Vector Optimization: basic concepts and numerical methods

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What is optimization

Mathematical optimization studies the problem of selecting a best element from a set X of available / feasible alternatives with regard to a criterion/ cost/ objective function f which is written as

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

Three examples

Queen Dido's problem.

Carthage City

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Find a territory bounded by a line which has the maximum area for a given perimeter.

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However, as it is inconceivable of a city touching the sea without seashore, the queen Dido set another objective for her territory to have a maximum seashore.

$\Rightarrow$ a half-circle "partially" meets her two objectives.

Three examples

House purchase problem.

Three houses A, B and C (of the same price that fits our budget).
Three criteria to evaluate: appearance, comfort and environment.
Here is the table of our evaluation (score from 0 to 5):

	A	B	C
Appearance	3	3	5
Comfort	4	4	4
Environment	5	4	3

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- The offer B is elliminated
- A and C are not comparable.
  At this stage, it is impossible to say which one is the best with regard to the three criteria.

Three examples

Power economic dispatch problem.

This problem involves allocation of generations to different thermal units to

- minimize the cost of generation

$$f_1 = \sum_{i=1}^{k} (a_i P_i^2 + b_i P_i + c_i)$$

- P_i is the generation of the ith generator

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- k is the number of the generators.

Three examples

- minimize the emission of gazes (sulfur dioxide and nitrogen oxides) causing atmospheric hazards

$$f_2 = \sum_{i=1}^{k} [10^{-2}(\alpha_i P_i^2 + \beta_i P_i + \gamma_i) + \xi_i exp(\zeta_i P_i)]$$

where $\alpha_i, \beta_i, \gamma_i, \xi_i, \zeta_i$ are coefficients emission characteristics.

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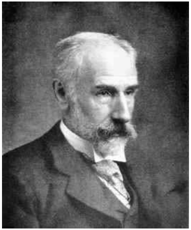
- under constraints

- $\sum_{i=1}^{k} P_i = P_D + P_{loss}$
(P_D - the total demand, P_{loss} - the real power loss in the transmission lines).
- $P_i^{min} \leq P_i \leq P_i^{max}$ (the limits on the loading of the ith generator).

How to make a choice



F.Y. Edgeworth (1845-1926), Irish economist

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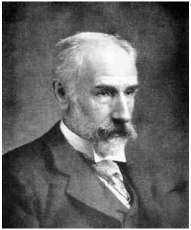

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Optimum for the multiutility problem within the context of two consumers, A and B:
"Find a point (x, y) such that in whatever direction we take an infinitely small step, A and B do not increase together but that, while one increases, the other decreases."

How to make a choice

V. Pareto (1848 - 1923), Italian economist

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Pareto optimum: "The optimum allocation of the resources of a society is not attained so long as it is possible to make at least one individual better off in his own estimation while keeping others as well off as before in their own estimation."

Partial order

Definition

Let R be a binary relation on E, that is R is a subset of $E \times E$. It is said to be a partial order on E if it is

- reflexive: $(x, x) \in R \ \forall \ x \in E$;
- transitive: $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$.
- antisymmetric: $(x, y), (y, x) \in R \Rightarrow x = y$.

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If E is a vector space, a partial order R is said to be compatible with the linear structure of the space if
$(x, y) \in R \Rightarrow (x + z, y + z), (tx, ty) \in R \; \forall \, z \in E, \; t > 0$.

Partial order

Cone representation of partial orders:

Theorem

If a partial order $R \subseteq E \times E$ is compatible with the linear structure, then the set

$$C := \{x \in E : (x, 0) \in R\}$$

is a pointed convex cone in E.

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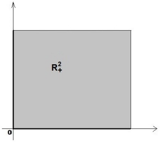
Conversely, if C is a pointed convex cone in E, then the relation R defined by

$$(x, y) \in R[x \geq_C y] \Leftrightarrow x - y \in C$$

is a partial order compatible with the linear structure in E.

Partial orders: examples

The Pareto cone $\mathbb{R}^k_+$ (the positive octant of $\mathbb{R}^k$).
This cone is convex, closed and pointed.



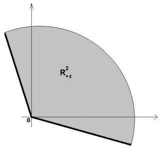
The Pareto order: $x \geq_{\mathbb{R}^k_+} y \Longleftrightarrow x_i \geq y_i, i = 1, \cdots, k.$

Partial orders:examples

The ε-extended Pareto cone.

$$\mathbb{R}^k_{+\epsilon} = \{x \in \mathbb{R}^k : (\epsilon e + e^i)x^T \geq 0, i = 1, ..., k\}$$

where $e = (1, ..., 1)$ and e^i is the ith unit vector.

