

# **The Stackelberg Games in Facility Location**

**Yury Kochetov**

**Sobolev Institute of Mathematics. Novosibirsk. Russia**

***The 3rd International Conference on Network Analysis***

***May 20 – 22, 2013 • Nizhny Novgorod, Russia***

# Contents

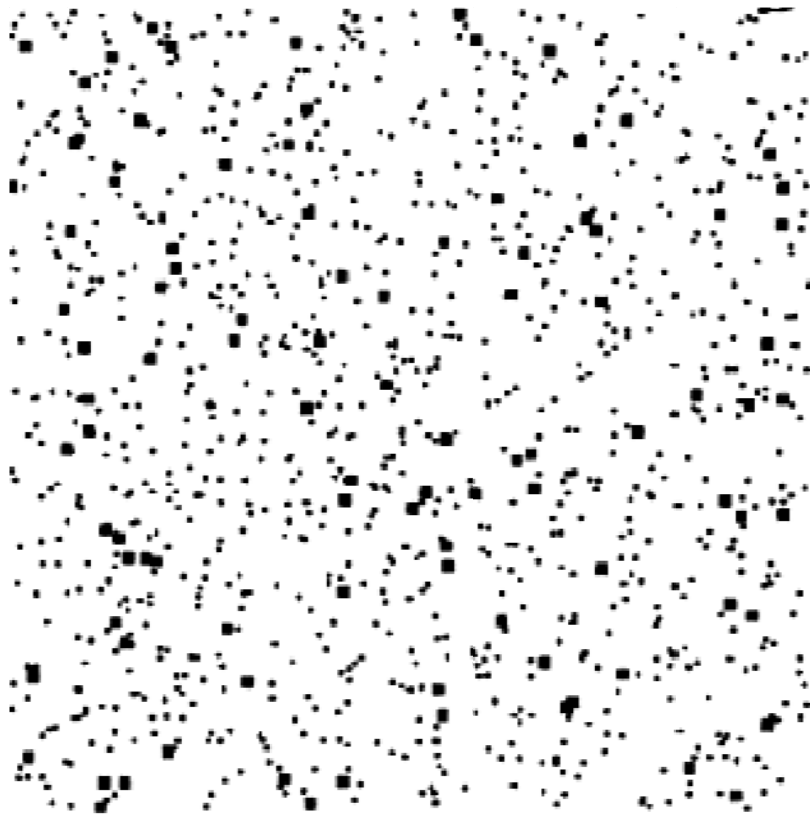
1. Introduction
2. The leader–follower location problem
3. Theoretical and empirical results
4. Extensions of the basic model
5. Application in telecommunication
6. Conclusions

# The $p$ -median problem

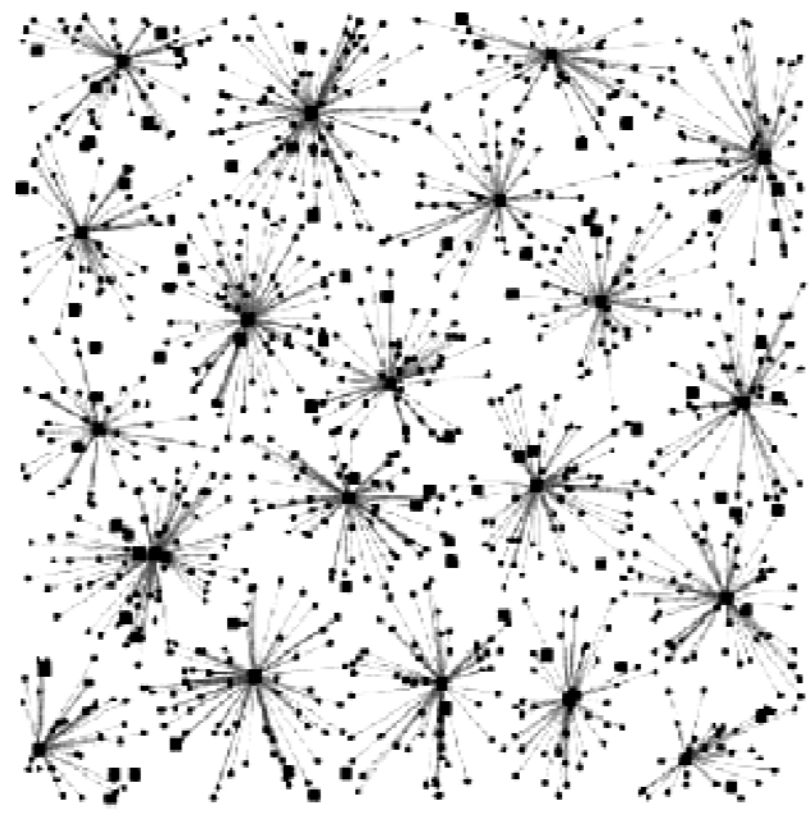
- Input:  $J$  is the set of clients;  
 $I$  is the set of potential facilities;  
 $c_{ij}$  is the distance for servicing client  $j$  from facility  $i$ ;  
 $p$  is the number of opening facilities;
- Goal: to find a set  $S \subset I$ ,  $|S| = p$  of opened facilities in such way to minimize the total distance from the facilities to clients:

