Simultaneous Equations With Mixed Dependent Variables

A. Larin & L. Leonova

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†
National Research University
Higher School of Economics
city of Nizhny Novgorod

Notation

System of equations

$$\begin{cases} w = \alpha_{wz} \cdot z + \beta_w' \cdot \mathbf{x} + \varepsilon_w \\ z = \alpha_{zw} \cdot w + \beta_z' \cdot \mathbf{x} + \varepsilon_z \end{cases}$$

Simultaneity: $(w \leftrightarrow z)$

Single equation estimates

- endogeneity ($E[\varepsilon_w | z, \mathbf{x}] \neq 0$ and $E[\varepsilon_z | w, \mathbf{x}] \neq 0$)

Structural to Reduced Form

Denote:

$$\mathbf{y} = \begin{pmatrix} w \\ z \end{pmatrix}, \; A = \begin{pmatrix} 1 & -\alpha_{wz} \\ -\alpha_{zw} & 1 \end{pmatrix}, \; \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_w \\ \boldsymbol{\beta}_z \end{pmatrix}, \; \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_w \\ \varepsilon_z \end{pmatrix}$$

Structural form

$$A \cdot \mathbf{y} = \boldsymbol{\beta} \cdot \mathbf{x} + \boldsymbol{\varepsilon}$$

Reduced form

$$\mathbf{y} = (A^{-1} \cdot \boldsymbol{\beta}) \cdot \mathbf{x} + A^{-1} \cdot \boldsymbol{\varepsilon} \quad \Rightarrow \quad \mathbf{y} = \mathbf{b} \cdot \mathbf{x} + \boldsymbol{\epsilon}$$

The Problem of Identification

- Estimates of structural parameters are from estimates of reduced form parameters

# of parameters in structural form $>$ # of parameters in reduced form

Restrictions on structural parameters are needed

- For example, we may set some of betas equal to zero:

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_{w1} & \beta_{w2} & \beta_{w3} & 0 \\ \beta_{z1} & \beta_{z2} & 0 & \beta_{z4} \end{pmatrix}$$

Continuous plus Discrete

- s_i - life satisfaction - ordered $\in \{1, 2, 3, 4, 5\}$
- w_i - job status - binary $\in \{0, 1\}$
- y_i - income - continuous $\in [0, +\infty)$

System for Latent Variables

Latent variables for s_i and w_i like in ordered and binary choice models

$$\begin{cases} s_i^* = \alpha_{sw} \cdot w_i^* + \alpha_{sy} \cdot y_i + \beta_s' \cdot \mathbf{x}_i + \varepsilon_i^s \\ w_i^* = \alpha_{ws} \cdot s_i^* + \alpha_{wy} \cdot y_i + \beta_w' \cdot \mathbf{x}_i + \varepsilon_i^w \\ y_i = \alpha_{ys} \cdot s_i^* + \alpha_{yw} \cdot w_i^* + \beta_y' \cdot \mathbf{x}_i + \varepsilon_i^y \end{cases}$$

Structural to Reduced Form

Structural form

$$\mathbf{A}\mathbf{z}_i = \beta' \mathbf{x}_i + \varepsilon_i$$

Reduced form

$$\mathbf{z}_i = \mathbf{b}' \mathbf{x}_i + \epsilon_i$$

Shocks splitting

In reduced form shocks may be statistically dependent.

$$\begin{cases} s_i^* &= & \mathbf{b}_s' \cdot \mathbf{x}_i & + & \epsilon_i^s \\ w_i^* &= & \mathbf{b}_w' \cdot \mathbf{x}_i & + & \epsilon_i^w \\ y_i &= & \mathbf{b}_y' \cdot \mathbf{x}_i & + & \epsilon_i^y \end{cases}$$

Shock regression: $\epsilon^s = \rho_{sy} \cdot \epsilon^y + u^s$, where $\rho_{sy} = Cov\{\epsilon^s, \epsilon^y\}/Var\{\epsilon^y\}$.

$$\begin{cases} s_i^* &= & (\mathbf{b}_s - \rho_{sy} \cdot \mathbf{b}_y)' \cdot \mathbf{x}_i & + & \rho_{sy} \cdot y_i & + & u_i^w \\ w_i^* &= & (\mathbf{b}_w - \rho_{wy} \cdot \mathbf{b}_y)' \cdot \mathbf{x}_i & + & \rho_{sy} \cdot y_i & + & u_i^w \\ y_i &= & \mathbf{b}_y' \cdot \mathbf{x}_i & & & + & \epsilon_i^y \end{cases}$$

Shocks u^s and u^w are independent of ϵ^y.

