Problem Set 2

Nizhny Novgorod HSE
Masters Growth and Development
Prof. Ahlin
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Answers must be written neatly by hand, or by computer. Students may work in groups, but final answers turned in must be the individual's – that is, each student must write and phrase answers in his own words. Justify all answers.

1. Consider the Mankiw/Romer/Weil 1992 model. The production function is $Y_t = A\gamma^t K_t^{\alpha} H_t^{\theta} N_t^{1-\alpha-\theta}$, or in per worker terms, $y_t = A\gamma^t k_t^{\alpha} h_t^{\theta}$, where $\gamma \geq 1$ is the (gross) growth rate of technology. The workforce grows at rate n: $N_t = (1+n)^t N_0$. Capital depreciates at rate δ. Thus, the two types of capital/worker evolve according to:

$\Delta k_t = sy_t - (n+\delta)k_t$, and

$\Delta h_t = qy_t - (n+\delta)h_t$,

where s and q are the (constant) rates of investment in physical and human capital, respectively.

a. Let $\gamma=1$. Derive the long-run (i.e. steady-state) capital/worker ratios. Derive the long-run level of income per worker in the economy. Write all answers in terms of s, q, A, n, d, α and θ.

b. In order to maximize long-run consumption per worker, what should s and q be set to?

c. Let $\gamma>1$. What are the growth rates of y_t, c_t ($=C_t/N_t$), k_t, and h_t in a balanced growth path? How does it depend on the policy parameters of the model, s, q, A, and n?

We next discard the assumption of a fixed investment rates, s and q. Assume instead the representative household maximizes a discounted infinite sum of per-period, per-person CRRA utility: $\sum_{t=0}^{\infty} \beta^t U(c_t)$, $U(c_t) = c_t^{1-\sigma}/(1-\sigma)$, $\sigma>0$. The discount factor is $\beta \in (0,1)$. The household's budget constraint is:

$c_t + (1+n)k_{t+1} + (1+n)h_{t+1} \leq y_t + (1-\delta)k_t + (1-\delta)h_t$.

d. Write down two Euler equations, one for physical capital and one for human capital. [The Euler equation equates the marginal utility from spending a dollar today with that from investing a dollar today in physical or human capital, and spending the proceeds from that investment tomorrow.] If both types of capital are invested in, what must their ratio be?

e. On a balanced growth path, what are the growth rates of c_t, y_t, k_t, and h_t?

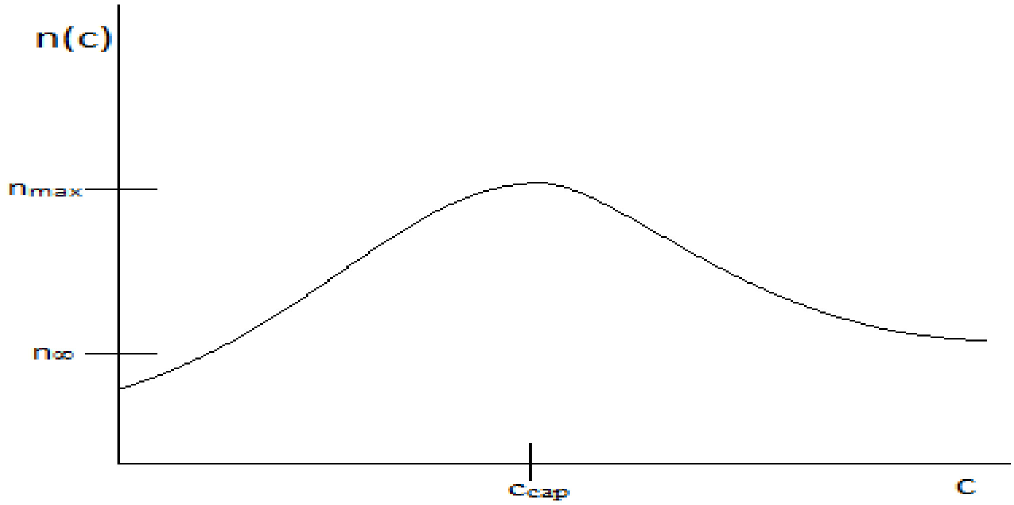
Let $\gamma=1$ for the remaining parts of the question.

f. Derive the long-run (i.e. steady-state) capital/worker ratios. Derive the long-run level of income per worker in the economy. How do n, β, and δ affect long-run income?

g. Give the endogenous expressions produced by this model for s and q. In other words, in the long-run, what fraction of income is invested in physical capital? Human capital? [Use the household budget constraint to decompose income into investment and consumption.] How do they depend on n, δ, α, θ, and β? Interpret these relationships, including any conflicts with the comparative static answers of part f.

2. Consider the Hansen/Prescott 2002 model, without capital. There are two technologies available. The Malthusian technology produces output according to function $Y_{Mt} = A_{Mt} N_{Mt}^{\mu} L_{Mt}^{1-\mu}$, where N_{Mt} is the labor allocated to the Malthusian technology, L_{Mt} is the land allocated to the Malthusian technology, A_{Mt} is the TFP of the Malthusian technology, and $0<\mu<1$. The Solow technology does not use land: $Y_{St} = A_{St} N_{St}$, where N_{St} is the labor allocated to the Solow technology and A_{St} is the TFP of the Solow technology. TFPs A_{Mt} and A_{St} grow according to growth factors (i.e. 1+growth rate) γ_M and γ_S, respectively.