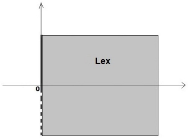


The corresponding order: $x \geq_{\mathbb{R}^k_{+\epsilon}} y \Leftrightarrow$
$\min\{x_i - y_i : i = 1, ..., k\} + \epsilon \sum_{i=1}^k (x_i - y_i) \geq 0.$

Partial order

The lexicographic cone:

$C_{lex} = \{$ vectors whose first nonzero component is strictly positive$\}$.

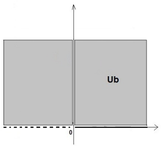


The lexicographic order: $x \geq_{Lex} y \Leftrightarrow$
$\exists j \in \{0, 1, ..., k\} : x_i = y_i \ \forall i = 1, ..., j \text{ and } x_{j+1} > y_{j+1}$.

Partial order

The ubiquitous cone:

$Ub = \{$ vectors whose last nonzero component is strictly positive$\}$.



The ubiquitous order: $x \geq_{Ub} y \Leftrightarrow$
$\exists\, j \in \{0, 1, ..., k\}$ such that $x_j > y_j$ and $x_i = y_i$ for $i = j+1, ..., k$.

Partial order

Conic extended cone. The ϵ-conic neighborhood of a cone C:

$$C_\epsilon = \{x \in \mathbb{R}^k : d(x, C) \leq \epsilon\|x\|\},$$

where $d(x, C)$ is the distance from x to C.
The order generated by C_ϵ is called the ϵ-conic extended order of the order "$\geq_C$".
In $\mathbb{R}^2$, the δ-conic extended order of the Pareto cone $\mathbb{R}^2_+$ coincides with the ϵ-extended Pareto order, where $\delta = \epsilon/\sqrt{\epsilon^2 + (1+\epsilon)^2}$.

Partial order

Correct cones: We say that a cone C in $\mathbb{R}^k$ is correct if $\mathrm{cl}\, C + C \setminus \ell(C) \subseteq C$ or equivalently $\mathrm{cl}\, C + C \setminus \ell(C) \subseteq C \setminus \ell(C)$. Here $\ell(C) = C \cap (-C)$.

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Some cases of correct cones.

- Every closed and convex cone is correct;
- If $C \setminus \ell(C)$ is open, then C is correct;
- If C consists of the origin and an intersection of half-spaces that are either open or closed, then C is correct.

Partial order

The Positive polar cone:

$$C' = \{\xi \in E' : \langle \xi, x \rangle \geq 0 \forall x \in C\}$$

The strictly positive polar cone:

$$C^+ = \{\xi \in E' : \langle \xi, x \rangle > 0 \forall x \in C, x \neq 0\}$$

Increasing sequence

Definition. A sequence $\{x^i\}_{i\geq 1}$ of elements in $\mathbb{R}^k$ is said to be increasing if $x^{i+1} \geq_C x^i$ for every $i = 1, 2, \ldots$ and it is strictly increasing if the above inequality is strict.

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Theorem

Assume that the order "$\geq_C$" is correct. Then the limit of a convergent strictly increasing sequence strictly dominates the terms of the sequence.

Monotone functions

Definition. $g : \mathbb{R}^k \to \mathbb{R}$ is increasing w.r. to "$\geq_C$" if

$$a >_C b \Rightarrow g(a) > g(b).$$

g is weakly increasing if it is increasing w.r. to "$\geq_{C^0}$", where $C^0 = \{0\} \cup \mathrm{int}(C)$.

Linear functions: g is linear increasing / weakly increasing $\Leftrightarrow g(x) = \langle \xi, . \rangle$ for some $\xi \in C^+$ / $\xi \in C' \setminus \{0\}$.

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Biggest weakly increasing functions: for $a \in \mathbb{R}^k, v \in \mathrm{int}\ (C)$,

$$h_{a,v}(x) = \sup\{t \in \mathbb{R} : x \in a + tv + C\}$$

is weakly increasing, and for every weakly increasing function g with $g(a) = 0$, we have inclusion of lower level sets

$$lev_g(a) \subseteq lev_{h_{a,v}}(a)$$

where $lev_g(a) := \{x \in \mathbb{R}^k : g(x) \leq g(a)\}$.