Joint Density

Likelihood for i-th observation:

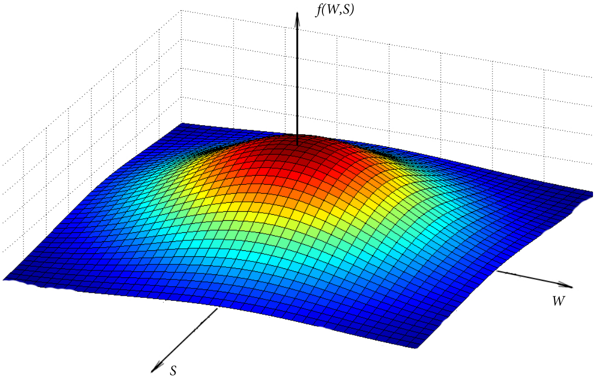
$$L_i = f(s_i, w_i, y_i|\mathbf{x}_i) = f(s_i, w_i|y_i, \mathbf{x}_i) \cdot f(y_i|\mathbf{x}_i)$$

- First part - from multivariate normal distribution
- Second part - from standard normal: $f(y_i|\mathbf{x}_i) = \phi(\epsilon_y/\sqrt{Var\{\epsilon_y\}})$

Bivariate Normal

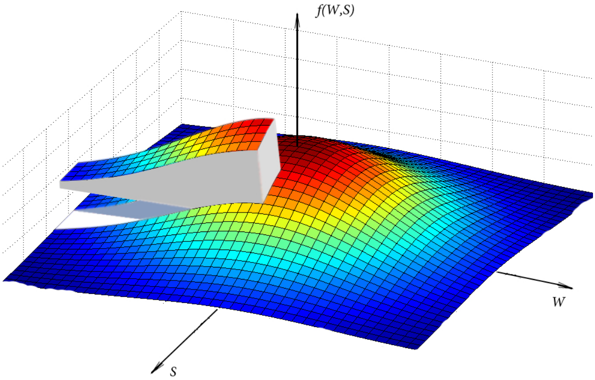
Joint density for $s_i = 3$ and $w_i = 0$:

$$\Pr\{s_i = 3, w_i = 0 | y_i, \mathbf{x}_i\} = \Pr\{\pi_l < u_i^w \leq \pi_h, u_i^w < c\} = \Phi_2(\pi_h, c) - \Phi_2(\pi_l, c)$$



Bivariate Normal

Each block corresponds to a pair of values $\{s_i, w_i\}$



Stata - cmp module

Likelihood function:

$$\log L = \sum_{i=1}^{n} \log L_i$$

Estimation is straightforward: $\widehat{\theta}_{ML} = \arg\max_\theta \sum_{i=1}^{n} \log L_i(\theta)$.

Sata: cmp module by David Roodman
for simultaneous eq. with mixed dependent variables of any type (binary, ordered, truncated and so on)

Individual effects

Panel data

- Time invariant individual effects are needed

Random effects:

$$\begin{cases} s^*_{it} & = & \mathbf{b}'_s \cdot \mathbf{x}_i & + & \gamma_s \cdot \xi_i & + & \epsilon^s_{it} \\ w^*_{it} & = & \mathbf{b}'_w \cdot \mathbf{x}_i & + & \gamma_s \cdot \xi_i & + & \epsilon^w_{it} \\ y_{it} & = & \mathbf{b}'_y \cdot \mathbf{x}_i & + & \gamma_s \cdot \xi_i & + & \epsilon^y_{it} \end{cases}$$

