approach may be applied. The nonlinear and dispersive terms with the required accuracy may be incorporated independently, as it was done for surface waves (Whitham 1974). The full dispersion term can be found at least for simplified stratification, but, generally speaking, it has an integral form (via inverse Fourier transformation). Consideration of the full nonlinearity of the wave processes leads to the nonlinear interaction of waves with different modal structures, and the number of the “coupled” evolution equations may be high. This is why a two-layer model of the density stratification having one mode only is popular in the analytical and numerical study of large-amplitude internal waves. The fully nonlinear part of the evolution equations can exactly be found in the hydrostatic (shallow-water) approximation of the two-layer model (Baines 1995; Slunyaev et al. 2003):

$$\frac{\partial \eta}{\partial t} + V_{nl} \frac{\partial \eta}{\partial x} = 0,$$

$$V_{nl}(\eta) = \sqrt{g} \frac{h_1 h_2}{h} + 3 \sqrt{g} \left[ \sqrt{h_1 h_2} (h_1 - h_2) - \sqrt{h_1 h_2} (\tilde{h}_1 - \tilde{h}_2) \right] \frac{h_1 - h_2}{h},$$

$$\tilde{h}_1 = h_1 + \eta, \quad \tilde{h}_2 = h_2 - \eta,$$

(24)

where indexes 1 and 2 numerate the upper and lower layers correspondingly, $g' = g \Delta \rho / \rho$ is the reduced gravity acceleration, $\Delta \rho = \rho_2 - \rho_1$ is the density jump which is small (the Boussinesq approximation), $\rho_j$ is the average density of the layers, $h = h_1 + h_2$ is the total depth and $\eta$ is the interfacial displacement. Together with the pure nonlinear and pure dispersive terms, mixed terms of nonlinear dispersion (see (15) or (21)) must be taken into consideration to improve the model. Ostrovsky and Grue (2003) suggested a heuristic form of the nonlinear dispersion, which is supposed to take into consideration the strongly nonlinear effects of long waves.
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Their ‘β-model’ is represented by a KdV-like equation for fully nonlinear and weakly dispersive interfacial waves, expressed in the original unscaled variables,

$$\frac{\partial \eta}{\partial t} + V_{nl} \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \left( \beta \left( \eta \right) \frac{\partial^2 \eta}{\partial x^2} \right) = 0, \quad \beta(\eta) = \frac{1}{6} V_{nl}(\eta) (h_1 + \eta) (h_2 - \eta) .$$

(25)

The nonlinear evolution equations described above are derived for unidirectional waves and are valid for media where the influence of lateral boundaries may be neglected and there are no counter-propagating waves. This is a usual approach for the consideration of oceanic and atmospheric internal waves. However, for the description of internal waves propagating in closed basins, such as lakes, bays and laboratory wave flumes, the reflection from the horizontal boundaries should be taken into account. Another example is the process of generation by external factors when the generated waves propagate in both directions from the source area. Then, two-directional (Boussinesq-type) models for internal waves can be derived. If the nonlinearity and dispersion are weak, these equations can be obtained from the hydrodynamic equations using the Galerkin’s procedure for any density and flow stratification (Ostrovsky 1978; Engelbrecht et al. 1988). Basic modes in this approach are the linear non-dispersive modes, and the Boussinesq equations are:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} \left[ \left( h + \mu \frac{N}{2} A \right) u \right] = 0, \quad \frac{\partial u}{\partial t} + \frac{c^2}{h} \frac{\partial A}{\partial x} + \mu N \left[ u \frac{\partial u}{\partial x} - \frac{1}{2h} \frac{\partial}{\partial t} (Au) \right] + \mu Dh \frac{\partial^3 A}{\partial x \partial t^2} = 0,$$

where $A(x, t)$ and $u(x, t)$ describe the isopycnal displacement and horizontal velocity, respectively; and $N = \alpha h/c$ and $D = \beta / ch^2$ are dimensionless parameters of nonlinearity and dispersion. The basic assumptions, as well as the vertical structure and boundary conditions, are absolutely similar to those introduced for the derivation of the KdV equation. Let us mention that the horizontal velocity field in a two-dimensional spatial domain is described to the leading order $\mu$ by the expression:

$$u'(x, y, t) = \mu cu(x, t) \frac{d\Phi}{dy} .$$

(27)

The system (26) can be transformed into more convenient forms similar to the Peregrine equations for surface waves (Peregrine 1967). If we consider unidirectional waves, the KdV eqn (25) can be readily derived from (26).

