GENERALIZED WIEDEMANN–FRANZ LAW AND ITS APPLICATION TO A STUDY OF THE SOLAR TRANSITION REGION PHENOMENA

P. A. Bespalov[1] and O. N. Savina[2]

[1]Institute of Applied Physics, Russian Academy of Sciences,
Ul'yanova 46, Nizhni Novgorod, 603600 Russia
[2] National Research University Higher School of Economics, 25/12
Bolshaia Pecherskaja Ulitsa 603155, Nizhny Novgorod, Russia

OUTLINE

1. ***Wiedemann–Franz law***
2. ***The anomalous thermal conductivity***
3. ***The anomalous electrical conductivity***
4. ***The generalized Wiedemann–Franz law for a plasma with ion-acoustic turbulence and Coulomb collisions***
5. ***The Solar transition region***
6. ***Model solution of the heat transfer equation for the lower corona and the Solar transition region***
7. ***Preliminary comparison of the calculation results with the experimental data about the Solar transition region***
8. ***Conclusion***

1. Wiedemann–Franz law

German physics G. Wiedemann and R. Franz in 1853 experimentally found the law, which connects electrical conductivity and thermal conductivity of the metals. They determined that the ratio of the coefficient of thermal conductivity χ to the conductivity σ at constant temperature is equal for all metals:

$$\frac{\chi}{\sigma} = \mathbf{const}$$

Danish physicist L. Lorenz showed in 1982 that this relation varies proportionally to the absolute temperature:

$$\frac{\chi}{\sigma} = \mathbf{L \cdot T}$$

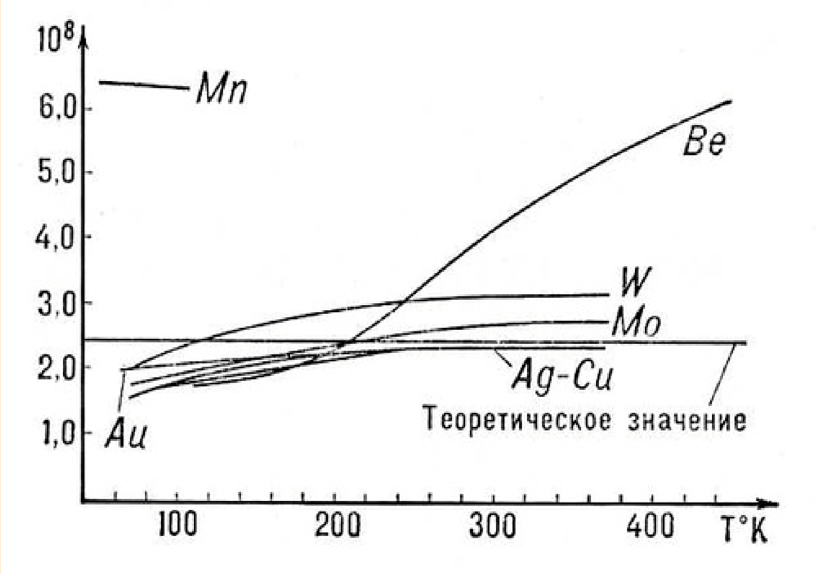
where L - Lorentz number, identical for all metals.

This regularity was explained by German physicist P. Drude, who considered electrons in the metal as gas. However, the expression for Lorentz number well consistent with the observations was obtained only with the aid of the quantum statistics, according to conclusions of which (Aschkroft and Mermim of 1979):

$$L = \frac{\pi^2}{3}\left(\frac{\kappa}{e}\right)^2 = 2.45 \cdot 10^{-8} W \cdot \Omega \cdot grad^{-2}$$

where κ is the Boltzmann constant, e is the electron charge. Wiedemann - Franz law agree with the experiments

In this report we will to discuss several sufficiently complex theoretical problems. First of all I want to remind you of Wiedemann - Franz law. According to this law the metals thermal conductivity coefficient is proportional to their electrical conductivity coefficient, to absolute temperature and universal constant.



Here are shown dependence of Lorentz numbers $L \cdot 10^8$ on the temperature for some metals