$$\min_{|S|=p} \left\{ \sum_{j \in J} \min_{i \in S} c_{ij} \right\}$$

**Example**     $|I| = 100$ ;  $|J| = 1000$



Instance



Solution

# The Leader-Follower Location Problem

- **Input:**
  - $J$  is the set of clients;
  - $I$  is the set of potential facilities;
  - $w_j$  is the demand of client  $j$ ;
  - $c_{ij}$  is the distance from client  $j$  to facility  $i$ ;
  - $p$  is the number of leader facilities;
  - $r$  is the number of follower facilities.

Each client patronizes the closest opened facility.

- **Output:** a set  $S \subset I$ ,  $|S| = p$  of opening facilities by the leader.
- **Goal:** maximize the market share of the leader anticipating that the follower will react to the decision by opening his own  $r$  facilities.

# Decision variables

$$x_i = \begin{cases} 1 & \text{if the leader opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if the follower opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if client } j \text{ patronizes a leader facility} \\ 0 & \text{if client } j \text{ patronizes a follower facility} \end{cases}$$

For given solution  $x$  we introduce the set

$$I_j(x) = \{i \in I \mid c_{ij} < \min_{l \in I} c_{lj} \mid x_l = 1\}$$

of facilities which allow the follower to “capture” client  $j$ .

# The Bi-Level 0-1 Linear Program

$$\max_x \sum_{j \in J} w_j z_j^*(x)$$

s.t.

$$\sum_{i \in I} x_i = p, \quad x_i \in \{0, 1\}, i \in I,$$

where  $z_j^*$  is optimal solution of the Follower problem

$$\max_{z, y} \sum_{j \in J} w_j (1 - z_j)$$

s.t.

$$1 - z_j \leq \sum_{i \in I_j(x)} y_i, \quad j \in J;$$

$$\sum_{i \in I} y_i = r, \quad x_i + y_i \leq 1, i \in I, \quad y_i, z_j \in \{0, 1\}.$$

# Theoretical and Empirical Results

- ♦  $\sum_2^P$ —hard problem even for Euclidean distances (I. Davydov, E. Carrizosa, Yu. Kochetov, 2012)
- ♦ The follower problem is NP—hard in the strong sense (I. Davydov, E. Carrizosa, Yu. Kochetov, 2012)
- ♦ Polynomially solvable cases (J. Spoerhase, H.C. Wirth, H. Noltemeir, 2007)
- ♦ The branch and cut method (M.C. Roboredo, A.A. Pessoa, 2012)
- ♦ An iterative exact method (E. Alekseeva, Yu. Kochetov, A. Plyasunov)
- ♦ Metaheuristics (E. Alekseeva et al. 2010; D. Serra, C. ReVelle, 1995; I. Davydov, 2012; J.A. Moreno Perez et al., 2009)



# Exact method

## Decision variables

$$x_i = \begin{cases} 1 & \text{if the leader opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$z_{ij} = \begin{cases} 1 & \text{if the leader facility } i \text{ is closest to client } j \\ 0 & \text{otherwise} \end{cases}$$

$D$  is the market share of the leader

Notations:

$\mathcal{F}$  is nonempty family of follower solutions.