At any time t, L_t and N_t give the total quantity of land and labor available in the economy, to be divided between the two technologies. $L_t=1$ for all t, i.e. land is in fixed supply; it is also supplied inelastically (for free) to the Malthusian technology, so $L_{Mt}=1$ for all t. Population N_t evolves according to growth factor (i.e. 1+growth rate) n(c), where c is consumption per person. The function n(c) is single-peaked, it peaks at n_{max} when $c=c_{cap}$, and it approaches n_{∞} as c gets large (i.e. $\lim_{c\to\infty} n(c) = n_{\infty}$). See graph:



For parts a.&b., ignore the Solow technology.

a. Characterize the Malthusian trap in this model. Specifically, at what level does consumption stagnate and what is the population growth rate in a Malthusian steady state where consumption and income per capita are not growing?

b. What condition must be satisfied for sustained growth to be possible for at least some initial conditions, with the Malthusian technology? What condition must be satisfied for sustained growth to be inevitable, regardless of initial conditions, with the Malthusian technology?

c. Derive the cost and marginal cost functions for both the Malthusian technology and the Solow technology. That is find $C_M(Y)$, $C_S(Y)$, $C_M'(Y)$, and $C_S'(Y)$, all as functions of the market wage, w.

d. Derive a necessary and sufficient condition for the Malthusian technology to be used exclusively in the Pareto efficient allocation (or competitive equilibrium) at time t. The condition may involve only the total labor endowment at time t, N_t, and the production function parameters.

e. Combining the results from above, what necessary and sufficient condition guarantees that an economy stuck in the Malthusian trap will eventually begin to use the Solow technology to some extent?

Optional Challenge Problem

3. An economy has a measure one of identical households; there is no population growth. Production of the representative household is $Y_t = A\gamma^t H_t^\alpha Q_t^\theta / E(Q_t^\theta)$, where $\alpha+\theta \leq 1$, $\gamma \geq 1$, H_t is human capital of the household, Q_t is political capital of the household, and $E(Q_t^\theta)$ denotes the average of the quantity Q_t^θ across all households in the economy. Thus, production depends positively on a household's human capital and political capital; but, political capital is unproductive in the aggregate, it only affects the *share* of total production that one household acquires (since $Q_t^\theta/E(Q_t^\theta)$ averages to one across the population). Further, *in equilibrium* given that households are identical, $Q_t^\theta = E(Q_t^\theta)$, so political capital will add nothing to any given household's production. (If $E(Q_t^\theta)=0$, then assume $Y_t = A\gamma^t H_t^\alpha$ if $Q_t^\theta=0$ and $Y_t=\infty$ if $Q_t^\theta>0$.) Assume both types of capital depreciate fully each period. Thus, the two types of capital evolve according to:

$H_{t+1} - H_t \equiv \Delta H_t = qY_t - H_t$, and $Q_{t+1} - Q_t \equiv \Delta Q_t = rY_t - Q_t$,

where q and r are the fractions of production devoted to investment in human and political capital, respectively, and $q+r<1$. Of course, $C_t=(1-q-r)Y_t$.

a. Let $\gamma=1$. Derive the long-run (i.e. steady-state) levels of income, consumption, human capital, and political capital. What values for q and r maximize long-run consumption?

b. Let $\gamma>1$. What are the growth rates of Y_t, C_t, H_t, and Q_t on a balanced growth path? How does it depend on the policy parameters of the model, q, r, and A?

For the remaining parts, we discard the assumption of fixed investment rates, s and q. Assume instead the representative household maximizes the sum $\sum_{t=0}^{\infty} \beta^t U(C_t)$, where $U(C_t)$ is a CRRA utility function: $U(c_t) = c_t^{1-\sigma}/(1-\sigma)$, $\sigma>0$. The household's resource constraint is:

$C_t + H_{t+1} + Q_{t+1} \leq A\gamma^t H_t^\alpha Q_t^\theta / E(Q_t^\theta)$

Further, assume $\beta\gamma^{(1-\sigma)/(1-\alpha)} < 1$; this will assure a well-defined maximization problem (non-infinite utility).

c. Write down two Euler equations, one for human capital and one for political capital. (Assume that a dollar invested in political capital tomorrow will affect the quantity Q_t^θ but not $E(Q_t^\theta)$, the latter because each household is small relative to the economy and takes all other households' behavior as given.) Show what the ratio of the two types of capital must be according the Euler equations.

d. On a balanced growth path, what are the growth rates of C_t, Y_t, H_t, and Q_t? On a balanced growth path, what are the rates of investment (analogous to q and r for parts a. and b.) in human and political capital? What is the fraction of output consumed, C_t/Y_t?

e. How do these three quantities – the part-d. analogs to q, r, and 1-r-q – compare to the case where $\theta=0$ (which can be thought of as the first-best), and why? Assuming $\gamma=1$, how do these three quantities compare to the "optimal" values or r, q, and 1-r-q in part a., and why?