For a two-layer fluid, strongly nonlinear Boussinesq-like equations can be derived in the same way as for surface waves (Miyata 1988; Choi and Camassa 1999; Ostrovsky and Grue 2003; Craig et al. 2004). The main idea here is that the wave field in each layer satisfies the linear Laplace’s equation for potential flow. As a result, we may use two boundary conditions (kinematic and dynamic) on the interface to derive the nonlinear equations for interface displacement and the difference
between the particle velocities above and below the interface. Next, the pressure (or potential) should be presented in the Taylor’s series in the layer thickness (the small parameter here is the ratio of water depth to wavelength – the dispersion parameter). The variational approach through the Hamiltonian or Lagrangian formulation is useful here to simplify the calculations. The derived Boussinesq-like equations are now called the Choi–Camassa equations (Choi and Camassa 1999):

\[
\frac{\partial (h_1 - \eta)}{\partial t} + \frac{\partial}{\partial x}[(h_1 - \eta)u_1] = 0, \quad \frac{\partial (h_2 + \eta)}{\partial t} + \frac{\partial}{\partial x}[(h_2 + \eta)u_2] = 0, \quad (28)
\]

\[
\frac{\partial (u_1 - u_2)}{\partial t} + u_1 \frac{\partial u_1}{\partial x} - u_2 \frac{\partial u_2}{\partial x} - g' \frac{\partial \eta}{\partial x} = D, \quad (29)
\]

\[
D = \frac{1}{3(h_1 - \eta)} \frac{\partial}{\partial x} \left[ (h_1 - \eta)^3 \left( \frac{\partial^2 u_1}{\partial x \partial t} + u_1 \frac{\partial^2 u_1}{\partial x^2} - \left\{ \frac{\partial u_1}{\partial x} \right\}^2 \right) \right]
\]

\[
- \frac{1}{3(h_2 + \eta)} \frac{\partial}{\partial x} \left[ (h_2 + \eta)^3 \left( \frac{\partial^2 u_2}{\partial x \partial t} + u_2 \frac{\partial^2 u_2}{\partial x^2} - \left\{ \frac{\partial u_2}{\partial x} \right\}^2 \right) \right], \quad (30)
\]

where \(u_1\) and \(u_2\) are the fluid velocities in the upper and lower layers of thickness \(h_1\) and \(h_2\), respectively, \(g'\) is the reduced gravity acceleration (we use the Boussinesq approximation for fluids with almost the same densities) and \(\eta\) is the interface displacement. If \(D = 0\), the system in (28) and (29) corresponds to the nonlinear shallow-water theory for a two-layer fluid, which remains valid for large-amplitude, but smooth, waves. The term \(D\) includes weak as well as nonlinear dispersion. Due to its smallness, some terms in expression (2) can be replaced through the non-dispersive system \((D = 0)\), which is why the equations of the nonlinear-dispersive theory may look different in papers by different authors. In fact, the system (28)–(30) is similar to the Green–Naghdi system for surface waves (Green and Naghdi 1976). As has been mentioned, this approach in the case of surface waves is extended now to fully nonlinear and almost fully dispersive waves (Madsen et al. 2003). A similar approach may be possible for waves in a two-layer fluid, but not available yet.