2. The anomalous thermal conductivity

Let us write the equations for a stationary temperature jump with developed ion-acoustic turbulence (Braginsky, 1963; Rudakov and Korablev 1966; Vedenov and Ryutov 1972; Bakhareva and Trakhtengerts 1983) and Coulomb collisions (Kovalev and Korolev 1981; Gomes and Mauas 1992) in the quasi-linear approximation outside any sources of heat

$$vx\frac{\partial f}{\partial z} = \widehat{St}(\varepsilon_{\vec{k}})f = \frac{\partial}{\partial x}\left(\frac{D}{v^3}\cdot\frac{\partial f}{\partial x}\right) + \nu\left(\frac{v_T}{v}\right)^3\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial f}{\partial x}\right),$$

$$D = \left(\frac{4\pi e}{m}\right)^2\int_0^\infty\int_{-1}^{+1}\varepsilon_{\vec{k}}y^2 k Re(1-x^2-y^2)^{-1/2}dydk,$$

$$\varepsilon_{\vec{k}} = \begin{cases} >0, & if\ \hat{\gamma}(f)_{\max} = 0; \\ 0, & if\ \hat{\gamma}(f) < 0, \end{cases}$$

$$\hat{\gamma}(f) = \pi^2\omega\left(\frac{\omega}{k}\right)^3\frac{m_i}{mn}\left\{-F\left(\frac{\omega}{k}\right) - \frac{m}{m_i}F_i\left(\frac{\omega}{k}\right) + \right.$$

$$\left. + \frac{ky}{2\omega}\int_0^\infty\int_{-1}^{+1}\frac{\partial f}{\partial x}Re(1-x^2-y^2)^{-1/2}dxdv\right\}.$$

For many applications it is important to know the coefficient of the turbulent plasma thermal conductivity. If the magnetic field is not substantial, plasma non-isothermal, and turbulence presents in form of ion-acoustic oscillations, one-dimensional in the coordinate space problem is reduced to the analysis of the following system of quasilinear equations. Here f is the distribution function of electrons, ε_k the power spectral density of ion acoustic oscillations, the last term in the kinetic equation corresponds to Coulomb collisions.

The ion-acoustic oscillations have the known dispersion relation:

$$\omega = \frac{kv_s}{(1+k^2r_D^2)^{1/2}}, \quad v_{Ti} < \frac{\omega}{k} < v_{Ti}\left(\frac{T}{T_i}\right)^{1/2}$$

where $v_s = \omega_{pi}r_D$, ω_{pi} is the plasma ion frequency and r_D is the Debye radius.
Similar equations are valid for some regimes of the plasma cloud turbulent dispersion (Bespalov and Trakhtengerts 1974; Bespalov and Efremova 1993)

3. The anomalous electrical conductivity

We will use the results of studying the well-known problem of the anomalous electrical conductivity for a nonisothermal plasma with developed ion-acoustic turbulence. The classical problem of the anomalous electrical conductivity can be solved using the system of equations (Rudakov and Korablev 1966; Vedenov and Ryutov 1972; Galeev and Sagdeev 1973; Kadomtsev, 1977):

$$-\frac{eE_z}{m}\cdot\frac{\partial f}{\partial v_z}=\widehat{St}(\varepsilon_{\vec{k}})f,$$

$$\varepsilon_{\vec{k}}=\begin{cases} >0, & \text{если} \quad \hat{\gamma}(f)_{max}=0; \\ 0, & \text{если} \quad \hat{\gamma}(f)<0, \end{cases}$$

On the other hand the problem of anomalous plasma conductivity with ion acoustic turbulence is known. Many famous scientists studied this problem in connection with the works on plasma heating. Here collision term and increment are determined by the same expressions as on the foregoing slide.

where $\vec{E}$ is the electric field, $\widehat{St}(\varepsilon_{\vec{k}})$ is operator that takes into account the scattering of suprathermal electrons by ion-acoustic waves with a spectral energy density $\varepsilon_{\vec{k}}$. In this case, the instability growth rate $\hat{\gamma}(f)$ defined by the equation the foregoing slide.

A relation that are important for the problem of the anomalous resistivity follows from preveous equations

$$j_z=-\frac{4}{3}\pi e\int_0^\infty v^3 f_1 dv=\sigma_\Sigma E_z,$$

$$\sigma_\Sigma=\sigma_{eff}+\sigma=\frac{16\pi e^2\int_0^\infty v^5 f dv}{3m\int_{-1}^1 D dx}+\frac{4\pi e^2}{3\nu\kappa T}\int_0^\infty v^5 f dv.$$

where we introduced the total and effective electrical conductivities.

4. *The generalized Wiedemann–Franz law for a plasma with ion-acoustic turbulence and Culomb collisions* *(Bespalov, Savina, 2007)*

We found the relationship between the solutions of two mentioned problems. We will now examine this question in more detail.

Let us assume that the problem of the plasma layer anomalous resistivity has a solution the first two terms of whose Legendre polynomial expansion can be written as,

$$f(x,v) = f_\circ(v) + f_1(v)x,$$

where $|f_1| \ll f_\circ$ under conditions of developed instability.

