Vector Optimization: seminar

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Linear program

(MOLP) in standard form

$$\begin{aligned} \textit{Maximize} \quad & Cx \\ \textit{subject to} \quad & Ax = b \\ & x \geqq 0, \end{aligned}$$

- A is a $m \times n$ real matrix and b is an m vector
- C is a $k \times n$ real matrix.

Definition

A feasible solution $\bar{x} \in X$ is a Pareto maximal solution if
$C\bar{x} \in \text{Max } C(X)$
or equivalently, a feasible solution $\bar{x}$ solves (MOLP) if there is no feasible solution $y \in X$ such that

$$C\bar{x} \leqq Cy \text{ and } C\bar{x} \neq Cy.$$

Existence

Theorem

Assume that the program (MOLP) has feasible solutions.

(MOLP) admits maximal solutions if and only if

$$C(X_\infty) \cap \mathbb{R}^k_+ = \{0\}.$$

In particular, if all asymptotic rays of X belong to the kernel of C, then (MOLP) has maximal solutions.

Example 1

Consider

$$\begin{array}{rrcl}
\textit{Maximize} & Cx = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -4 & 0 \end{pmatrix} x & & \\
 & x_1 + x_2 - x_3 & = & 5 \\
\textit{subject to} & x_1 - x_2 & = & 4 \\
 & x_1 \geqq 0, x_2 \geqq 0, x_3 \geqq & & 0.
\end{array}$$

- Compute X_∞ (the asymptotic cone of the feasible set)
- Compute $C(X_\infty)$
- Make a conclusion on the existence of efficient solutions.

Example 1

Answer:

- $X_\infty = \left\{ \begin{pmatrix} t \\ t \\ 2t \end{pmatrix} : t \geqq 0 \right\}.$
- $C(X_\infty) = \left\{ \begin{pmatrix} 3t \\ -6t \end{pmatrix} : t \geqq 0 \right\}$
- $C(X_\infty)$ has only the zero vector in common with the positive octant, hence the problem has maximal solutions.

Example 2

Consider

$$\begin{array}{rrcl} \textit{Maximize} & Cx = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x. & & \\ & x_1 + x_2 - x_3 & = & 5 \\ \textit{subject to} & x_1 - x_2 & = & 4 \\ & x_1 \geqq 0, x_2 \geqq 0, x_3 & \geqq & 0. \end{array}$$

- Compute $C(X_\infty)$
- Make a conclusion on the existence of efficient solutions.

Example 2

Answer:

- $C(X_\infty) = \left\{ \begin{pmatrix} 0 \\ 2t \end{pmatrix} : t \geqq 0 \right\}$
- $C(X_\infty)$ has a nonzero vector in common with the positive octhant, hence the problem has no maximal solutions.

Normal cone

$$N_X(a) = \{u : \langle u, x - a \rangle \leq 0 \; \forall x \in X\}$$

Existence

Theorem

Let $\bar{x}$ be a feasible solution of (MOLP).
Then $\bar{x}$ is a maximal solution if and only if the normal cone $N_X(\bar{x})$ to X at $\bar{x}$ contains some vector $C^T\lambda$ with λ a strictly positive vector of $\mathbb{R}^k$.

Example

Consider the bi-objective program

$$\begin{array}{ll} \text{Maximize} & \begin{pmatrix} 1 & 0 & 1 \\ -2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \text{subject to} & \begin{aligned} x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 &= 4 \\ x_1 \geqq 0, x_2 \geqq 0, x_3 &\geqq 0. \end{aligned} \end{array}$$

Consider a feasible solution $\overline{x} = (9/2, 1/2, 0)^T$.

- Compute the normal cone to X at $\overline{x}$.
- Show that there is a strictly positive vector λ such that $C^T\lambda \in N_X(\overline{x})$.
- Make a conclusion on the existence of efficient solutions.

Example

The normal cone to the feasible set at $\overline{x}$ is the positive hull of the hyperplane of basis $\{(1,1,-1)^T,(1,-1,0)^T\}$ (the row vectors of the constraint matrix) and the vector $(0,0,-1)^T$ (the constraint $x_3 \geqq 0$ is active at this point).
In other words, this normal cone is the half-space determined by inequality

$$x_1 + x_2 + 2x_3 \leqq 0. \tag{1}$$

The image of the positive octant of the value space $\mathbb{R}^2$ under C^T is the positive hull of the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}.$$

We deduce that the vector v_2 lies in the interior of the normal cone to the feasible set at $\overline{x}$. Hence that normal cone does contain a

Example 2

Same question for the feasible solution $\bar{z} = (5, 1, 1)^T$.

Example 2

The normal cone to the feasible set at $\bar{z}$ is the hyperplane determine by equation

$$x_1 + x_2 + 2x_3 = 0.$$

Direct calculation shows that the vectors v_1 and v_2 lie in different sides of the normal cone at $\bar{z}$. Hence there does exist a strictly positive vector λ in $\mathbb{R}^2$ such that $C^T\lambda$ is contained in that cone. Consequently the solution $\bar{z}$ is maximal too.

Corley's dual

Dual (VD)

$$\begin{array}{rl} \text{Minimize} & Yb \\ \text{subject to} & YA \geqq C \end{array}$$

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- Show the weak duality relation $Cx \leqq Yb$

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Dual (VD)

$$\begin{array}{rl} \text{Minimize} & Yb \\ \text{subject to} & YA \geqq C \end{array}$$

- This problem is linear
- Show the weak duality relation $Cx \leqq Yb$
- Find conditions for zero duality gap $Cx = Yb$.

Example: Primal program

Consider a multiobjective problem

$$\begin{aligned} &\text{Maximize} && \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\text{subject to} && \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & && x_1 \geqq 0, x_2 \geqq 0. \end{aligned}$$

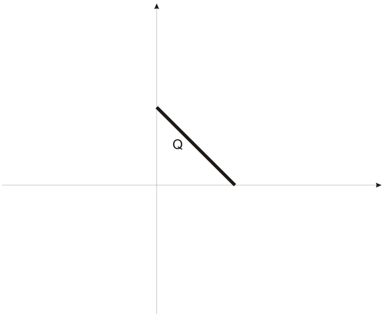

Figure : The feasible set of the primal problem

Corley's dual

$$\text{Minimize} \quad \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \geqq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

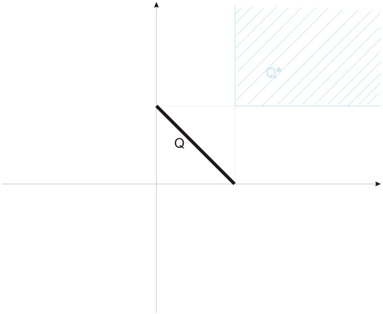

Figure : The value set of Corley's dual