For  $y \in \mathcal{F}$  we define the set  $I_j(y)$  of the facilities which allow to the leader saving client  $j$ :

$$I_j(y) = \{i \in I \mid c_{ij} \leq \min_{l \in I} c_{lj} \mid y_l = 1\}$$

# The Single Level Reformulation

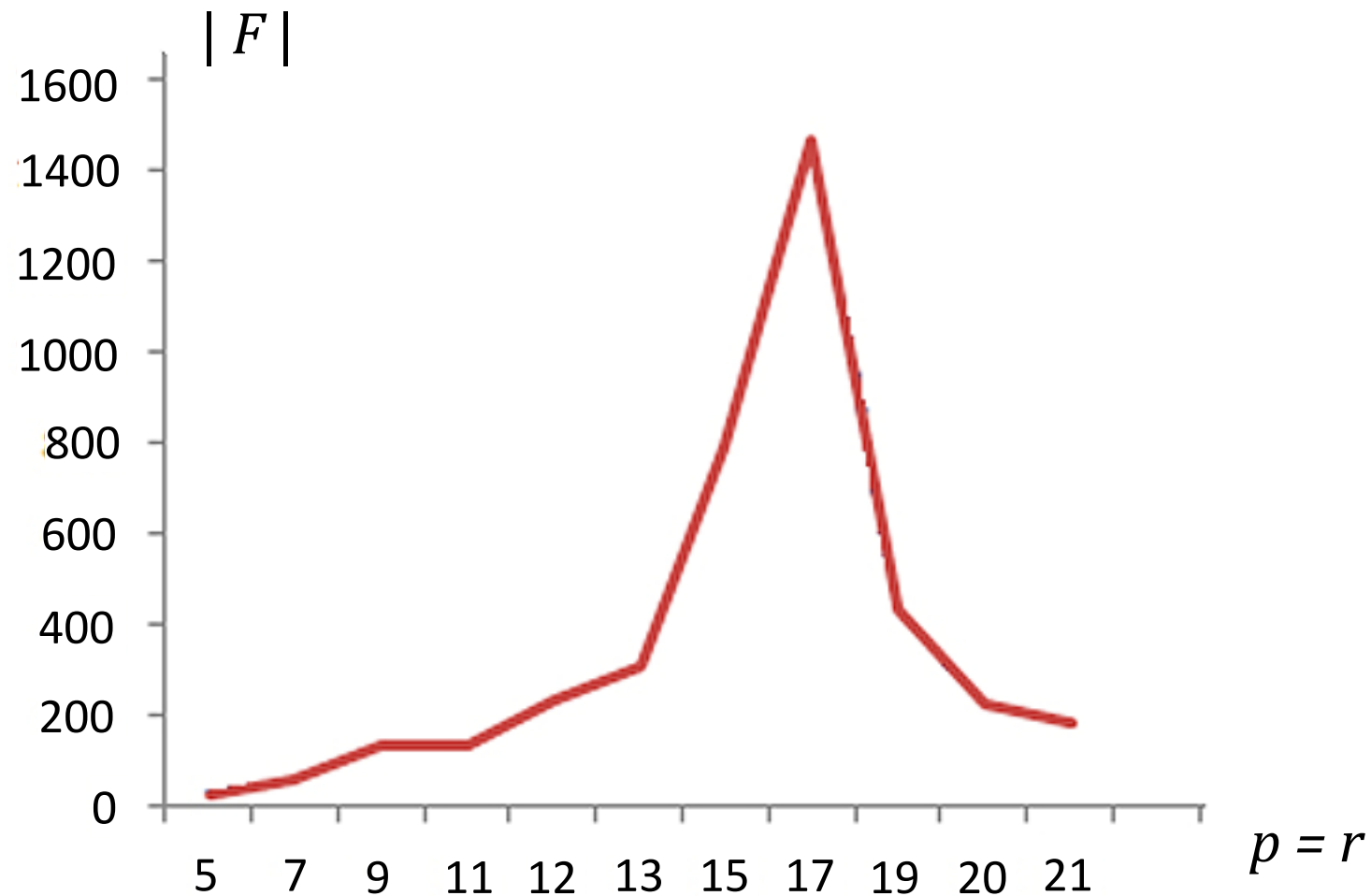
$$\begin{aligned} & \max D \\ \text{s.t.} \quad & \sum_{j \in J} \sum_{i \in I_j(y)} w_j z_{ij} \geq D, \quad y \in \mathcal{F} \\ & \sum_{i \in I} z_{ij} = 1, \quad j \in J \\ & x_i \geq z_{ij}, \quad i \in I, j \in J \\ & \sum_{i \in I} x_i = p \\ & x_i, z_{ij} \in \{0,1\}, \quad D \geq 0 \end{aligned}$$

If  $\mathcal{F}$  contains all follower solutions, we have an equivalent reformulation.