Let us now return to the problem of the anomalous thermal conductivity for the same layer of plasma between the two grounded planes with different temperatures. We will restrict our analysis to a sufficiently thin layer of the temperature jump compared to the mean free path. We will seek a solution of this problem in the form:

$$f(z,x,v) = \Phi(v)z + f_\circ(v) + \frac{4}{3}[f_1(v) + \frac{v}{4}\cdot\frac{\partial f_1}{\partial v}]x,$$

where z is counted off from the center of the thin temperature jump. Here, $f_\circ$ and $f_1(v)$ are the same functions as those in the solution of the problem of the anomalous electrical conductivity. We verified that the function $\Phi(v)$ could be chosen in such a way that this equation will give the first two terms of the Legendre polynomial expansion of the solution to problem of the anomalous thermal conductivity.

First, for any differentiable bounded function f , the following condition is satisfied

$$u_z = (2\pi/n)\int_0^\infty \int_{-1}^1 fxv^3 dx dv = 0 .$$

Second, after substituting f into equation for the growth rate of ion-acoustic instability, the growth rate is found to be the same as that in the problem of the anomalous electrical conductivity. Identical will be the power spectral densities of the ion-acoustic oscillation in both layers

Thirdly, let us define the dependence $\Phi(v)$ so that, kinetic equation would be carried out with an accuracy to the first two terms of expansion in terms of Legendre's polynomials.

With the aid of this condition we obtain ($f_1(v) \neq 0$ for the speeds more then thermal):

$$\Phi(v) = -\frac{2}{v^4}\left(\int_{-1}^{1} D dx\right)\left(f_1 + \frac{v}{4}\cdot\frac{\partial f_1}{\partial v}\right).$$

We verified, that this function is the solution of the anomalous thermal conductivity problem if we select function Φ *in the form.*

Let us now write the expression for the heat flux

$$q_z = \pi m \int_0^\infty \int_{-1}^{+1} x v^5 f dx dv =$$

$$= -\frac{2\pi m \partial/\partial z(\int_0^\infty \int_{-1}^{+1} v^9 f dx dv)}{9\int_{-1}^{+1} D dx} =$$

The heat flux is reduced to the Fourier formula

$$= -\frac{m^2 \sigma_\Sigma \partial/\partial z[(\int_0^\infty \int_{-1}^{+1} v^9 f dx dv)(\int_0^\infty \int_{-1}^{+1} v^7 f dx dv)^{-1}]}{12e^2[(\int_0^\infty \int_{-1}^{+1} v^5 f dx dv)(\int_0^\infty \int_{-1}^{+1} v^7 f dx dv)^{-1}]} =$$

$$= -\frac{4\delta\sigma_\Sigma}{9e^2}\langle\frac{mv^2}{2}\rangle\frac{\partial}{\partial z}\langle\frac{mv^2}{2}\rangle,$$

where δ is a numerical coefficient of the order of unity; the brackets $\langle...\rangle$ denote an averaging. Given definition is reduced to the Fourier formula

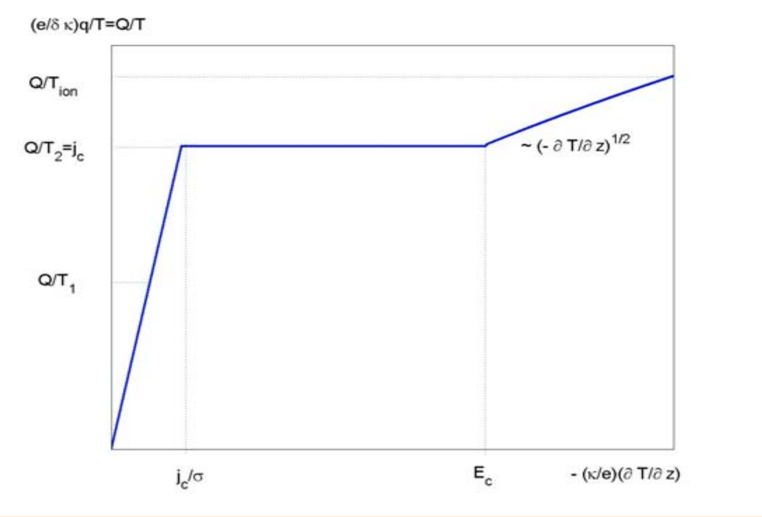
$$q_z = -\chi_\Sigma \frac{\partial T}{\partial z},$$

where we introduced the total and effective thermal conductivities

$$\chi_\Sigma = \frac{\kappa^2}{e^2} T\delta\sigma_\Sigma, \quad \sigma_\Sigma = \sigma + \sigma_{eff}$$

The effective electrical conductivity itself in the problem of the anomalous electrical conductivity is known to be a function of E_z . *Comparing two problems, we can verify that in equation for* χ , E_z *in* σ_{eff} *should be replaced by the expression* $-(3\kappa/2e)\partial T/\partial z$.

