Problem Set 1 Answers

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Masters Growth and Development
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Throughout the problem set, A is constant.

1a. At what average rate will income per capita in the USSR have to grow in order to overtake (i.e. to equal) the industrialized nations' income per capita in exactly 30 years? Assume the industrialized nations' income per capita is growing at 2%.

$y_{USSR} = 500e^{30g} = 5000e^{30(0.02)} = y_{Industrial}$
$30g = \ln 10 + 30(0.02)$
$g = 0.02 + \ln 10/30 = 9.7\%$.

b. If the USSR sustains the growth rate of part a., how long *after* it has overtaken the industrialized nations' GDP per capita will it take for it to attain **double** the industrialized nations' GDP per capita? Again, assume the industrialized nations' GDP per capita is growing at 2% per year.

Suppose T–t years after overtaking, the USSR will double the industrialized countries GDP per capita. Let y_t be the GDP per capita level when they are equal. Then:

$$y_{T,USSR} = y_t\, e^{0.097(T-t)} = 2y_t\, e^{0.02(T-t)} = 2*y_{T,Industrial}$$
$$(T-t)(0.097-0.02) = \ln 2$$
$$(T-t) = \ln 2/(0.077) = 9.0$$

So at this pace, 9 years after overtaking, USSR will double the industrialized countries' GDP per capita. It would thus take 30 years to catch and 9 more years to double.

c. What fraction of national output must the USSR devote to building new capital goods in order to attain the growth rate of part a.? What fraction would be left for consumer items? [Hint: another word for the fraction of output devoted to building new capital goods is the investment rate, i.e. the ratio I_t/Y_t. And, remember that savings equals investment, so the investment rate equals the savings rate.]

Note that the fraction of production devoted to capital goods instead of consumption goods is just s, and the fraction devoted to consumption goods is 1–s. One can see this because, in the basic growth model, $I_t = S_t = sY_t$. Thus s can be interpreted as the fraction of national income *saved* and **also** as the fraction of national production devoted to *investment goods*. Similarly, $C_t = Y_t - S_t = Y_t - sY_t = (1-s)Y_t$; so (1–s) is the fraction of output devoted to *consumption goods*.

Use the key equation g= sA–d–n, knowing that A=0.5, d=8%, n=2%, g=12%.

$$9.7\% = s*0.5 - 8\% - 2\%$$
$$s = (9.7\% + 8\% + 2\%)/0.5 = 39.4\%$$
$$1-s = 60.6\%$$

So 39.4% of national output should be devoted to building new capital goods and 60.6% is left for consumer items.

d. At what rate are the industrialized countries saving if they are growing at 2% per year?

As in part c, use g = sA–d–n:

$$2\% = s*0.6 - 8\% - 2\%$$
$$s=(2\%+8\%+2\%)/0.6=20\%$$

So in the industrialized countries, the saving rate is 20%.

e. What would you calculate the ratio of consumption per capita in the USSR to consumption per capita in the industrialized countries when the USSR overtakes the industrialized countries (i.e. when GDP per capita is equal)? Assume the savings rates of parts c. & d. What would the ratio be when the USSR reaches double the industrialized nations' GDP per capita?

The key equation is $c_t = (1-s)y_t$. We need to calculate $c_{t,USSR}/c_{t,industrial}$ and $c_{T,USSR}/c_{T,industrial}$, where t is the year the countries are equal and T is the year that USSR doubles the rest of the world. So we need:

$c_{t,USSR}/c_{t,industrial} = (1-s_{USSR})\, y_{t,USSR} / [(1-s_{industrial})\, y_{t,industrial}] = \frac{1-s_{USSR}}{1-s_{industrial}} \cdot \frac{y_{t,USSR}}{y_{t,industrial}} = \frac{0.606}{0.8} \cdot \frac{1}{1} = 0.76.$

Note that $y_{t,USSR}$ and $y_{t,industrial}$ are equal, by definition, since t is the year they are equal. Thus their ratio is 1. So, when the two countries have equal incomes, the USSR is still consuming a lot less – 76% of the industrialized nations' consumption – because it is devoting so much of production to investment goods. However, in this model over time they will eventually reap the rewards of forced savings:

$c_{T,USSR}/c_{T,industrial} = (1-s_{USSR})\, y_{T,USSR} / [(1-s_{industrial})\, y_{T,industrial}] = \frac{1-s_{USSR}}{1-s_{industrial}} \cdot \frac{y_{T,USSR}}{y_{T,industrial}} = \frac{0.606}{0.8} \cdot \frac{2}{1} = 1.52.$

Note that $y_{T,USSR}$ equals twice $y_{T,industrial}$, by definition since T is the year they double, and thus the ratio is 2/1. A mere 9 years later the Soviet Union is enjoying 52% more consumption, in this model, and on their way to an ever-increasing gap in consumption.