### 3 Solitons and their properties

Solitons are famous examples of the manifestation of nonlinear effects in internal wave dynamics. Discovered first for surface waves at the beginning of the 19th century, later they were found in many other physical problems (see Chapter 1). These surface solitary waves are described by the KdV and Boussinesq models, which when derived for internal waves became the first practical approximations for the observed internal solitary waves. Let us begin our consideration of solitons from the Gardner equation, bearing in mind that the KdV equation is a particular case of the Gardner eqn (20) (its low-amplitude limit). The Gardner equation contains the cubic nonlinear term \((\alpha_1)\), and the existence of solitons depends drastically on
its sign. The solitary wave for the Gardner equation can be written in a most general form as

\[ A(s, \tau) = \frac{6\gamma^2/\alpha}{1 + R \cosh(\gamma(s - s_0 - V\tau))}, \quad R^2 = 1 + \frac{6\alpha_1\beta\gamma^2}{\alpha^2}, \quad V = \beta\gamma^2, \quad (31) \]

where the wave amplitude is

\[ a = \frac{6\gamma^2/\alpha}{1 + R}. \quad (32) \]

The solitary wave amplitude \( a \) (or its scale coefficient \( \gamma \)) is a free parameter; another free parameter is the initial position of the wave \( s_0 \). It is convenient for our purposes to employ the dimensionless version of the Gardner equation (GE). After the changes (assuming that \( \alpha \neq 0 \)),

\[ \eta(x, t) = \frac{[\alpha_1]}{\alpha} A(s, \tau), \quad x = \sqrt{\frac{\alpha^2}{6\beta[\alpha_1]}} s, \quad t = \beta \cdot \left( \frac{\alpha^2}{6\beta[\alpha_1]} \right)^{3/2} \tau, \quad (33) \]

the Gardner eqn (20) transforms to the canonical form

\[ \frac{\partial \eta}{\partial t} + 6\eta(1 + p\eta)\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0. \quad (34) \]

Parameter \( p = \pm 1 \) denotes the sign of the coefficient \( \alpha_1 \). When it is positive \( (p = 1) \), we refer to eqn (34) as GE+ and as GE− in the opposite case \( (p = -1) \). When \( \alpha_1 = 0 \), the Gardner soliton (20) transforms to the KdV soliton and is given by the formula:

\[ A_+(s, \tau) = \frac{a}{\cosh^2 \left( \frac{a\alpha}{12\beta} (s - s_0 - V\tau) \right)}. \quad (35) \]

Here, the soliton velocity, \( V = c + aa/3 \), is expressed through the soliton amplitude \( a \). The soliton shapes are given in Fig. 4 for various signs of cubic nonlinearity in the normalized eqn (34). The KdV soliton is shown in Fig. 3a by dotted lines. The KdV solitons have only one polarity (defined by the sign of the quadratic nonlinearity coefficient \( \alpha \)). It is explicitly seen that a soliton becomes narrower when it is higher, thus intensive solitons have a broader spectrum and require the correct consideration of not only higher nonlinear effects but also higher-order dispersive effects. In the simple case of a two-layer fluid, the solitons are up-crested if the pycnocline is placed below the middle level and vice versa. The solitons of the GE retain the polarity of KdV solitons, but they are broader (see Fig. 3a). The amplitude of such solitons is bounded by the limiting value, \( a_{\lim} = -\alpha_1/\alpha \) (in dimensionless variables \( a_{\lim} = 1 \)). The limiting soliton has a flat crest and its slopes are shock-like waves often called \textit{kinks}. In fact, kinks can be considered as independent
nonlinear structures if the distance between them is large. This ‘table’ soliton plays an important role in wave dynamics, becoming a pedestal for other solitons, and changes significantly the process of disintegration of a pulse disturbance (as will be discussed in the next section).