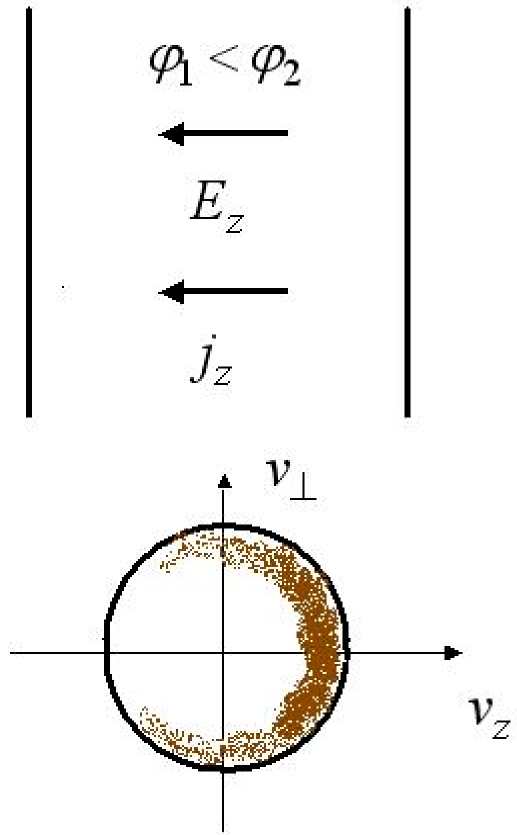
The derived expression formally differs from the standard Wiedemann–Franz law (see, e.g., Ashcroft and Mermin 1976) only by the numerical coefficient. A significant functional difference between two equations is that equation contains a non-linear characteristic of the medium that has been studied in many papers —the anomalous electrical conductivity of a plasma with ion-acoustic turbulence $\sigma_{eff}(E_z)$.



This figure shows the typical dependence of electric current density on electric field. Knowing this dependence we immediately obtain the connection of heat flux from the temperature and its gradient in the different regimes of thermal conductivity.
The connection of two examined problems in the compressed form is explained on the following slide.

$$-\left(\frac{e}{\delta\kappa}\right)\frac{q}{T}=\begin{cases}8.4\cdot 10^{-2}\,\frac{(\kappa T)^{3/2}\kappa}{e^{3}m^{1/2}}\frac{\partial T}{\partial z}, & \left(\frac{e}{\delta\kappa}\right)\frac{q}{T}\leq j_c;\\ 1.7en\left(\frac{\kappa T}{m_i}\right)^{1/2}, & -\frac{\kappa}{e}\frac{\partial T}{\partial z}\leq E_c;\\ 1.7en\left(\frac{\kappa T}{m}\right)^{1/2}\frac{\kappa^{1/2}}{e^{1/2}(8\pi n\kappa T)^{1/4}}\left|\frac{\partial T}{\partial z}\right|^{1/2}, & E_c\leq -\frac{\kappa}{e}\frac{\partial T}{\partial z}.\end{cases}$$

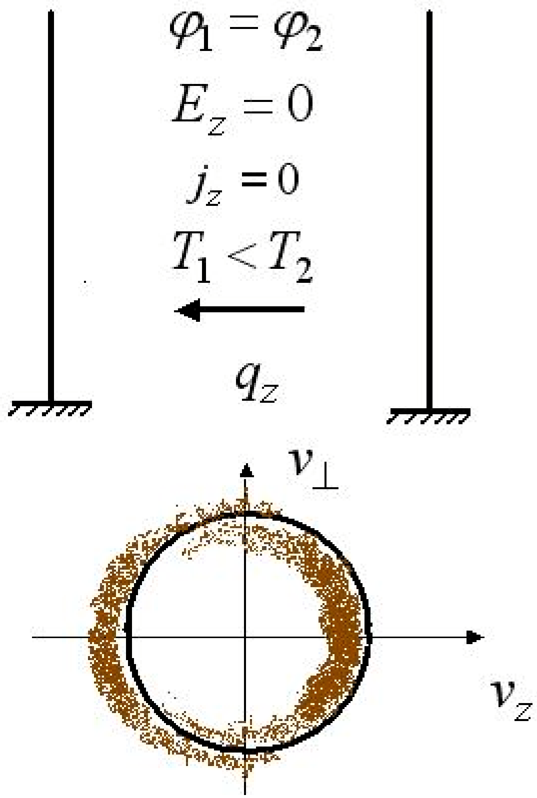
The anomalous electrical conductivity in the layer of the plasma