2a. Solve for Soviet long-run GDP per capita (y^*) as a function of its savings rate.

$y = 400k^{1/3}$, n = 2%, d = 8%.

In the long run, $g_k = 0$, i.e. $sAf(k^*)/k^* = (n+d)$

$sA/(n+d) = k^*/ f(k^*) = k^{*2/3}$

$k^* = [sA/(n+d)]^{3/2}$

$y^* = Af(k^*) = A\, [sA/(n+d)]^{3/2*(1/3)} = A\, [sA/(n+d)]^{1/2} = A^{3/2}\, [s / (n+d)]^{1/2}$

Plugging in numbers (except s):

$y^* = (400)^{3/2}\, s^{1/2} /(.02+.08)^{1/2} = 8000 * s^{1/2} * 3.1623 = 25298.22 * s^{1/2}$

b. What fraction of national output should be devoted by the Soviet Union to building new capital goods in order to overtake, i.e. **equal**, the industrialized nations' GDP per capita in the long-run? What fraction is left for consumer items?

To equal the industrialized countries' GDP per capita in the long run, the Soviet Union must reach \$10,000:

y^*_{USSR} = \$10,000; using our formula for y^* for USSR from part a:

$10{,}000 = 25298.22\, s^{1/2}$

$s = (10{,}000/25298)^2 = 15.625\%$

The fraction for consumption is $1-s = 1-15.625\% = 84.375\%$

c. What fraction of national output should be devoted by the Soviet Union to building new capital goods in order to surpass, i.e. **double**, the industrialized nations' GDP per capita in the long-run? What fraction is left for consumer items?

To double the industrialized countries' GDP per capita in the long run, the Soviet Union must reach \$20,000:

y^*_{USSR} = \$20,000; using our formula for y^* for USSR from part a:

$20{,}000 = 25298.22\, s^{1/2}$

$s = (20{,}000/25298)^2 = 62.5\%$

The fraction for consumption is $1-s = 1-62.5\% = 37.5\%$

d. In the long run, what is the ratio of Soviet GDP per capita to GDP per capita in the industrialized countries, and what is the ratio of Soviet consumption per capita to consumption per capita in industrialized countries if the Soviet Union achieves the goal of part b? of part c? (That is, find four ratios.)

In part b., $y^*_{USSR} = y^*_{industrial}$ = \$10,000, so

$y^*_{USSR}/ y^*_{industrial} = 1$, and

$c^*_{USSR}/ c^*_{industrial} = (1-s_{USSR})\, y^*_{USSR} / [\, (1-s_{industrial})\, y^*_{industrial}] = (1-.15625)/(1-.1) = 0.9375.$

In part c., $y^*_{USSR} = 2 * y^*_{industrial}$ = \$20,000, so

$y^*_{USSR}/ y^*_{industrial} = 2$, and

$c^*_{USSR}/ c^*_{industrial} = (1-s_{USSR})\, y^*_{USSR} / [\, (1-s_{industrial})\, y^*_{industrial}] = (1-.625)*2/(1-.1) = 0.8333.$

