The Black-Scholes Option Pricing Formula

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Abstract

Applied mathematics is distinguished by iconic formulas such as

$$\begin{aligned} F &= G\frac{m_1 m_2}{r^2}, \\ E &= mc^2, \\ H(X) &= -\sum_x p(x)\log_2 p(x), \\ w_0 &= z_0\mathbf{N}(d_1) - \kappa e^{-\rho\tau}\mathbf{N}(d_2). \end{aligned}$$

The first two formulas are deterministic, while the latter two are stochastic in the sense that they involve estimated or approximate values of some unpredictable quantity.

The third formula is Shannon's entropy or uncertainty formula in the theory of communication. The fourth is the Black-Scholes formula giving the present value w_0 of an option to purchase, at time τ (in the future), a share whose future price is unpredictable but whose present value is known to be z_0. The function $\mathbf{N}$ is the cumulative distribution function of the standard normal probability distribution. The parameters d_1 and d_2 depend on the *volatility* or fluctuations over time of the share price.

An *option* to buy a share is a contract conferring (on the holder of the option) the *right*, but not the *obligation*, to purchase a particular share (the *underlying* asset). Options are one of a class of *derivative assets* which are financial instruments whose price is dependent upon or *derived from* one or more underlying assets. The value of a derivative depends on the fluctuations in the the value of the underlying asset. The most common underlying assets include stocks, bonds, commodities, currencies, interest rates and market indexes. Futures

contracts, forward contracts, options and swaps are common types of derivatives.

Derivatives provide insurance or protection against adverse movements in the value of the underlying assets. Alternatively they can be used to make bets (or wagers) on future changes in the value of the underlying assets.

The fourth formula above, discovered in 1973 by Fischer Black, Myron Scholes and Robert Merton, purported to solve the problem of deciding how much to pay for such insurance or wager. This discovery gave rise to a vast financial industry into which a great part of the wealth of the world is now poured. The industry, and the mathematical pricing theory on which it is based, are controversial and problematic. Also, mathematically reliable pricing theory is quite difficult even for otherwise competent finance practitioners.

Rigorous pricing involves the mathematical theory of probability pioneered by A.N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitreichnung* (1933), which in turn is based on the theories of measure and integration produced by Borel and Lesbesgue.

But in the 1950's a theory of integration based on Riemann sums rather than measure theory was developed independently by R. Henstock and J. Kurzweil, and this approach enables probability theory, and pricing theory, to be presented in a more accessible way.

References

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