The case $p = 1$ ($\alpha_1 > 0$) may be fulfilled, for instance, in the case of a three-layer stratification (Talipova et al. 1999). The solitons remain similar to the KdV case, if they are positive (we suppose here $\alpha$ to be positive), see Fig. 3b. The soliton amplitude is not bounded in this model and a new solitary branch of opposite polarity exists. The latter have amplitudes larger (in modulus) than $a_{\text{alg}} = -2/\alpha_1$ ($a_{\text{alg}} = -2$ in the dimensionless variables). The soliton with amplitude $a_{\text{alg}}$ has power-law tails (it is shown by the dotted line in Fig. 3b), and is called the algebraic soliton. The algebraic soliton is structurally unstable (Pelinovsky and Grimshaw 1997), but the other members of this branch demonstrate the usual vitality of solitons. When the solitons are very large (of either polarity), their dynamics is very close to the case of the modified KdV equation (when $\alpha \equiv 0$ in the Gardner equation).

Both the Gardner and the KdV equations are integrable, and the Cauchy problem can be solved exactly by analytical methods (see Lamb 1980 and Chapter 2). The solitons show an exceptional property of elastic collision among themselves and other waves. The multisoliton solutions can be obtained in an explicit form; in the case of the KdV equation, there are two kinds of soliton collisions. If the interacting solitons are very different (amplitude ratio $a_2/a_1 > 3$), then the fast one passes over the other and it is called overtaking collision (see Fig. 4a). If the solitons have close amplitudes, the process of exchange takes place: the fast soliton gets slower and gives its energy to the other one; there is always some distance between the solitons in such a type of collision (Fig. 4b), and they do not form a single-crested wave in contrast to the first type of interaction. It is obviously seen from Fig. 4 that the

Figure 3: Solitons of different approximated models: (a) KdV solitons with amplitudes 0.1, 0.2, 0.5, 1, 2 (dotted lines) and GE– solitons with amplitudes 0.1, 0.2, 0.5, 0.9, 0.99, 0.999 (solid lines); (b) GE+ solitons with amplitudes 0.5, 1, 2, 3, 4, −2 (the algebraic soliton shown as the dashed line), −3, −4. The dimensionless form of eqn (34) is used.
Figure 4: The processes of interaction of KdV solitons: (a) overtaking (the amplitude ratio is 4) and (b) exchange (the amplitude ratio is 2). The dimensionless form of eqn (34) with $p = 0$ is used.

Figure 5: Collision of a soliton of amplitude 0.1 with a flat-crested soliton of amplitude 0.999 within the framework of the GE equation. Solitons get a phase shift during the collision, but their shapes and velocities remain unchanged after the interaction.

When the cubic nonlinearity coefficient is negative ($p = -1$, the GE equation), the soliton amplitude varies from 0 to 1. The interaction of solitons with small and moderate amplitudes is similar to the KdV case (Fig. 4). But if one soliton has an amplitude close to the limiting value, the smaller soliton actually changes its polarity climbing up onto the ‘table’ crest (Fig. 5). Each collision with the kinks leads to a change of the polarity of the smaller soliton and a phase shift, but the speed and the shape remain the same (Slyunyaev and Pelinovskii 1999).

In the case of the GE + equation (the positive cubic nonlinearity), the interaction of the solitons of the same polarity is qualitatively the same as in the KdV
Figure 6: Collision of solitons of different polarities within the framework of the GE+. The amplitudes are 0.2 (of the faster one; it goes from the left and is almost invisible when it does not interact with the second soliton) and −2.1 (of the slower one).

case (Slyunyaev 2001). The interaction of the solitons of opposite polarities is more complicated. Figure 6 gives such an example of the collision of two solitons with different polarities. The negative soliton is much larger than the positive one, which is almost invisible in the figure at the initial moment, but the larger wave almost changes its sign at the moment of the interaction (the solid line). A large phase shift is obtained by the negative soliton and the interaction is clearly seen in Fig. 6.