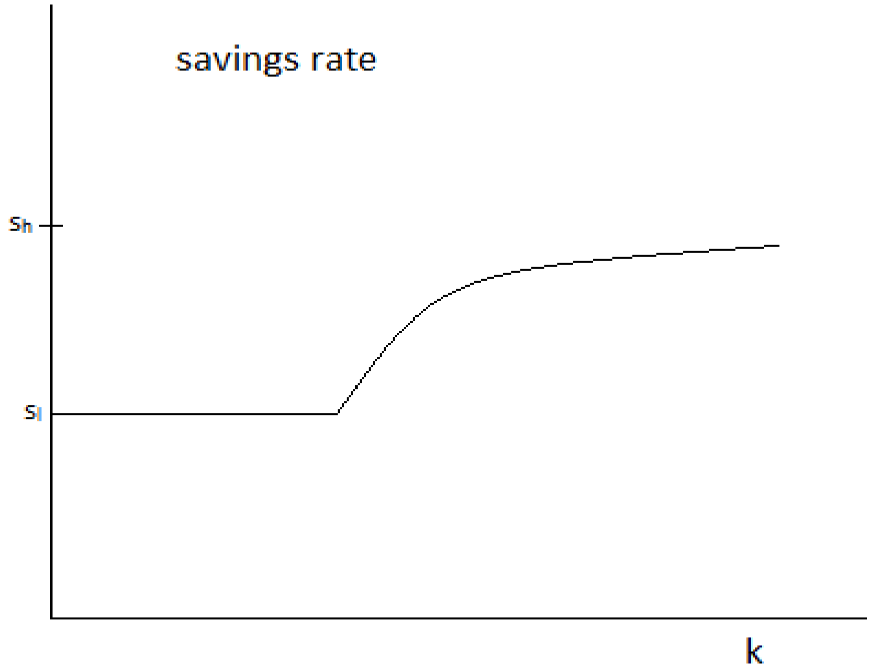
e. Comparing the outcomes of part d. of this question to the same question answered using the AK model (problem 1), which model produces a more optimistic outlook for achieving, by saving and investing at high rates, the Soviet goals of overtaking and surpassing industrialized countries' living standards? Remember that living standards are best measured by consumption here.
Interestingly, the more ambitious goal here of doubling the industrialized nations GDP (which produced good results if the AK model is true, see 1e. above), would here have a negative effect on what really matters in the model, consumption per person. Savings would be so high in order to maintain a high level of capital per person that little would be left to consume, and in fact, living standards measured by consumption would be less than 85% the industrialized nations' level. That is, the more ambitious income goal could actually lead to lower consumer well-being in the short- **and** long-run, in the Solow model. The promised eventual rise in consumption would never materialize if there are significant diminishing returns to capital. Even the goal of equaling industrialized income would not produce equal living standards. By contrast, the AK model is quite optimistic – a mere 9 years after overtaking industrial income levels, it will reach twice their income levels and 50% more consumption. In the AK model, high savings always pays off given enough time. So, clearly the AK model is the more optimistic – one can achieve nearly any goal with enough savings/investment and enough time.

3. Assume the framework of the AK model, with the following modification. A household's savings rate, specifically its marginal propensity to save, is low when its income is below $1000. It then switches to a higher rate when income exceeds $1000. Specifically:

- n = 1%
- d = 5%
- A = 0.5
- MPS_{low} = 0.1
- MPS_{high} = 0.25

Assume there is no income inequality. What are the growth dynamics in this economy? Specifically, answer the following questions.
a. Derive the average savings rate in the economy as a function of capital per worker, k.
Savings(y) = 0.1*y + (0.25-0.1)*max{y-1000,0}
This reflects the fact that 0.1 of every unit of income gets saved, plus an additional 0.15 of every unit of income over 1000.
Savings_rate(y) = Savings(y)/y = 0.1 + (0.25-0.1)*max{1-1000/y,0}
Since y=Ak, Savings_rate(k) = 0.1 + (0.25-0.1)*max{1-1000/Ak,0}
= 0.1 + (0.25-0.1)*max{1-2000/k,0}.
You can also write this as:

$$s = \begin{cases} 0.1 \quad if\ k \leq 2000 \\ 0.25 - \dfrac{300}{k} \quad if\ k \geq 2000 \end{cases}$$



b. What is the average income level above which sustained growth will occur, and below which it will not occur?
The key condition is that g_k = sAf(k)/k – (n+d) = 0. In other words, we want to find where the actual investment curve (sAf(k)/k) intersects break-even investment (n+d). See the graph from the notes.
s(k) A = n+d
0.25 – 300/k = (n+d)/A = 0.12
$k_{critical}$ = 300/0.13 = 2308
$y_{critical}$ = A$k_{critical}$ = $1154

c. What is the asymptotic (i.e. long-run) growth rate for an economy that starts with enough income to sustain growth? How does the income growth rate evolve over time for such an economy (i.e. does it increase or decrease)?
Asymptotically, the savings rate approaches 0.25, and we have the simple AK model where g=s_hA-d-n = 6.5%.
The growth rate increases over time. One can see this because g_k = g_y = s(k)A – (n+d), and s(k) increases in k (see above). Of course, it approaches 6.5% in the long run, but always on a monotonically increasing path.

d. What is the long-run income for an economy that starts poor enough?
The growth rate for an economy that starts below $y_{critical}$ will be negative, since s(k)A – (n+d)<0 in this region by construction. Eventually the economy will have k<2000, and thus will grow at s_lA – (n+d) = –1%. Growth at this negative rate will leave the economy approaching an income of 0 in the long-run.

e. How would income inequality affect growth prospects in this model? Specifically, holding fixed the average income level, would higher inequality raise or lower the average savings rate of the economy? A discussion or diagram without a formal proof is fine.
Note that the savings function (household savings as a function of income, not savings rate as a function of income) is weakly convex: it is piecewise linear, with an upward kink at y=$1000. Thus, a mean-preserving spread in income weakly raises average savings. (This is a basic result in microeconomics, that a mean-preserving spread in y raises the average of a convex function of y.) But a mean-preserving spread in income is exactly a rise in income inequality. Thus, in this model higher income inequality raises savings, and thus growth. The intuition is that high-income households save at high rates, and thus money in their hands is good for the economy-wide growth rate.