Pareto maximality

Definition

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Notations: $\mathrm{IMax}(A)$ or $\mathrm{IMax}(A|C)$; $\mathrm{Max}(A)$ or $\mathrm{Max}(A|C)$

Pareto maximality

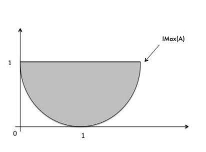
The sets $\mathrm{IMin}(A)$ and $\mathrm{Min}(A)$ are defined similarly.

$$\begin{aligned}\mathrm{IMin}(A|C) &= \mathrm{IMax}(A|-C)\\ \mathrm{Min}(A|C) &= \mathrm{Max}(A|-C).\end{aligned}$$

Pareto maximality

Example

Consider the set A (the lower half-disc)

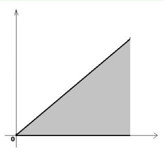


Then $\mathrm{IMax}(A|\mathbb{R}^2_+) = \mathrm{Max}(A|\mathbb{R}^2_+) \neq \emptyset$.

Pareto maximality

Example

With this new ordering cone



$\mathrm{IMax}(A|C) = \emptyset$
$\mathrm{Max}(A|C) = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = 1, 1.5 \leq x \leq 2\}.$

Pareto maximality

Equivalent definition of Pareto maximality:
Let A be a nonempty set in $\mathbb{R}^k$. Then

- $a \in \mathrm{IMax}(A) \Leftrightarrow a \in A$ and $A \subseteq a - C$;
- $a \in \mathrm{Max}(A) \Leftrightarrow a \in A$ and $A \cap (a + C \setminus \{0\}) = \emptyset$.
- If $\mathrm{IMax}(A|C)$ is nonempty, then it is a singleton and $\mathrm{IMax}(A|C) = \mathrm{Max}(A|C)$.

Maximality with respect to extended orders

Extended order: " $\geq_{\widetilde{C}}$" (where $\widetilde{C}$ is a pointed convex cone) is an extended order of the order " $\geq_C$" if

$$x \geq_C y \Rightarrow x \geq_{\widetilde{C}} y$$

(equivalently $C \subseteq \widetilde{C}$.)

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Theorem

Let A be a nonempty set and $\widetilde{C}$ a pointed and convex cone containing C. The following assertions hold

- $\mathrm{IMax}(A|C) = \mathrm{IMax}(A|\widetilde{C})$ *if* $\mathrm{IMax}(A|C)$ *is nonempty.*
- $\mathrm{Max}(A|C) \supseteq \mathrm{Max}(A|\widetilde{C})$

Maximality of sections

Section: A section of A at x: $A_x := A \cap (x + C)$.

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Theorem

For every $x \in \mathbb{R}^k$ we have

$$\mathrm{Max}(A_x) \subseteq \mathrm{Max}(A) = \bigcup_{y \in A} \mathrm{Max}(A_y).$$

Proper maximality and weak maximality

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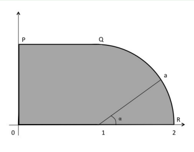
- $a \in \mathrm{PrMax}(A|C)$ if $a \in \mathrm{Max}(A|\widetilde{C})$ for some pointed and convex cone $\widetilde{C}$, such that $cl(C)\backslash\{0\} \subseteq \mathrm{int}(\widetilde{C})$.

Proper maximal points

Example

Consider the set A



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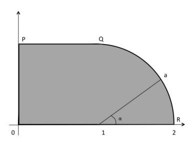

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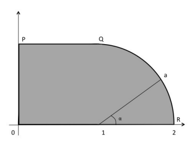

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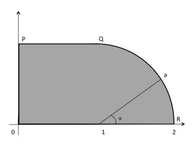


$\mathrm{WMax}(A) = [P, Q] \cup \overset{\frown}{QR}$.
$\mathrm{Max}(A) = \overset{\frown}{QR}$.
$\mathrm{PrMax}(A) = \mathrm{Max}(A) \setminus \{Q, R\}$.

Maximality

Relationship between proper maximality, maximality and weak maximality.