# Iterative Exact Method

0. Choose an initial subfamily  $F \in \mathcal{F}$  and put  $D^* := 0$ .
1. Solve the problem with  $F$  instead of  $\mathcal{F}$  and find  $x(F)$  and upper bound  $D(F)$ .
2. Solve the follower problem for  $x(F)$  and find  $y(F)$  and lower bound  $D(y)$ .
3. If  $D^* < D(y)$  then  $D^* := D(y)$ .
4. If  $D^* = D(F)$  then STOP.
5. Include  $y(F)$  into the subfamily  $F$  and go to 1.

The total number of iterations depending on the parameters  $p$  and  $r$ ,  
 $n = m = 50$ , class *Euclidean*



# The Leader-Follower Facility Location and Design Problem

**Leader** enters in a market by opening own facilities.

**Follower** already has own facilities and reacts by opening new facilities, closing existing ones, and adjusting the attractiveness of its existing facilities.

**Each client** patronizes a facility proportionally to the attractiveness of the facility and inversely proportionally to the distance between client and the facility (Huff's gravity-based rule).

**The objective** of each firm is to find out the optimal location and attractiveness of the facilities in such a way that its own profit is maximized.

# Parameters

$J = \{1, \dots, n\}$  is the set of clients;

$I = \{1, \dots, m\}$  is the set of candidate facilities of the leader;

$K = \{1, \dots, r_1\}$  is the set of existing facilities of the follower;

$L = \{1, \dots, r_2\}$  is the set of candidate facilities of the follower;

# Parameters

$w_j$	buying power of client $j$
$c_i$	unit attractiveness cost of leader's facility $i$
$e_l$	unit attractiveness cost of follower's facility $l$
$b_k$	unit cost of changing attractiveness of follower's facility $k$
$f_i$	fixed cost of opening facility $i$ by the leader
$\tilde{f}_l$	fixed cost of opening facility $l$ by the follower
$t_k$	revenue of closing an existing facility $k$
$U_i$	maximal attractiveness of leader's facility $i$
$\overline{M}_l$	maximal attractiveness of follower's facility $l$
$\overline{A}_k$	maximal attractiveness of existing follower's facility $k$
$\underline{A}_k$	current attractiveness of existing follower's facility $k$

# Decision Variables

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is opened by the leader} \\ 0 & \text{otherwise} \end{cases}$$

$Q_i$  is attractiveness of facility  $i$  of the leader;

$$z_k = \begin{cases} 1 & \text{if existing facility } k \text{ is kept open by the follower} \\ 0 & \text{otherwise} \end{cases}$$

$A_k$  new attractiveness of existing facility  $k$ ;

$$y_l = \begin{cases} 1 & \text{if new facility } l \text{ is opened by the follower} \\ 0 & \text{otherwise} \end{cases}$$

$M_l$  is attractiveness of new facility  $l$  of the follower.



# The gravity based rule

$Q_i/d_{ij}^2$  is the utility of facility  $i$  with attractiveness  $Q_i$  for client  $j$ ;

$\sum_{k \in K} \frac{A_k}{\bar{d}_{kj}^2} + \sum_{l \in L} \frac{M_l}{\tilde{d}_{lj}^2}$  is the total utility of the follower facilities for client  $j$ ;

The probability that client  $j$  visit a facility  $i$  is expressed as

$$p_{ij} = \frac{Q_i / d_{ij}^2}{\sum_{i \in I} Q_i / d_{ij}^2 + \sum_{k \in K} A_k / \bar{d}_{kj}^2 + \sum_{l \in L} M_l / \tilde{d}_{lj}^2}$$

# Bi-Level Model

$$\max_{x, Q} \sum_{j \in J} w_j \sum_{i \in I} p_{ij} - \sum_{i \in I} f_i x_i - \sum_{i \in I} c_i Q_i$$

$$\text{s.t.} \quad Q_i \leq U_i x_i, \quad i \in I;$$

$$Q_i > 0, \quad x_i \in \{0, 1\}, \quad i \in I;$$

$$\begin{aligned} \max_{z, y, A, M} \quad & \sum_{j \in J} w_j (1 - \sum_{i \in I} p_{ij}) + \sum_{k \in K} t_k (1 - z_k) - \sum_{k \in K} b_k (A_k - \bar{A}_k z_k) - \\ & \sum_{l \in L} e_l M_l - \sum_{l \in L} \tilde{f}_l y_l \end{aligned}$$

$$\text{s.t.} \quad A_k \leq \bar{A}_k z_k, \quad k \in K;$$

$$M_l \leq \bar{M}_l y_l, \quad l \in L;$$

$$A_k \geq 0, \quad M_l \geq 0, \quad z_k, y_l \in \{0, 1\}, \quad k \in K, l \in L.$$