4. Compare the effect on long-run income of a fifty percent increase in the savings rate (for physical capital), **s**, e.g. from 20% to 30%, in the Solow model vs. the H-augmented Solow model. In the case of the

Solow model, assume $Y=AK^{\alpha}N^{1-\alpha}$, while in the H-augmented Solow model, $Y=AK^{\alpha}H^{\alpha}N^{1-2\alpha}$ (with $\alpha<1/2$ in both cases). For example, $Y=AK^{1/3}N^{2/3}$ in the Solow model, and $Y=AK^{1/3}H^{1/3}N^{1/3}$ in the H-augmented Solow model. What do you think accounts for the difference?

For the Solow model, y^* comes from setting

$sAf(k)/k = n+d$

$k/f(k) = sA/(n+d)$

$k^{1-\alpha} = sA/(n+d)$

$k^* = [sA/(n+d)]^{1/(1-\alpha)}$

$y^* = Ak^{*\alpha} = A^{1/(1-\alpha)}s^{\alpha/(1-\alpha)}(n+d)^{-\alpha/(1-\alpha)}$

If s increases 50% to 1.5s, then income goes up to

$y^{*\prime} = A^{1/(1-\alpha)}(1.5s)^{\alpha/(1-\alpha)}(n+d)^{-\alpha/(1-\alpha)} = 1.5^{\alpha/(1-\alpha)}A^{1/(1-\alpha)}s^{\alpha/(1-\alpha)}(n+d)^{-\alpha/(1-\alpha)}$

$= 1.5^{\alpha/(1-\alpha)}\, y^*$.

For the H-augmented Solow model, y^* comes from simultaneously setting

$sAf(k,h)/k = n+d$

$qAf(k,h)/h = n+d$

Combining, one sees that in steady state, $h = k(q/s)$

Using this and the first equation,

$k/f(k,h) = sA/(n+d)$

$k/[k^{\alpha}h^{\alpha}] = sA/(n+d)$

$k/[k^{\alpha}(kq/s)^{\alpha}] = sA/(n+d)$

$k^{1-2\alpha} = s^{1-\alpha}q^{\alpha}A/(n+d)$

$k^* = [s^{1-\alpha}q^{\alpha}A/(n+d)]^{1/(1-2\alpha)}$

$h^* = (q/s)k^* = [s^{\alpha}q^{1-\alpha}A/(n+d)]^{1/(1-2\alpha)}$

$y^* = Ak^{*\alpha}h^{*\alpha} = A^{1/(1-2\alpha)}s^{\alpha/(1-2\alpha)}q^{\alpha/(1-2\alpha)}(n+d)^{-2\alpha/(1-2\alpha)}$

If s increases 50% to 1.5s, then income goes up to $y^{*\prime} = 1.5^{\alpha/(1-2\alpha)}\, y^*$.

Summarizing, in the Solow model $\frac{y^{*\prime}}{y^*} = 1.5^{\frac{\alpha}{1-\alpha}}$ and in the H-augmented Solow model $\frac{y^{*\prime}}{y^*} = 1.5^{\frac{\alpha}{1-2\alpha}}$. Thus, the effect in the H-augmented Solow model is greater than in the Solow model by a factor of

$1.5^{\frac{\alpha}{1-2\alpha}}/1.5^{\frac{\alpha}{1-\alpha}} = 1.5^{\frac{\alpha^2}{(1-2\alpha)(1-\alpha)}}>1$.

If $\alpha=1/3$, this is $1.5^{1/2} = 1.225$. Thus, the output effect is 22.5% greater in this example.

Even though capital's share is the same in both models in this example, a fixed-percentage increase in the savings rate has a greater impact in the H-augmented model. One can think of this as a result of a greater overall share of accumulable factors, physical and human capital together (2α rather than α). Accumulating more physical capital alleviates diminishing returns in human capital and allows more human capital to be accumulated, and vice versa. Thus there is a kind of feedback effect across the two types of capital.

<u>Optional Challenge Problems</u>

5. Consider a mix of the AK model and the Solow model. In particular, let $Y=AK + BK^{\alpha}N^{1-\alpha}$. All other aspects of the economy are the same as in the AK and Solow models.

a. What is the condition for sustained long-run growth to be possible? Is this similar to the AK or Solow model?