The discrete spectrum of the associated scattering problem defines localized solitary solutions of integrable equations (such as the KdV, see Chapter 2) or the Gardner equation (Miles 1979; Grimshaw et al. 2002a). The discrete spectrum of the KdV and GE− equations lies on an axis (the negative real axis for the KdV equation in the notation given in Chapter 2) of the spectral plane. The case of the GE+ equation turns out to be richer: the discrete spectrum may be complex (complex conjugated to describe real solutions). Pairs of complex conjugated spectral values define other localized nonlinear solutions of the GE+ equation – nonlinear wave packets or ‘breathers’ (Pelinovsky and Grimshaw 1997):

\[
\eta = 2 \frac{\partial}{\partial x} \tan^{-1} \left[ \frac{l \cosh (\Psi) \cos (\theta) - k \cos (\Phi) \sinh (\varphi)}{l \sinh (\Psi) \sin (\theta) + k \sin (\Phi) \cosh (\varphi)} \right],
\]  

(36)

where θ and ϕ are ‘traveling’ phases,

\[ \theta = k(x - wt) + \theta_0, \quad \varphi = l(x - vt) + \varphi_0, \quad w = -k^2 + 3l^2, \quad v = -3k^2 + l^2. \]

In contrast to solitons, this wave is described by two important ‘energetic’ parameters

\[ \Phi + i\Psi = \tan^{-1} [l + ik], \]
Figure 7: Breathers of the GE+.

and two initial phases ($\theta_0$ and $\phi_0$). A typical image of a breather is given in Fig. 7. A breather may look like the collision of two solitons of opposite polarities (Fig. 7a; compare with Fig. 6), or resemble a localized wave group containing a large number of individual waves (Figs 7b and 9b), which preserves its individuality. Although solitons always propagate faster than linear waves, a breather may move slower or faster depending on its parameters.

The Gardner equation remains a very simple model and is convenient for investigation since it is integrable. The Gardner, and even the KdV equation, may give a good approximation of many observed solitons in nature and in the laboratory. (see, for example, Ostrovsky and Stepanyants 1989, 2005). But these equations remain approximate and need assumptions of weak nonlinearity and dispersion to be satisfied. The fully nonlinear weakly dispersive models for interfacial waves in a two-layer fluid were discussed in Section 2. In fact, such models reduce to ordinary differential equations of the second order describing solitary waves, but cannot usually be solved analytically. Qualitatively, the soliton shape in the fully nonlinear models is the same as in the framework of the GE− (although it is valid for weakly nonlinear interfacial waves in a two-layer fluid): the solitary wave amplitude is bounded, and its exact value within the fully nonlinear model is (Choi and Camassa 1999)

$$a_{\text{lim}} = \frac{h_2 - h_1 \sqrt{\rho_2 / \rho_1}}{1 + \sqrt{\rho_2 / \rho_1}},$$  

and the speed of the limiting ‘table’ soliton is

$$c_{\text{lim}}^2 = g h \frac{1 - \sqrt{\rho_2 / \rho_1}}{1 + \sqrt{\rho_2 / \rho_1}}, \quad h = h_1 + h_2.$$  

In the Boussinesq limit (layers with near densities), the limiting value for the soliton amplitude is

$$a_{\text{lim}} = (h_2 - h_1)/2,$$
and the speed of the ‘table’ soliton is

$$c_{\text{lim}} = \frac{c_h}{2}, \quad c_h = \sqrt{g'h}.$$  \hfill (40)

So, the crest of the limiting soliton is situated exactly in the middle of the total water depth. The limiting amplitudes and velocities of solitons given by some models are compared in Fig. 8. Solid lines represent the case of the Gardner equation, dashed lines correspond to the extended Gardner equation (with the first four nonlinear terms taken from the fully nonlinear eqn (24)), and crests represent the ‘β-model’ of Ostrovsky and Grue (2003). It is seen from Fig. 8a that the crests of limiting solitons are located near the middle of the fluid for a wide range of values of $h_1$ for all the models. If $h_1$ and $h_2$ differ much, the approximate models underestimate the limiting amplitude. Curves of limiting speeds of solitary waves look quite different for the Gardner and extended Gardner equations when $h_1$ and $h_2$ are very different (Fig. 8b). It is evident that the limiting velocity approaches the exact value $c_h/2$ when the model is improved. The dashed line in Fig. 8b shows the linear velocity $c$.