$$f(x,v) = f_\circ(v) + f_1(v)x$$

$$\varepsilon_{\bar{k}}$$

The anomalous thermal conductivity in the same layer of the plasma

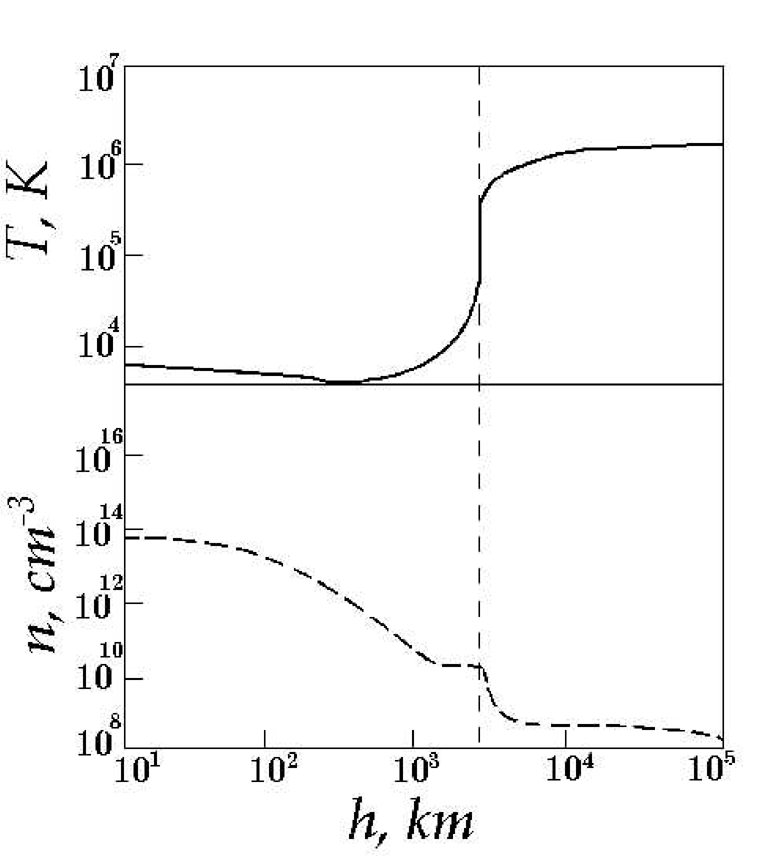


$$f(z,x,v) = f_\circ(v) + \frac{4}{3}\left(f_1(v) + \frac{v}{4}\frac{\partial f_1}{\partial v} \right)x + \Phi(v)z$$

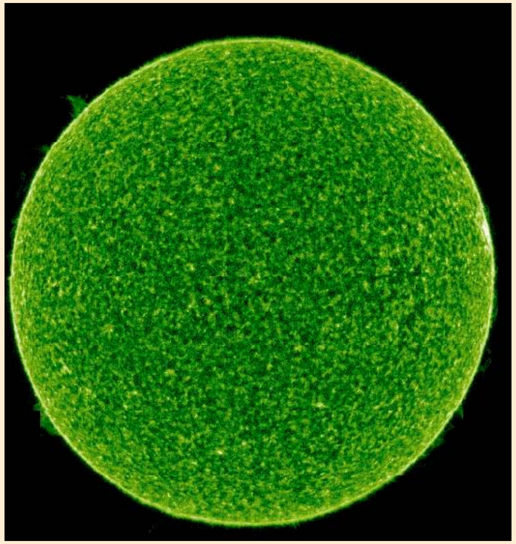
$$\varepsilon_{\bar{k}}$$

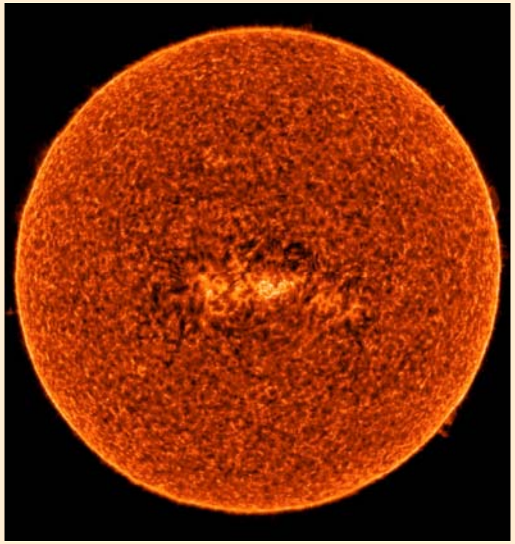
5. The Solar transition region

We applied the obtained results for explaining the properties of narrow temperature jump in the transition region of solar atmosphere between the corona and the chromosphere. In this region temperature falls from $5 \cdot 10^5$ *to* $5 \cdot 10^4$ *grad at a distance smaller than of one hundred kilometers, and according to some data even by several kilometers.*



Gibson «Quit Sun», 1977





The images from the SUMER instrument on the SOHO Mission.