Integrating effects out

Observations are correlated in time and multi-dimensional integration is needed

$$L_i = f(s_{i1}, w_{i1}, y_{i1}, \ldots, s_{iT}, w_{iT}, y_{iT} | \mathbf{x}_{i1}, \ldots, \mathbf{x}_{iT})$$

But correlation is just due to individual effects.
Integrate them out:

$$L_i = \int_{-\infty}^{+\infty} f(s_{i1}, w_{i1}, y_{i1} | \xi_i, \mathbf{x}_{i1}) \ldots f(s_{iT}, w_{iT}, y_{iT} | \xi_i, \mathbf{x}_{iT}) \cdot f(\xi_i) \mathrm{d}\xi_i$$

Likelihood with random effects

Likelihood function for i-th observation is:

$$L_i = \int_{-\infty}^{+\infty} \left[\prod_{t=1}^{T} f(s_{it}, w_{it}, |y_{it}, \xi_i, \mathbf{x}_{it}) \cdot f(y_{it}|\xi_i, \mathbf{x}_{it}) \right] \cdot f(\xi_i) \mathrm{d}\xi_i$$

Computing integrals

Note that

$$L_i = \int_{-\infty}^{+\infty} G(\xi_i) \cdot f(\xi_i) \mathrm{d}\xi_i$$

is an expectation of function $G(\xi_i)$:

$$L_i = E_{\xi_i} [G(\xi_i)]$$

Monte Carlo to approximate integrals.

Numerical algorithms

Numerical algorithm (*in Matlab*) to maximize *logL*.

Initial values - from model without individual effects.
Maximum search for 20 000 obs. and 40 parameters takes about 3 days.

Identification and restrictions

Restrictions

- Number of reduced form parameters is number of structural parameters minus 6
- At least 6 restrictions are needed
- Zero coefficients (some x-s that affect, for example, job status but not income)

Maximize $\log L$ by structural parameters.

Covariance Matrix of Estimators

Covariance matrix of structural parameters estimates

- Inverse negative Hessian of log L
- From gradients of log L_i
- Robust - "sandwich" of first two

Numerical approx. for derivatives.

Wald test

In structural form

$$\begin{cases} s_i^* = \alpha_{sw} \cdot w_i^* + \alpha_{sy} \cdot y_i + \beta_s' \cdot \mathbf{x}_i + \varepsilon_i^s \\ w_i^* = \alpha_{ws} \cdot s_i^* + \alpha_{wy} \cdot y_i + \beta_w' \cdot \mathbf{x}_i + \varepsilon_i^w \\ y_i = \alpha_{ys} \cdot s_i^* + \alpha_{yw} \cdot w_i^* + \beta_y' \cdot \mathbf{x}_i + \varepsilon_i^y \end{cases}$$

to estimate first equation separately from other two w_i and y_i should be independent of ε_i^s.

- Null hypothesis: $E[\varepsilon_i^s | w_i, y_i] = 0$
- Wald test for significance of the model $E[\varepsilon_i^s | w_i, y_i] = \delta_{sw}\epsilon_i^w + \delta_{sy}\epsilon_i^y$.
- Delta method to compute cov. matrix of $(\delta_{sw}, \delta_{sy})'$.

Treating Endogenous Variables

A lot of ways to treat endogenous variables when computing marginal effects. What to freeze?

Effects of income on life satisfaction
- change only in income (all other variables and shocks kept fixed).
- change in income and all other variables and shocks are free.

Unobserved Variables

To compute average marginal effects the following is needed:

- values of latent variables
- values of individual effects
- values of shocks

Generate pseudo sample according to parameters estimates.

Effects of Discrete Dependent Variable

Job status is discrete and marginal effect of latent variable $\partial \Pr\{s_{it} > 3\}/\partial w_{it}^*$ has poor interpretation.

Marginal effect of job status change:

$$\Pr\{s_{it} > 3 | w_{it} = 1\} - \Pr\{s_{it} > 3 | w_{it} = 0\}$$

Thanks For Your Attention!