More sophisticated models may be used for the description of internal solitons (see Section 2), which improve the quantitative description of the dynamics and introduce new effects (the most important seems to be the inelasticity of the soliton interaction, generally due to the loss of integrability, but they are energetically small). An interesting approximate approach to the study of large-amplitude soliton interaction analysis has been developed by Gorshkov et al. (2004) based on soliton representation as a particle.

Actually when the pycnocline is relatively close to the middle of the fluid, the Gardner equation provides a good approximation of the fully nonlinear

![Figure 8](image-url)
models. The checking of observed and experimental internal solitons with different improved models, including fully nonlinear numerical simulations, may be found, for instance, in Mirie (1985), Michallet and Barthelemy (1998), Grue et al. (1999), Vlasenko et al. (2000), Ostrovsky and Grue (2003) and Small and Hornby (2005).

4 Evolution of initial disturbances

Usually, short-scale large-amplitude waves are generated by long-scale waves such as tides. This process may be considered theoretically as the Cauchy problem for the initial disturbance; it is classical if studied within the frameworks of the KdV equation (see, for instance, Lamb 1980). A negative initial disturbance of the Gardner eqn \( \text{GE} \) in the KdV limit \( (\rho = 0) \) evolves into a dispersive train, as shown in Fig. 9a, which may be represented via Painleve functions that are nonlinear analogues of the Airy function. Any positive finite disturbance evolves into one or many solitons and quasi-linear waves forming a decaying dispersive tail (Fig. 10a). Depending on the actual shape of the initial perturbation, the number of solitons and the distribution of energy between the solitary and dispersive parts of the waves may be different. Due to the property of integrability, the number of solitons is conserved in the process of wave propagation, although the dispersive tail vanishes with time. Thus, solitons represent the asymptotic behaviour of the wave field.

This bewitching process of soliton formation, and then the recurrence of initial state in the periodic domain, was observed first by Zabusky and Kruskal in 1965 and led to the discovery of the exceptional role of solitons in integrable systems. Due to the linear relation between amplitudes and velocities of KdV solitons, they are generated in order, see Fig. 10a. The largest (fastest) goes first and the slower and lower solitons follow it. The amplitude of the leading soliton may be almost twice the amplitude of the initial impulse.

The initial-value problem for the Gardner equation can also be solved exactly due to its integrability (Miles 1979; Grimshaw et al. 2002a). If the spectrum of the associated linear scattering problem is found, then it survives in time and the

![Figure 9](image_url)

Figure 9: Evolution of the initial negative box-like impulse of amplitude 1.2 within the frameworks of (a) the KdV equation and (b) the GE + equation.
Figure 10: Evolution of the initial positive pulse of amplitude 1.2 within the frameworks of (a) the KdV equation and (b) the GE– equation. Note the different scales of time and field in the figures.

result of the evolution may be formally found at any moment of time. The discrete spectrum of the scattering problem corresponds to the localized solutions, such as solitons and breathers (Lamb 1980).

Naturally, the low-amplitude limit of the GE shows qualitatively similar results to the KdV evolution of an initial disturbance. This remains true for purely positive disturbances of the GE+ equation and negative ones in the GE– equation. In contrast to the KdV case, an intense negative impulse may evolve into a set of solitons and breathers as shown in Fig. 9b, when the positive cubic nonlinearity is taken into account (the GE+ case). Two negative solitons propagating to the right are observed in the figure. Furthermore, breathers are also generated and two of these may be seen clearly. One of the breathers contains few individual waves and slowly propagates to the left in Fig. 9b, while the other contains a large number of individual waves and moves to the left faster.