Again, we can see how capital k_t evolves using the equation

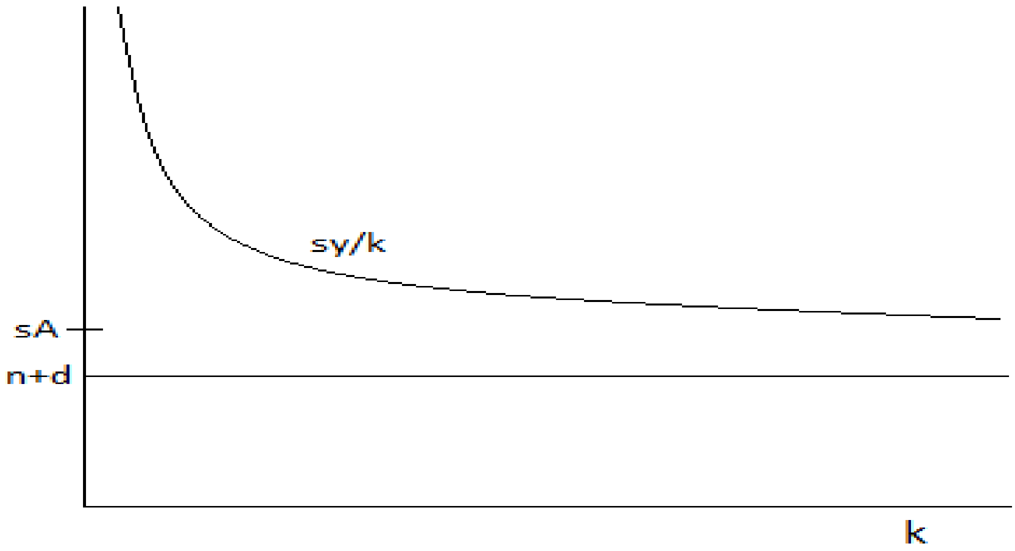
$g_k = s\, y/k – (n+d)$

Here, y ($=F(k,1)$) $=Ak + Bk^{\alpha}$.

So $g_k = s(A + B/k^{1-\alpha}) – (n+d)$

If sustained long-run growth is occurring (by which we mean y growing without bound), then k is growing without bound. So, the asymptotic growth rate of capital in this case would be $sA – (n+d)$, since $\lim_{k\to\infty}B/k^{1-\alpha} = 0$. This must be positive for long-run growth to occur, so the condition is $sA>n+d$, exactly as in the AK model. One can also see this in the following graph; if $sA>n+d$, the actual investment curve is always above the break-even investment curve. On the other hand, if $sA<n+d$, the actual investment curve

intersects the break-even investment curve exactly once, and this defines a steady-state level of capital k^*, exactly as in the Solow model, which is found by setting $g_k = 0$ and solving for k.



b. Assuming a growing economy, how does the growth rate of k evolve over time (i.e. does it increase or decrease)? Is this similar to the AK or Solow model?
Since $g_k = s(A + B/k^{1-\alpha}) - (n+d)$, one can see that g_k decreases over time if k is growing, since $\partial[B/k^{1-\alpha}]/\partial k<0$. Thus growth (in capital stock, at least) starts higher and slows down over time, as in the Solow model. The difference with the Solow model is that here, if sA>n+d, growth does not slow down toward zero, but toward some quantity greater than zero.

c. If two countries are identical except that one has a small capital stock and the other a large one, how do the growth rates (of k) of these two countries compare? Is this similar to the AK or Solow model?
For the same reason as in part b., i.e. since g_k decreases in k, one can see that the country with the smaller capital stock k will grow more rapidly (in terms of k) than the country with the larger capital stock k. This is similar to the Solow model, which predicts conditional convergence – all else equal, poor countries grow faster than rich ones.