Solar Maximum Mission and the Solar and Heliospheric Observatory(SOHO)
The Transition Region and Coronal Explorer (TRACE) mission.

The left image (green) is emission from Carbon IV at temperatures of about 100,000°C. The right image (red) is emission from Sulfur VI at temperatures of about 200,000°C.

6. Model solution of the heat transfer equation for the lower corona and the solar transition region (Bespalov and Savina 2008)

The results of our calculations were obtained by assumption that the heat flux is constant and equal to (Gibson 1973)

$$q = 5 \cdot 10^5 \, erg \cdot cm^{-2} \cdot s^{-1}.$$

As the boundary conditions for the heat conduction equation in the corona at $z = z_1$, *we will take the temperature* $T_1 = 2 \cdot 10^6 K$ *and the electron density* $n_1 = 10^8 cm^{-3}$. *It is easy to verify that the boundary conditions correspond to a heat flux below the critical one. Therefore, to determine the co-ordinate dependence of the temperature, we should use equation, which corresponds to the conservation of the heat flux transferred by electron Coulomb collisions.*

If plasma is isothermal, then the heat flux transferred by ions is lower approximately by a factor of $(m_i / m)^{1/2}$. We will disregard the heat flux transferred by ions. Hence

$$\frac{\partial T}{\partial z} = -\frac{1.2 \cdot e^4 m^{1/2} q}{\delta \kappa^{7/2} T^{5/2}}$$

the temperature distribution does not depend on the height profile of the density. In accordance with this equation, the temperature in the lower corona decreases*:*

$$\frac{T(z)}{T_1} = [1 - \frac{(z - z_1)}{(\Delta z)_1}]^{2/7},$$

where

$$(\Delta z)_1 = 0.11 \frac{(\kappa T_1)^{7/2}}{m^{1/2} e^4 q} \approx 5 \cdot 10^{10} \, cm.$$

The temperature will decrease until the constant heat flux becomes critical at some depth. We showed that this position exactly coincides with the upper edge of the transition region. The functional dependence of heat flux on the gradient of temperature changes in the transition region. Therefore temperature falls almost exponentially with the scale determined by Debye radius.

Here, we do not set the goal of determining the density distribution in the lower corona. Therefore, for a preliminary determination of the density distribution, let us write the local condition for pressure balance:

$$P = 2n\kappa T = const\,.$$

The electron plasma density in the lower corona changes in accordance with this equation and accepted boundary conditions:

$$\frac{n(z)}{n_1} = \frac{T_1}{T(z)}.$$

The temperature will decrease until the constant heat flux becomes critical at some depth and the transition region. corresponding to developed ion-acoustic turbulence. According to these equations, we have

$$\frac{T(z)}{T_1} = [1 - \frac{(z - z_1)}{(\Delta z)_1}]^{2/7} = \frac{q^2}{(1.7 \cdot \delta n_1)^2 (\kappa T_1)^3},$$

From this relation, we can find $z = z_2$ for which the constant heat flux reaches its critical value. Numerically solving the system of equations yields $T = T_2 = 5.5 \cdot 10^5\,K$, $n = n_2 = 4 \cdot 10^8\,cm^{-3}$, and $z_2 - z_1 = 4.95 \cdot 10^{10}\,cm$. These parameters are close to those obtained above and we will take them as the basis in calculating the parameters of the transition region..

Developed ion-acoustic turbulence takes place in the region of the temperature jump at $z_2 < z$. Therefore, as the dependence of the heat flux on the temperature and its gradient should be used equation points to a jump in temperature gradient at $z = z_2$. The subsequent nearly exponential decrease in temperature and increase in density are defined by equations