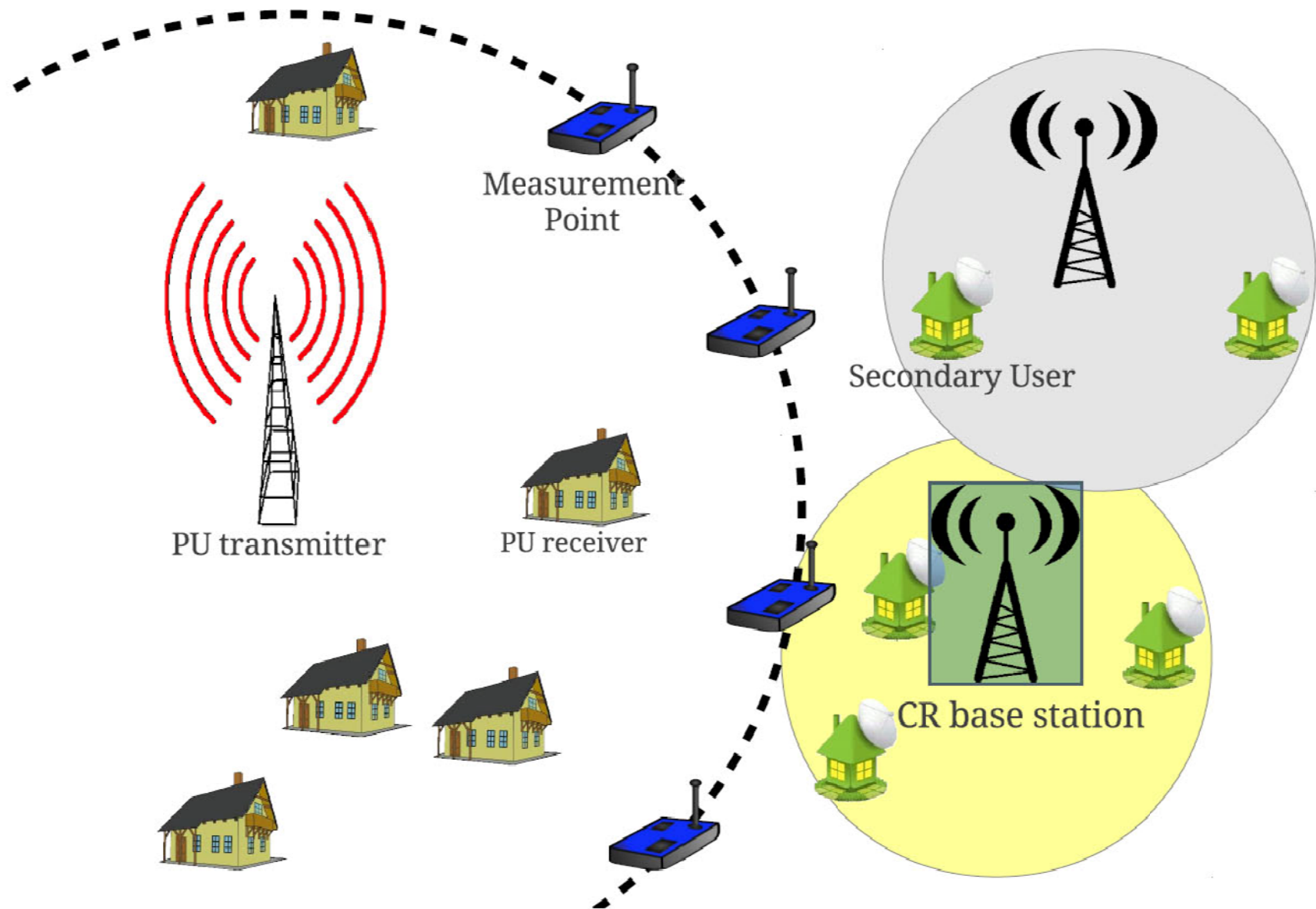
# Strategic Planning in Cognitive Radio Networks

We consider a primary network operating on a set of frequency bands.

A cognitive radio operator (the leader) wants to deploy a durable secondary network by opportunistically using the unused capacity of the primary network. To this end, the operator places a set of own base stations and tunes the correspondent transmission power so as to maximize the profit drawn from the served clients.