d. How does an increase in the savings rate s affect the long-run growth rate of the economy? What condition does this depend upon? Is this similar to the AK or Solow model?
Consider an increase from s to s'. Iff s'A>n+d, then the economy ends up with sustained growth. In this case, the long-run growth rate of k is s'A – (n+d), and so is the long-run growth rate of income.
[To see this, note that $\dot{y} = A\dot{k} + \frac{\alpha B}{k^{1-\alpha}}\dot{k}$, so
$\frac{\dot{y}}{y} = \frac{A\dot{k}+\frac{\alpha B}{k^{1-\alpha}}\dot{k}}{Ak+Bk^{\alpha}} = \frac{\dot{k}}{k}\left(\frac{A+\frac{\alpha B}{k^{1-\alpha}}}{A+\frac{B}{k^{1-\alpha}}}\right)$. Note that the term in parentheses goes to 1 as $k\rightarrow\infty$.]
Similarly, the long-run growth rate of income under the previous savings rate was max{0,sA – (n+d)}; the zero is in case sustained growth was not achieved under the original savings rate. Clearly, the long-run growth rate went up. This is the same prediction as in the AK model – a higher s gives a higher long-run growth rate.
On the other hand, if s'A<n+d, the long-run growth rate is still zero after the increase in the savings rate (see graph from part a.) It was clearly zero when the savings rate was s, so there is no change in the long-run growth rate from an increase in s. The increase in s only raises the long-run *level* of income, not *growth rate*, as in the Solow model.

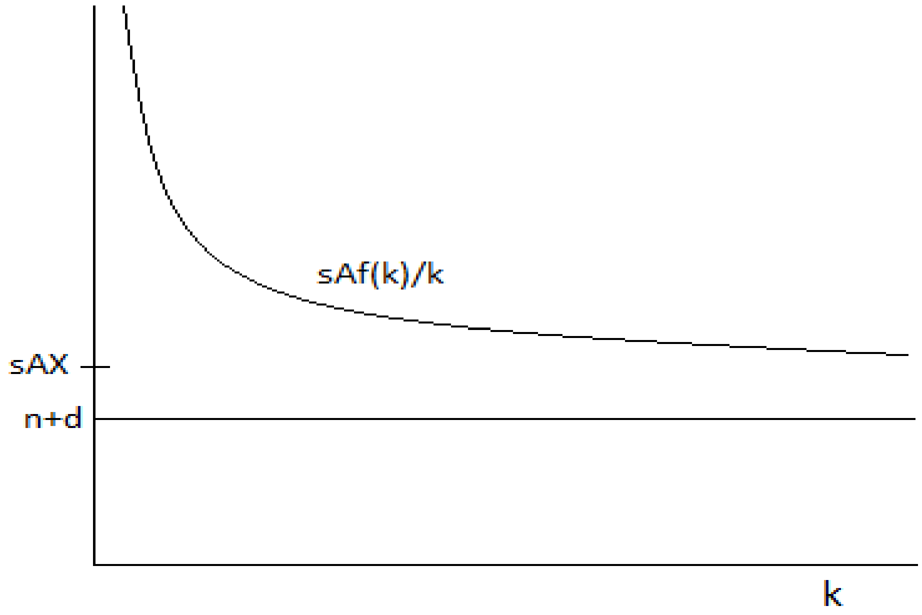
e. In what ways does this model capture the best predictions of both the AK and Solow models?
This model (proposed by Jones and Manuelli, 1990) has endogenous growth, where the fundamental parameters of the model determine the growth rate in non-trivial ways. But it also has the convergence/catch-up growth phenomenon that the Solow model captures so well, as growth slows down as

an economy gets richer, which also implies that all else equal, poor countries grow faster than rich ones. Essentially, this is a Solow model with a positive long-run growth rate that does not just depend on technological progress (outside the model), but key parameters of the model, e.g. savings rate (which can be endogenized as functions of preference parameters), population growth rate (which can also be endogenized), etc.