Theorem

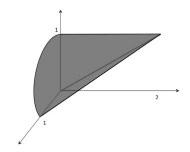
For a nonempty set A in $\mathbb{R}^k$ one has the inclusions

$$\mathrm{PrMax}(A) \subseteq \mathrm{Max}(A) \subseteq \mathrm{WMax}(A).$$

Proper maximal point and weak maximal point

Example

A is the convex hull of the quarter disc $D := \{(x,0,z) \in \mathbb{R}^3_+ : x^2 + z^2 \leq 1\}$ and the point $(0,2,1)$.



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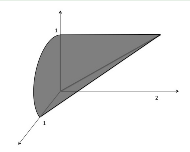

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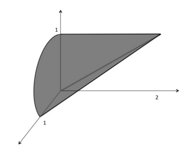

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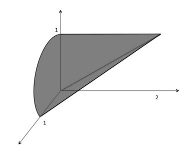

$[(0, 0, 1); (0, 2, 1))$ is weak maximal, but not maximal
$\mathrm{Max}(A)$ is not closed
The point $(1, 0, 0)$ is maximal but not proper maximal

A set without proper maximal points

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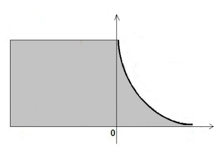

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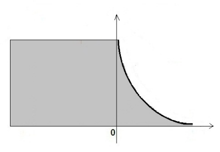

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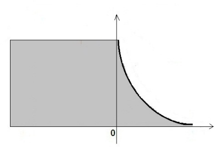

$\mathrm{Max}(A)$ is the graph of the function $\frac{1}{x}, x > 0$.
$\mathrm{PrMax}(A) = \emptyset$

Maximal points of free disposal hulls

Free disposal set. $A \subseteq \mathbb{R}^k$ is free disposal if $A = A - C$.
$A - C$ is called the free disposal hull of A. It is the smallest free disposal set that contains A.

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Theorem

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$A - C$ is called the free disposal hull of A. It is the smallest free disposal set that contains A.

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- $\mathrm{WMax}(A) = A \cap \mathrm{WMax}(A - C)$.

The main theorems

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Generalization to order-complete sets

Order complete set. A set $A \subseteq \mathbb{R}^k$ is said to be $C-$complete (resp. strongly $C-$complete) if it has no covering of the form

$$\{(x^\nu + \text{ cl } C)^c : \nu \in \mathbb{N}\} \quad \left(\text{resp. } \{(x^\nu + C)^c : \nu \in \mathbb{N}\}\right)$$

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REMARK:
every strongly $C-$complete set is $C-$complete. The converse is not always true. When C is a closed cone these two concepts coincide.

Existence

Example

Consider the set

$$A = \{(\nu, 0) \in \mathbb{R}^2 : \nu \in \mathbb{N}\}$$

and equip $\mathbb{R}^2$ with the ubiquitous cone *Ub*.
Then A is *Ub*-complete, but not strongly *Ub*-complete.

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Theorem

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If the cone C is correct and the set A admits a nonempty C-complete section, then the set A has maximal points.

In particular, if the cone C is correct and the set $A - C$ has a nonempty compact section, then A has maximal points.

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Existence condition by monotone functions.

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Assume that the strictly positive polar cone C^+ of C is nonempty. A nonempty set A has a maximal point if and only if it has a compact section.

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In particular, if the closure of the cone C is pointed, then every compact set has maximal points.
REMARK: a more subtle proof shows that the conclusion remains true without any condition on the cone C. This fact is, however, not true in infinite dimension.

Existence

Linear case:

Theorem

If A is a polyhedral convex set and C is a polyhedral cone, then $x \in \mathrm{Max}(A \mid C)$ if and only if these is some $\xi \in C^+$ such that

$$\langle \xi, x \rangle \geq \langle \xi, x' \rangle, \text{ for all } x' \in A.$$

Consequently the set $\mathrm{Max}(A \mid C)$ consists of faces of A.

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- $x \in \mathrm{WMax}(A|C)$;
- $x \in \mathrm{Max}(A|C)$ *if it is a unique maximum of* $\langle \xi, \cdot \rangle$ *on* A;
- $x \in \mathrm{PrMax}(A|C)$ *if* $\xi \in (cl(C))^+$.

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- *If* $x \in \mathrm{Max}(A|C)$, *then there is some* $\xi \in C' \backslash \{0\}$ *such that* x *is a maximum of* $\langle \xi, \cdot \rangle$ *on* A.