The operator has to:

- ♦ ensure that the deployment of the secondary network does not impair the primary network;
- ♦ pay for each base station under the budget constraint;
- ♦ find a solution which will be robust face to the arrival of a possible competitor (the follower).



# Leader Problem

$$LP : \max \left\{ \sum_{j \in N} w_j \sum_{i \in B_s} \sum_{c \in C} x_{ijc} - \lambda \sum_{i \in B_s} \sum_{c \in C} x_{ic} \right\}$$

$$s.t. \quad \sum_{i \in B_s} \sum_{c \in C} (x_{ijc} + y_{ijc}^*) \leq 1 \quad j \in J$$

$$\sum_{i \in B_s} \sum_{c \in C} x_{ic} \leq K$$

$$\sum_{c \in C} x_{ic} \leq 1 \quad i \in B_s$$

$$P_{ic} \leq P_{max} x_{ic} \quad (i, c) \in B_s \times C$$

$$x_{ijc} \leq x_{ic} \quad (i, j, c) \in B_s \times J \times C$$

$$\sum_{i \in B_s} P_{ic} h_{imc} \leq \bar{I}_{mc} \quad (m, c) \in B_m \times C$$

$$P_{ic} g_{ijc}^i \geq \bar{\gamma} \cdot \sum_{l \in B_s, l \neq i} P_{lc} g_{ljc}^i + \bar{\gamma} \cdot \sum_{r \in B_s} Q_{rc}^* g_{rjc}^i +$$

$$+ \bar{\gamma} \cdot \sum_{k \in B_p} H_{kc} l_{kjc}^i + \bar{\gamma} N_c - \Gamma(1 - x_{ijc}) \quad (i, j, c) \in B_s \times J \times C$$

$$x_{ic}, x_{ijc} \in \{0,1\}, \quad P_{ic} \geq 0$$

# Follower Problem

$$\begin{aligned}
 FP : \max & \left\{ \sum_{j \in N} w_j \sum_{i \in B_s} \sum_{c \in C} y_{ijc} - \lambda \sum_{i \in B_s} \sum_{c \in C} y_{ic} \right\} \\
 s.t. & \sum_{c \in C} (x_{ijc} + y_{ijc}) \leq 1 \quad i \in B_s \\
 & \sum_{c \in C} y_{ic} \leq 1 \quad i \in B_s \\
 & y_{ijc} \leq y_{ic} \quad (i, j, c) \in B_s \times J \times C \\
 & Q_{ic} \leq Q_{max} y_{ic} \quad (i, c) \in B_s \times C \\
 & \sum_{i \in B_s} Q_{ic} h_{imc} \leq \bar{I}_{mc} - I_{mc} \quad (m, c) \in B_m \times C
 \end{aligned}$$

$$\begin{aligned}
Q_{ic}g_{ijc}^i &\geq \bar{\gamma} \cdot \sum_{r \in B_s, r \neq i} Q_{rc}g_{rjc}^i + \bar{\gamma} \cdot \sum_{l \in B_s} P_{lc}g_{ljc}^i + \\
&+ \bar{\gamma} \cdot \sum_{k \in B_p} H_{kc}l_{kjc}^i + \bar{\gamma}N_c - \Gamma(1 - y_{ijc}) \quad (i, j, c) \in B_s \times J \times C
\end{aligned}$$

$$Q_{ic}g_{ijc} \geq Q_{rd}g_{rjd} - \Gamma(1 - y_{ijc}) \quad (i, r) \in B_s, j \in J, (c, d) \in C$$

$$Q_{ic}g_{ijc} \geq P_{ld}g_{ljd} - \Gamma(1 - y_{ijc}) \quad (i, l) \in B_s \quad j \in J \quad (c, d) \in C$$

$$y_{ic}, y_{ijc} \in \{0,1\}, \quad Q_{ic} \geq 0$$






**Theorem 1.** The leader problem is  $\Sigma_2^P$ -hard.

**Theorem 2.** The follower problem is NP-hard in the strong sense.

We design a hybrid stochastic tabu search algorithm for this Stackelberg game. At each step, we solve the mixed integer program derived from the follower problem by CPLEX software.

# Conclusions

-  Bilevel facility location models are presented
-  Recent results are reviewed
-  New interesting models can be obtained:
  - using detail models for user behavior;
  - continuous locations;
  - prices and others.