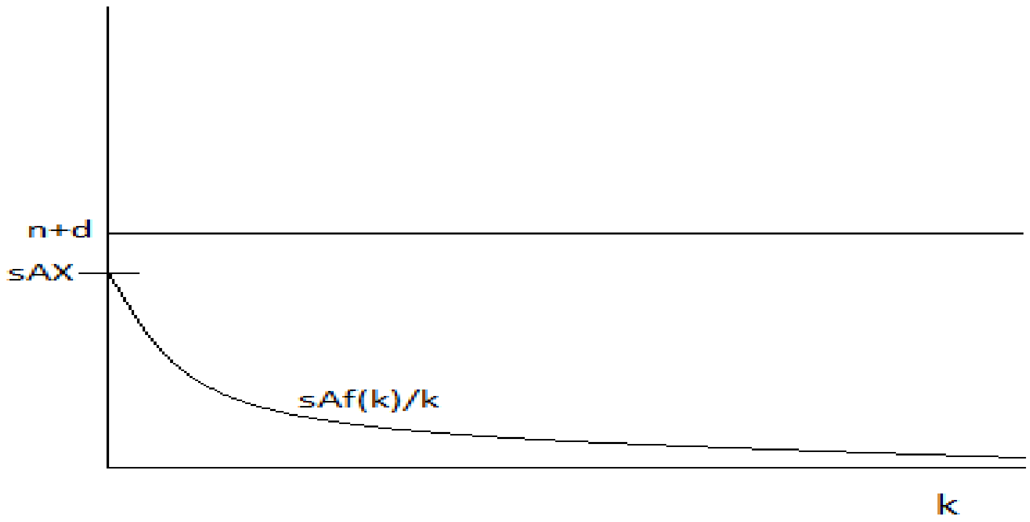
6. Consider the AK or Solow model, except generalize the production function to be of the constant-elasticity-of-substitution form:
$Y = A\{a(bK)^{\psi} + (1\text{-}a)[(1\text{-}b)L]^{\psi}\}^{1/\psi}$,
where $0<a<1$, $0<b<1$, and $\psi<1$. As $\psi\to1$, the production function approaches linearity, i.e. capital and labor are perfect substitutes. As $\psi\to-\infty$, the production function approaches a fixed proportion (Leontief) one, with no substitutability. As $\psi\to0$, the production function approaches Cobb-Douglas, with elasticity of substitution equal to 1.
a. Verify the production function exhibits constant returns to scale.
Need to show $AF(\lambda K, \lambda N) = \lambda AF(K,N)$.
$A\{a(b\,\lambda K)^{\psi} + (1\text{-}a)[(1\text{-}b)\,\lambda L]^{\psi}\}^{1/\psi} = A\{\lambda^{\psi}\, a(bK)^{\psi} + \lambda^{\psi}\,(1\text{-}a)[(1\text{-}b)L]^{\psi}\}^{1/\psi}$
$= A(\lambda^{\psi})^{1/\psi}\{a(bK)^{\psi} + (1\text{-}a)[(1\text{-}b)L]^{\psi}\}^{1/\psi} = \lambda A\{a(bK)^{\psi} + (1\text{-}a)[(1\text{-}b)L]^{\psi}\}^{1/\psi}$, as needed.

b. Does $f(k)/k \to0$ as $k\to\infty$? What does it depend upon? Does $f(k)/k \to\infty$ as $k\to0$? What does it depend upon?
$f(k) = F(k,1) = \{a(bk)^{\psi} + (1\text{-}a)(1\text{-}b)^{\psi}\}^{1/\psi}$.
$f(k)/k = k^{-1}\{a(bk)^{\psi} + (1\text{-}a)(1\text{-}b)^{\psi}\}^{1/\psi} = \{k^{-\psi}\,[a(bk)^{\psi} + (1\text{-}a)(1\text{-}b)^{\psi}]\}^{1/\psi} = \{ab^{\psi} + (1\text{-}a)(1\text{-}b)^{\psi}k^{-\psi}\}^{1/\psi}$
As a preliminary fact, note that $f(k)/k$ is strictly decreasing in k.
Returning to the question, there are two cases.
First, if $\psi>0$, then clearly $f(k)/k\to\infty$ as $k\to0$, since the term in brackets goes to infinity. However, as $k\to\infty$, $f(k)/k \to a^{1/\psi}b > 0$, since the second term in brackets goes to zero.
Second, if $\psi<0$, then clearly $f(k)/k \to0$ as $k\to\infty$, since the term in brackets goes to infinity, and is raised to a negative power. However, as $k\to0$, $f(k)/k \to a^{1/\psi}b < \infty$.
Thus, in each case only one or the other Inada condition holds, but not both.
[If $\psi=0$, we are in the Cobb-Douglas case, where we know both Inada conditions hold.]

c. Under what conditions is sustained growth possible? Be as specific as possible.
Recall $g_k = sAf(k)/k – (n+d)$. Long-run sustained growth is clearly not possible if $f(k)/k \to0$ as $k\to\infty$, which is true in the Solow model and in this model with $\psi\leq0$. However, if $\psi>0$, sustained growth is possible; the condition needed is that $\lim_{k\to\infty} sAf(k)/k > (n+d)$, i.e. $sAa^{1/\psi}b > n+d$.
In sum, we need $\psi>0$ and $sAa^{1/\psi}b > n+d$.
In this case, we have the following graph (defining $X\equiv a^{1/\psi}b$):