Vector problem

Consider (MOP) (vector/ multiple objective optimization problem):

$$\begin{array}{ll} \text{Max} & f(x) \\ \text{subject to} & x \in X \end{array}$$

with $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^k$.
x solves (MOP) (efficient / maximal solution)

$$\Leftrightarrow f(x) \in \text{Max}(f(X)|C).$$

Vector problem

A typical condition for existence of efficient solutions:

Theorem

Assume that $\mathrm{cl}(C)$ is pointed, X is a nonempty compact set and f is a continuous function. Then the problem (MOP) has efficient solutions.

Scalarization

A frequently used method in the study of (MOP) is to convert it into a scalar optimization problem of the form (P)

$$\begin{array}{ll} \max & g \circ f(x) \\ \text{subject to} & x \in X \end{array}$$

where $g : f(X) \to \mathbb{R}$ is a scalarizing function.

General case.

Relationship between the solution set $S(P)$ of problem (P) and the solution set $S(MOP)$ of (MOP).

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- If (MOP) is linear, then $\exists \xi^i \in (clC)^+, i = 1, ..., m$ such that

$$S(MOP) = \bigcup_{i=1}^{m} S(P_{\xi^i}).$$

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$$PrS(MOP) = \cup_{\xi \in (clC)^+} S(P_\xi) \subseteq WS(MOP) = \cup_{\xi \in C' \setminus \{0\}} S(P_\xi).$$

- For a given $v \in \mathrm{int}(C)$, $\quad x \in WS(MOP) \Leftrightarrow x \text{ solves } (P_{h_{f(x),v}})$.

Classical method

AIM OF SOLUTION METHODS:
Find Max($f(X)$) or a representative part of it.

Weighting Method:

This method consists of choosing weights $p_1, ..., p_k \geq 0$, not all zero and solving the associated scalar problem (P) by known techniques:

$$(P) \qquad \begin{array}{ll} \max & \sum_{i=1}^{k} p_i f_i(x) \\ \text{subject to} & x \in X \end{array}$$

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- *If $p_i > 0, i = 1, ..., m$ then any optimal solution of (P) is an efficient solution of (MOP).*
- *If $p_i \geq 0, i = 1, ..., k$ and not all are zero, then any optimal solution of (P) is a weakly efficient solution of (MOP). If in addition the set $f\ (\mathrm{argmin}\,(P))$ is a singleton, then it is an efficient solution.*

Classical method

Here is the algorithm.

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STEP 1. Choose $\xi > 0$ and $m \geq 1$ such that $\frac{1}{m} \leq \xi$. Choose $\lambda = \frac{1}{m}(m_1, ..., m_k)$ with $m_i \in \{0, 1, ..., m\}$ such that $m_1 + ... + m_k = m$.

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STEP 2. Solve (P_λ). If $m_i > 0$, $i = 1, ..., k$, store an optimal solution x^λ and its value $f(x^\lambda)$. If $m_i = 0$ for some i, solve (P^*_λ) and store an optimal solution x^λ and its value $f(x^\lambda)$.

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STEP 3. Choose another λ in Step 1 and go to Step 2 unless no λ left.

Classical method

REMARKS:

• The method generates a set of maximal solutions and a set of maximal values corresponding to an ϵ-net of weighting vectors (in the sense that for every $\xi \in \mathbb{R}^k_+$ with $\sum_{i=1}^k \xi_i = 1$, there is λ of that family such that $\|\xi - \lambda\| \leq \epsilon$).

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• For nonconcave problems and even for linear problems, the generated set of maximal values may be a very small part of the maximal value set of the problem even if ϵ tends to zero.

Classical method

ε-constraint method. In this method one maximizes one objective, while other objectives are considered as constraints. Choose $\ell \in \{1, ..., k\}, L_j \in \mathbb{R}, j = 1, ..., k, j \neq \ell$, and solve the scalar problem (P_ℓ):

$$\begin{array}{rc} \text{Max} & f_\ell(x) \\ \text{subject to} & f_j(x) \geq L_j, j = 1, ..., k, j \neq \ell \\ & x \in X. \end{array}$$

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Remark: If L_j are big, then (P_ℓ) may have no feasible solutions. If L_j are too small, then an optimal solution of (P_ℓ) may be not efficient.

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- $x^0 \in S(X, f)$ *if and only if it is optimal for all* (P_ℓ), $\ell = 1, \ldots, k$ *and* $L^{-\ell} = (f_1(x^0), \ldots, f_{\ell-1}(x^0), f_{\ell+1}(x^0), \ldots, f_k(x^0))$.