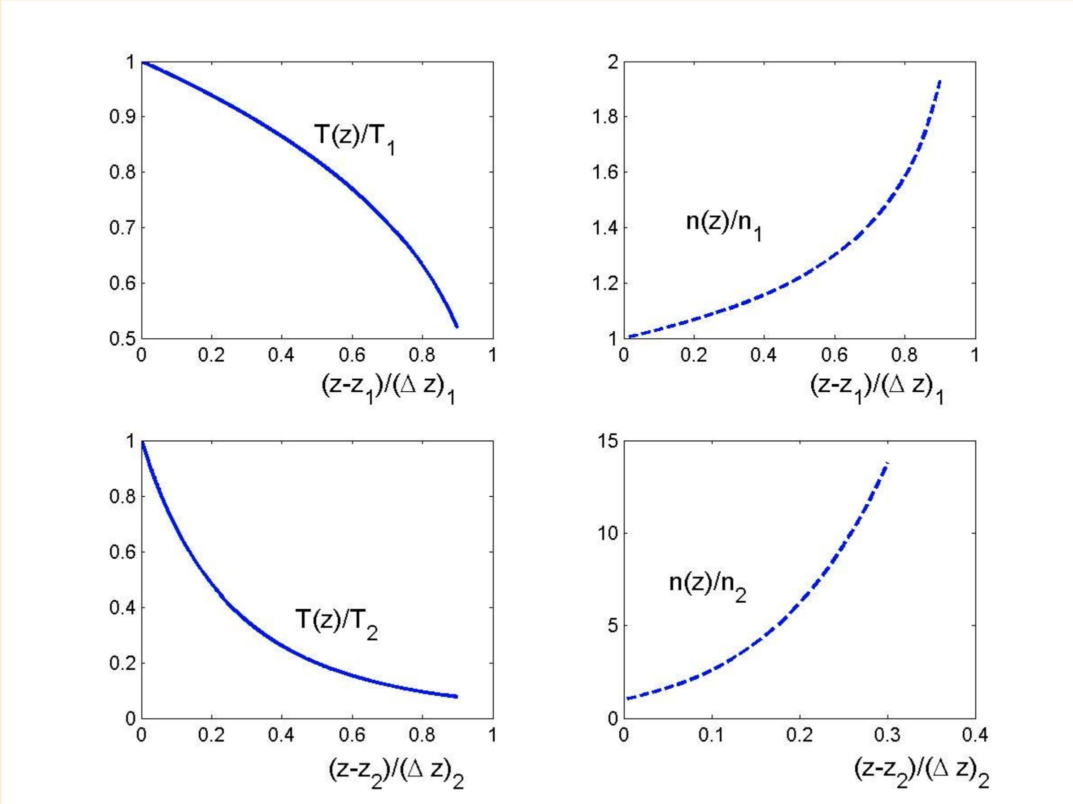
$$\frac{\partial T}{\partial z} = -q^2 \frac{(2\pi)^{1/2} me}{\delta^2 \kappa^{7/2} n_2^{3/2} T_2^{15/4}} T^{5/4}, \qquad \frac{T(z)}{T_2} = [1 + \frac{(z - z_2)}{(\Delta z)_2}]^{-4},$$

Consequently, in the transition region

where r_{D2} is the Debye radius at the boundary of the transition region.

$$(\Delta z)_2 = 4.4 \frac{m^{1/2}}{n_2^{1/2} e} (\frac{\kappa T_2}{m})^{7/2} (\frac{n_2 m}{q})^2 \approx 10 \cdot r_{D2}.$$

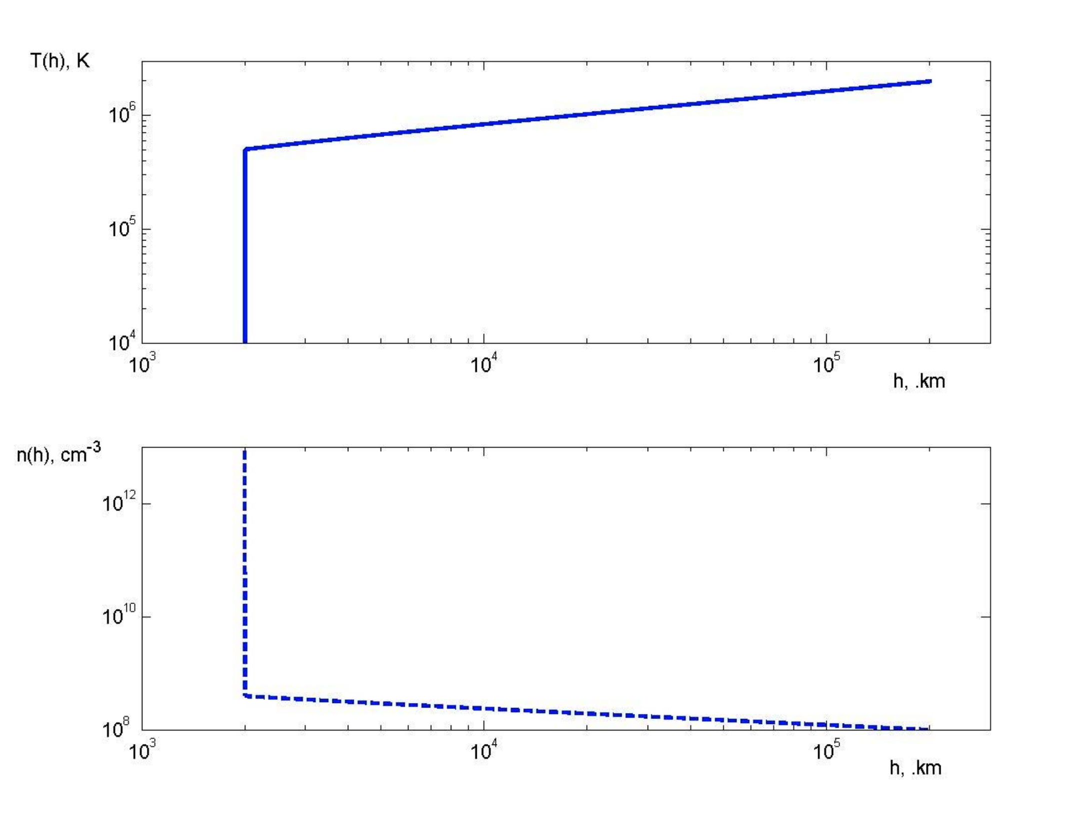
7. Preliminary comparison of the calculation results with the experimental data about the Solar atmosphere transition region *(Bespalov and Savina 2008)*