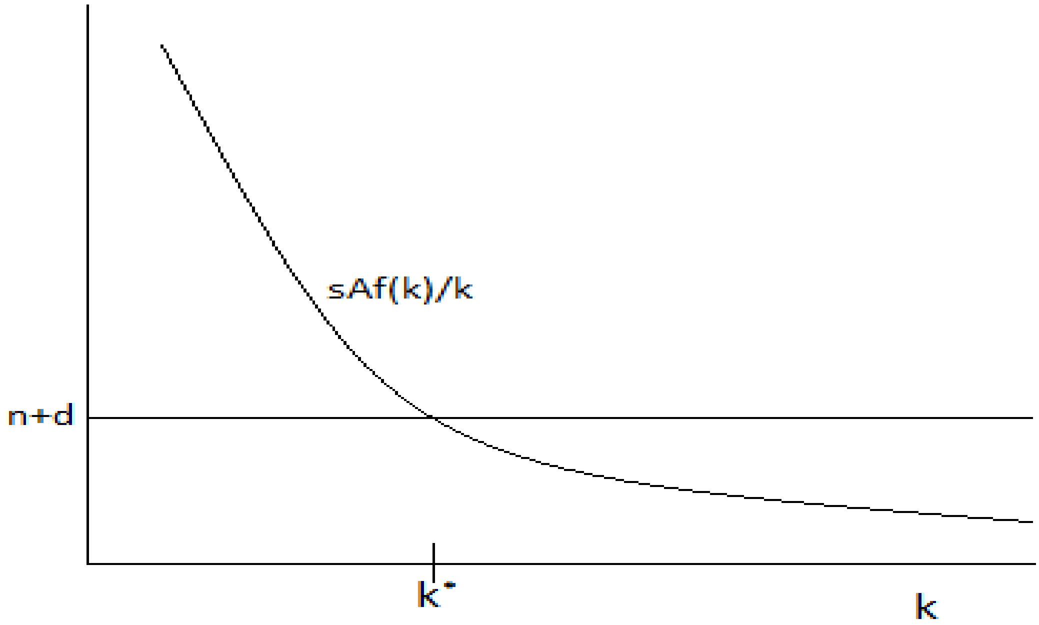
d. Under what conditions is long-run income zero? Be as specific as possible.
Recall $g_k = sAf(k)/k - (n+d)$. Long-run income of zero is clearly not possible if $f(k)/k \to\infty$ as $k\to 0$, which is true in the Solow model and in this model with $\psi\geq 0$; this condition implies that countries with very little capital are growing. However, if $\lim_{k\to 0} sAf(k)/k < (n+d)$, i.e. $\psi<0$ and $sAa^{1/\psi}b < n+d$, then growth is negative even at very low levels of capital (and even lower for higher levels of capital, since g_k declines in k). In this case, the economy is approach zero capital and zero income. Graphically:



e. If the conditions in c.&d. are not met, what is the steady state income level? Does it increase or decrease with s, A, and n, respectively?
If the conditions in c.&d. are not met, then the actual investment curve starts above the break-even investment line, and goes below it for high enough levels of capital, so that there is a unique intersection of the two curves. As in the Solow model, where the curves intersect, capital per person k_t is no longer changing and is thus at its steady state level. This is found by solving $g_k = 0$:
$sA\{ab^{\Psi} + (1-a)(1-b)^{\Psi}k^{-\Psi}\}^{1/\Psi} = n+d.$
$(1-a)(1-b)^{\Psi}k^{-\Psi} = [(n+d)/(sA)]^{\Psi} - ab^{\Psi}$
$k^{\Psi} = (1-a)(1-b)^{\Psi}/\{[(n+d)/(sA)]^{\Psi} - ab^{\Psi}\}$
$k^{*} = (1-a)^{1/\Psi}(1-b)/\{[(n+d)/(sA)]^{\Psi} - ab^{\Psi}\}^{1/\Psi}$

$$y^* = Af(k^*) = A\{a(bk^*)^\psi + (1\text{-}a)(1\text{-}b)^\psi\}^{1/\psi} = A\left[\frac{a(1-a)[b(1-b)]^\psi}{\left(\frac{n+d}{sA}\right)^\psi - ab^\psi} + (1-a)(1-b)^\psi\right]^{1/\psi}$$

One can show that y^* increases in s and A, and decreases in n, as in the Solow model. This also clear from shifting the curves in the following graph, knowing that y^* increases in k^*:



f. Why is sustained growth sometimes possible even with diminishing returns to capital?

It is not just diminishing returns to capital that gives the Solow model its prediction that growth cannot be sustained by capital accumulation alone – the Inada conditions are also important for this neo-classical (e.g. Solow) model prediction. Here, the production function has diminishing returns to capital, but with $\psi>0$, the returns to capital approach some positive number rather than zero. The returns to capital can remain high enough to allow for indefinitely sustained capital accumulation. Clearly important for diminishing returns to capital to end up not too severe is that labor can more easily substitute for capital.