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STEP 1. Solve

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for $i = 1, ..., k$. Let $x^1, ..., x^k$ be optimal solutions.

STEP 2. Find $f_j(x^j), i, j \in \{1, ..., k\}$ and determine

$$M_i = \max\{f_i(x^1), ..., f_i(x^k)\}$$
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STEP 3. Choose $r = 1, 2, ...$ and solve (P_ℓ) with

$$L_j = M_j - \frac{t}{r-1}(M_j - m_j), t = 0, ..., r-1.$$

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REMARKS:

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- If at a solution of (P_ℓ), all the constraints are binding, then this solution is maximal for (MOP).
- If $f_1, ..., f_s$ are binding, $f_{s+1}, ..., f_k(\neq f_\ell)$ are nonbinding, one solves (P*)

$$\max \quad \sum_{j=s+1,\dots,k,j\neq\ell} f_j(x)$$

$$\text{subject to} \quad x \in X, f_i(x) = f(x^*)_i, i = 1, ..., s, i \neq \ell,$$

to obtain an efficient solution.

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  to obtain an efficient solution.
- In the last step for each t the problem (P_ℓ) (or the corresponding (P*)) provides an efficient solution and hence a maximal value of (MOP). With r large one may generate a good representative subset of the maximal value set of the problem.

New method

Free disposal outer approximation method. Assume that $f(X)$ is a compact set in the interior of the Pareto cone. The Edgeworth-Pareto hull (E-P hull for short) of $f(X)$:

$$f(X)^{\diamondsuit} := \left(f(X) - \mathbb{R}^k_+\right) \cap \mathbb{R}^k_+,$$

It is said to be finitely generated if there are a finite number of vectors $v^1, ..., v^m$ such that $f(X)^{\diamond}$ coincides with $\{v^1, ..., v^m\}^{\diamond}$.

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STEP 1. For $i = 1, ..., k$ solve

$$(P_0) \qquad q_i^0 = \max_{x \in X} f_i(x).$$

Put $\ell = 1,\ W_1 = \{(q_1^0, ..., q_k^0)\},\ E_0 = \emptyset,\ V_0 = W_1,\ E = \emptyset,\ S = \emptyset.$

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STEP 2. For $q \in W_\ell \setminus E_{\ell-1}$. Solve

$$(P_{q,e}) \qquad t_q = \max h_{q,e}(f(x)), \ \text{ subject to } x \in X.$$

Compute

$$\begin{aligned} E_\ell &= E_{\ell-1} \cup \{q \in W_\ell \setminus E_{\ell-1} : t_q = 0\} \\ V_\ell &= W_\ell \setminus E_\ell. \end{aligned}$$

and set

$$S = S \cup \{x \in X : f(x) = q, q \in E_\ell\}$$

New method

STEP 3. If $V_\ell = \emptyset$, stop. Otherwise for $q \in V_\ell$ solve

$$(SP_q) \qquad \begin{array}{l} \max \sum_{i=1}^{k} f_i(x) \\ \text{subject to } x \in X, f(x) \geq q + t_q e. \end{array}$$

Set

$$\begin{aligned} S &= S \cup \{x \in X : x \text{ solves } (SP_q)\} \\ E &= E \cup \{f(x) : x \text{ solves } (SP_q)\}. \end{aligned}$$

New method

STEP 3. If $V_\ell = \emptyset$, stop. Otherwise for $q \in V_\ell$ solve

$$(SP_q) \qquad \begin{array}{l} \max \sum_{i=1}^{k} f_i(x) \\ \text{subject to } x \in X, f(x) \geq q + t_q e. \end{array}$$

Set

$$\begin{aligned} S &= S \cup \{x \in X : x \text{ solves } (SP_q)\} \\ E &= E \cup \{f(x) : x \text{ solves } (SP_q)\}. \end{aligned}$$

STEP 4. Determine $W_{\ell+1}$ by

$$W_{\ell+1} = \mathrm{Max}(A_{\ell+1})$$

with $A_{\ell+1} = W_\ell^\diamondsuit \cap \{y \in \mathbb{R}^k_+ : h_{q,e}(y) \leq t_q, q \in V_\ell\}$.
Put $\ell = \ell + 1$ and return to STEP 2.