The transformation of initially positive intensive impulse disturbances happens differently in the case of the KdV and the GE– equations (Fig. 10). In the latter case, the appearance of the leading ‘table’ soliton is clearly seen, while the generation of smaller-amplitude solitons is suppressed (Grimshaw et al. 2002a). Each slope of the initial impulse generates solitary waves: one set of these has positive polarity and forms behind the back slope of the initial impulse, but other solitons are generated on the crest of the forming wide soliton behind the front of the impulse. Initially, these solitons are negative and change their signs as they go down from the wide soliton (this happens due to the larger velocity of the ‘table’ soliton, see Fig. 10b). The process of collision of the wide soliton with a smaller one was described in Section 3 and is shown in Fig. 5 and also in Fig. 10b. The presence of two groups of solitons destroys the usual order of solitary waves, which collide according to the difference in their speeds. Due to the limitation in soliton amplitude and speed provided by the GE–, the generated solitons are typically less and slower (note the different scales in Fig. 10a and b).

The knowledge of these scenarios gives a good qualitative idea about the processes of soliton generation. Disintegration of an internal bore-like disturbance is
often considered to lead to further evolution of internal waves; in Boussinesq-like and fully nonlinear models. This was studied by Lamb and Yan (1996), Vlasenko et al. (2000, 2002) and Lamb (2002). For example, the study of a certain oceanic area (Malin Shelf in the Atlantic) demonstrates a weak difference from the description given by the Gardner equation (Small and Hornby 2005).

5 Solitons in variable media

The multiscale asymptotic approach to the derivation of the KdV-type evolution equations takes into account the vertical structure of the internal wave fields through a modal representation. This simplification may turn out to be a strong point of the models of this type. Besides the first evident benefit (which is the simplicity of the models from an analytical and computational point of view – they are 1+1 dimensional), these models may also take into account the horizontal variability of the path of the internal waves (such as variable stratification, depth, etc). Such models may be derived after a suitable modification of the asymptotic technique considered in Section 2 (see Zhou and Grimshaw 1989; Pelinovsky et al. 1994; Holloway et al. 2002). They require the variation of environmental parameters along the wave path to be slow compared with the internal wave scale and result in variable coefficients of the evolution equations.

The background conditions, indeed, change significantly during the internal solitary waves propagation over the continental shelf, which results in the transformation of solitary waves (Liu et al. 1998; Holloway et al. 2002; Zhao et al. 2003; Grimshaw et al. 2004). The most startling and most easily observed effect is the change of soliton polarity. This process is very often observed in the coastal zone, both directly and from satellite data (see, for instance, Zhao et al. 2003) and may be simply understood in terms of the KdV theory. Indeed, the soliton polarity is defined by the sign of the quadratic nonlinearity, which strongly depends on stratification and may change its sign (this is evident for the two-layer stratification, see Section 2). This term is the most important in the process of soliton transformation. The degeneration of the quadratic nonlinear term requires the consideration of a higher-order nonlinearity, leading to a generalized Gardner equation (Grimshaw et al. 1998, 1999).

\[
\frac{\partial A}{\partial T} + c \frac{\partial A}{\partial X} + \alpha(T)A \frac{\partial A}{\partial T} + \alpha_1 A^2 \frac{\partial A}{\partial T} + \beta A^3 \frac{\partial^3 A}{\partial T^3} = 0
\]

(41)

represents a first-order model describing this process, where all the coefficients except \( \alpha \) can be regarded as being constant and \( \alpha = 0 \) at the ‘turning’ point. This equation is not integrable, but conserves the mass

\[
M = \int_{-\infty}^{\infty} A(X, T) dX = \text{constant}
\]

(42)

and energy (momentum) integrals

\[
E = \frac{1}{2} \int_{-\infty}^{\infty} A^2(X, T) dX = \text{constant}.
\]

(43)
These conserved quantities may be used for the description of the adiabatic stage of the soliton evolution (Grimshaw et al. 1999; Nakoulima et al. 2004). When the soliton polarity is defined by the sign of the quadratic nonlinearity (the GE− case or the low-amplitude limit of GE+), a soliton cannot pass the turning point, adiabatically changing its parameters and preserving its own identity, instead it undergoes drastic changes. Then a small-amplitude soliton transforms into a dispersive wave packet in the vicinity of the turning point, and a negative soliton of a lesser amplitude is formed later (see Fig. 11b). If a wide (‘table’) soliton passes the turning point, the generated soliton of opposite polarity is again wide, but its mass integral is less due to the radiation of the dispersive tail (Fig. 11b).