The electron density at the boundary of the jump increases from the polar to equatorial latitudes and lies within the range $n_2 = (2.3 \div 6.3) \cdot 10^8\ cm^{-3}$ (Gallagher et al. 1999). The temperature at the boundary of the temperature jump is $T_2 = 5 \cdot 10^5\ K$. For definiteness, we will take protons as the ions, although, in reality, the ion mass is several times larger.

Substituting all parameters into formulas yields an estimate of $q = (2.1 \div 5.6) \cdot 10^5\ erg \cdot cm^{-2} \cdot s^{-1}$.

Note that we do not know any direct proofs of the existence of ion-acoustic turbulence in the Solar transition region. Turbulence of this type was invoked in connection with the problem of suprathermal particle acceleration and with the solution of the question about the conversion of electrostatic oscillations into electromagnetic ones (Zheleznyakov et al. 1996).



So, we determined the relationship between the coefficients of anomalous thermal and electrical conductivities for a certain class of turbulent plasma tasks.
The results of the model calculations of the high-altitude temperature profile in the lower corona and the Solar transition region explain rather well the known experimental data.

The given graphs many years serve as the object of serious searches. There is no doubt that higher than transition regions are significant heat sources. Actually heat balance in the Solar atmosphere depends on the absorption of hydromagnetic waves, braking of the particle fluxes, ohmic heating and radiation losses. On each of these questions there is an extensive literature.

The local temperature minimum observed in the chromosphere cannot be explained without including the radiative losses. Shmeleva and Syrovatskii (1973) showed that the ultraviolet radiation of the medium affects the height profile of the temperature in the solar atmosphere. Taking into account this effect and the ordinary thermal conductivity, the authors managed to explain the possibility of the formation of a temperature jump with a scale length of the order of $4 \cdot 10^6 \div 4 \cdot 10^7\ cm$. The experimental results of more recent observations give an appreciably smaller thickness of the Solar transition region. The scale length of the temperature variations in the lower corona obtained by Shmeleva and Syrovatskii (1973), $10^8\ cm$, is close to its observed value.

A fundamental problem is to explain the details of the temperature jump between the chromosphere and the corona. The temperature in the Solar transition region increases from a value of less than $10^4 K$ to $5 \cdot 10^5 K$ at a height of $2000 km$ above the convection zone (Gibson 1973; Aschwander). The jump has a small thickness compared to the particle mean free path, no more than $100 km$ and, according to some data, of the order of several kilometers.

The existence of a temperature jump was associated with the height variations in the ionization state of the medium when the non-Maxwellian distribution function of the particles responsible for the ionization of the medium was taken into account (Dupre 1980a, 1980b). Subsequently, experimenters pointed out that their data were difficult to explain in terms of the ionization equilibrium model (Doschek et al. 1997). A model of the transition region that included the effects of gravity, thermal conduction, heating, and radiative cooling was considered by Woods et al. (1990), who obtained their main results by assuming a nearbalance between heating and cooling, i.e., far from the main temperature jump. The radiative losses were measured, calculated, and taken into account in many papers (see, e.g., Fontenla et al. 1999). In recent years, researchers have returned to a discussion of the possibility that the magnetic field affects the processes near the temperature jump. The magnetic field in the Solar transition region outside active regions is weak ($1 \div 5 G$) and highly nonuniform.

Observations of the Solar transition region showed that the hydrogen temperature rises relatively slowly compared to the electron temperature even at slightly larger heights (Marsch et al. 2000). This suggests that the ion temperature in the Solar transition region is much lower than the electron one.

8. Conclusion

The relationship between the coefficients of anomalous thermal and electrical conductivities is examined for a certain class of turbulent plasma tasks.

It is checked, that the heat flux through the Solar transition region correspond precisely to critical value.

The results of the model calculations of the high-altitude temperature profile in the lower corona and the Solar transition region explain rather well the known experimental data.