New method

REMARKS:

New method

REMARKS:

• The set $W_1^{\diamondsuit}$ (see Step 1) is the E-P hull of the vertex $(q_1^0, ..., q_k^0) \in \mathbb{R}^k$. It is the first finitely generated E-P hull outer approximation of the set $f(X)^{\diamondsuit}$.

New method

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• The aim of the subsequent steps is to generate decreasing (by inclusion) sequence of finitely generated E-P approximations of $f(X)^{\diamondsuit}$ meanwhile generating a set of maximal values of (MOP).

New method

REMARKS:

• In Step 2, one considers a vertex q of W_l. If $q \in [f(X)]^\diamondsuit$ which corresponds to the case $t_q = 0$, then it is a maximal value of (MOP). If q is outside of $[f(X)]^\diamondsuit$, the problem (P_q) will provide a new maximal value to add to the collection E. If all vertices of W_ℓ belong to $[f(X)]^\diamondsuit$, then the E-P hull $W_\ell^\diamondsuit$ coincides with $[f(X)]^\diamondsuit$ and the algorithm stops. Otherwise one determines new vertices for a smaller approximation of $f(X)^\diamondsuit$ in Step 4.

New method

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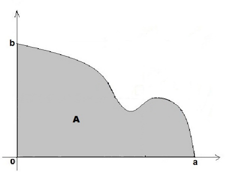
• In Step 2, one considers a vertex q of W_l. If $q \in [f(X)]^{\diamondsuit}$ which corresponds to the case $t_q = 0$, then it is a maximal value of (MOP). If q is outside of $[f(X)]^{\diamondsuit}$, the problem (P_q) will provide a new maximal value to add to the collection E. If all vertices of W_ℓ belong to $[f(X)]^{\diamondsuit}$, then the E-P hull $W_\ell^{\diamondsuit}$ coincides with $[f(X)]^{\diamondsuit}$ and the algorithm stops. Otherwise one determines new vertices for a smaller approximation of $f(X)^{\diamondsuit}$ in Step 4.

• One may prove that the sequence $(W_\ell)^{\diamondsuit}$ approaches to the set $[f(X)]^{\diamondsuit}$ as l tends to ∞, and the collection E approaches the closure of the maximal value set of (MOP).

New method

Example

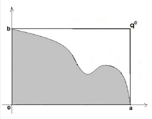
We wish to generate the maximal value set of A.



New method

Example

We wish to generate the maximal value set of A.

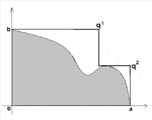


The first polyhedron A_1 approximating A is the box $[0aq^0b]$, where q^0 is found by solving (P_0).

New method

Example

We wish to generate the maximal value set of A.

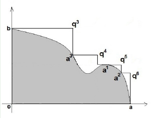


At Step 2, we solve (P_{q^0}) and obtain the nonconvex polyhedron A_2, the E-P hull of $W_2 = \{q^1, q^2\}$.

New method

Example

We wish to generate the maximal value set of A.



The third nonconvex polyhedron A_3 is the E-P hull of the set $W_3 = \{q^3, q^4, q^5, q^6\}$.
The collection E of maximal values contains
a^1 after the first iteration, and
a^1, a^2 and a^3 after the second iteration.

Reading

Classical books:

V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making: Theory and Methodology*, North-Holland, New York, 1983.
D. T. Luc, *Theory of Vector Optimization*, LNEMS 319, Springer-Verlag, Germany, 1989.
Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press INC., New York, 1985.
R. E. Steuer, *Multiple-Criteria Optimization: Theory, Computation, and Application*, John Wiley and Sons, New York, 1986.
P. L. Yu, *Multiple-criteria Decision Making: Concepts, Techniques and Extensions* , Plenum Press, New York, 1985.
M. Zeleny, *Linear Multiobjective Programming*, Springer-Verlag, New York, 1974.

Reading

More recent books:

R. I. Bot, S. M. Grad and G. Wanka, Duality in Vector Optimization, Springer-Verlag Berlin Heidelberg, 2009.

M. Erhgott, *Multicriteria Optimization*, Springer, Berlin, 2005.

A. Gopfert, R. Hassan, C. Tammer and C. Zalinescu, *Variational Methods in Partially Ordered Spaces*, Springer, New York, 2003.

J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*, Springer, Berlin, 2004.

K. Miettinen, *Nonlinear Multiobjective Optimization*, Kluwer Academic Publishers, Boston, 1999.