In the case $\alpha_1 > 0$ (the GE+ equation), cubic nonlinearity supports intensive solitons of both polarities (this is close to the case of modified KdV equation with the only nonlinear term being the cubic nonlinear term, that is $\alpha \equiv 0$) resulting in a relatively small change of the soliton shape and amplitude while passing the transition zone, as shown in Fig. 12. If the initial soliton is small, its transformation is similar to that shown in Fig. 11.

![Figure 11: Transformation of a soliton passing the turning point: small-amplitude case (a) and a wide soliton (b) (numerical simulation). The model is the Gardner eqn (41) with $c = 0$, $\alpha = 1 - t/T$, $\alpha_1 = -0.08$ and $\beta = 1$.](image-url)
Breathers may also be generated by a soliton due to change of the nonlinear coefficients of the Gardner equation, and if $\alpha_1 > 0$ after the turning point (Grimshaw et al. 1999; Nakoulima et al. 2004), the effect of soliton transformation becomes multifarious.

The opposite case of a sudden change of environmental parameters may also be considered within these models. Then the Cauchy problem is solved (see Section 4) where the wave field just before the transition zone gives the initial conditions, and the coefficients of the equation are changed to the new (after the transition) values. Typically, this leads to the so-called soliton fission.

6 Conclusion

Internal solitons are fascinating, nonlinear geophysical objects that allow detailed investigation using sophisticated measuring devices developed in recent years. Their most extensive theoretical description is based on the solution of the primitive, fully three-dimensional equations of hydrodynamics with the essential consideration of vertical stratification due to the salinity, temperature distribution and inhomogeneous shear flows. It is a very expensive approach that needs a powerful computer and a lot of information about the environmental conditions, and may be difficult to interpret. At present, only simple model simulations are being carried out with this approach. Another common approach to study these internal waves considers the modal structure supported by the stratification (the modes may be obtained through numerical solution of the appropriate boundary-value problem,
or simplified stratifications, such as the two-layer fluid, may be employed leading to analytical solutions) and evolution equations that describe the wave motion in the horizontal plane. This approach is less expensive from the point of view of numerical computations and uses averaged information about the stratification (thus, is less affected by the sea variability and errors of measurements). The nonlinear evolution equations describing internal waves may be simple (and even integrable), but the quality of the description of the internal waves is reasonable and, often, no less accurate than natural measurements. The KdV equation is the first, and a popular, model of nonlinear internal waves since 1966 (Benney 1966). Considering the real variability of sea stratification, this equation should be replaced by the Gardner equation, which is supposed to be the simplest model describing internal solitons. The evolution equation may be improved by introducing other terms of higher orders, using modified asymptotic techniques or employing specific stratifications that allow one to formulate a highly (fully) nonlinear theory; heuristic methods may also be useful in this way. These more accurate models usually show qualitatively similar results and the difference may be confidently observed only in laboratory experiments.

On the basis of the Gardner equation and its generalizations, the description of the observed internal solitons on the ocean shelves becomes possible. This model includes the smooth horizontal variability of the stratification with ease (which is a strong effect on a sloping shelf). Ray methods allow one to take into account the second horizontal dimension (the wave refraction), while, along a ray, the variable-coefficient Gardner equation is solved. The coefficients may be found on the basis of the available atlas of sea stratification. The existence of exact solutions of the integrable version of the equation helps to verify the numerical model and interpret the obtained results. All these above-mentioned facts allow us to get a realistic description of the measured internal solitons over the shelves within this relatively simple model.

Acknowledgements

This work was supported by grants INTAS 03-51-3728, RFBR 04-05-39000, 05-05-64333, 06-05-64232 and programmes for young researchers MK-1358.2005.1 (for O.P.) and the Russian Science Support Foundation (for A.S.).

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