Derivation of the Consumption Euler Equation

Alexander Larin

National Research University — Higher School of Economics
Nizhny Novgorod

Research Seminar
"Microeconomic Assumptions of Macroeconomic Models"

March 10, 2015

About Euler Equation

- First-order condition (FOC) for the optimal consumption dynamics
- Shows how household choose current consumption c_t, when explicit consumption function is non available
- The most popular version

$$\mathrm{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1} \right] = 1. \quad (1)$$

- Here
  - $\mathrm{E}_t[\cdot] \equiv \mathrm{E}[\cdot | I_t]$ — all the information available in period t
  - c_t — real consumption
  - $R_{t,t+1}$ — real interest rate between t and $t+1$
  - β — subjective discount
  - γ — relative risk aversion coefficient
  - $1/\gamma$ — elasticity of intertemporal substitution

Household Optimization Problem

- Expected lifetime utility

$$U_t = \mathrm{E}_t \left[u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \ldots \right] \rightarrow max_{c_t, c_{t+1}, c_{t+2}, \ldots}. \quad (2)$$

- Preferences (time-separable, labor-separable, constant relative risk aversion)

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} + \ldots. \quad (3)$$

- Budget constraint

$$A_{t+1} = (A_t + y_t - c_t) R_{t,t+1}. \quad (4)$$

- Here, A_t is the stock of assets and y_t denotes labor income

Derivation. Calculus of Variations Approach

- Full differential of U_t

$$\mathrm{d}U_t = \mathrm{E}_t \left[u'(c_t)\mathrm{d}c_t + \beta u'(c_{t+1})\mathrm{d}c_{t+1} + \beta^2 u'(c_{t+2})\mathrm{d}c_{t+2} + \ldots \right]. \quad (5)$$

- Assume we are in optimum so that

$$\mathrm{d}U_t / \mathrm{d}c_t = 0. \quad (6)$$

- Consider the case when household transfer money from $t+1$ to t only so that

$$\mathrm{d}c_{t+1} = -R_{t,t+1}\mathrm{d}c_t. \quad (7)$$

- From equations 5, 6, 7

$$\mathrm{E}_t \left[u'(c_t) - \beta u'(c_{t+1}) R_{t,t+1} \right] = 0, \quad (8)$$

$$\mathrm{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t,t+1} \right] = 1, \quad (9)$$

$$\mathrm{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1} \right] = 1. \quad (10)$$

Bellman Equation

- Let's rewrite household optimization problem in a recursive form (Bellman equation)
$$V_t(A_t) = max_{c_t}\left\{u(c_t) + \beta \mathrm{E}_t\left[V_{t+1}(A_{t+1})\right]\right\}. \tag{11}$$
- Budget constraint
$$A_{t+1} = (A_t + y_t - c_t)R_{t,t+1}. \tag{12}$$
- FOC
$$u'(c_t) + \beta \mathrm{E}_t\left[V'_{t+1}(A_{t+1})\frac{\partial A_{t+1}}{\partial c_t}\right] = 0, \tag{13}$$
- Easy to show that in optimum $V'_{t+1}(A_{t+1}) = u'(c_{t+1})$ (Envelope theorem).
- From the budget constraint $\partial A_{t+1}/\partial c_t = -R_{t,t+1}$.
- Substituting this into equation 13, we get

$$u'(c_t) = \beta \mathrm{E}_t\left[u'(c_{t+1})R_{t,t+1}\right], \tag{14}$$
$$\mathrm{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}R_{t,t+1}\right] = 1, \tag{15}$$
$$\mathrm{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}R_{t,t+1}\right] = 1. \tag{16}$$

Liquidity constraint

- Borrowing constraint: $A_t + y_t - c_t \geq 0$
- Calculus of variations
  - Benefit from transferring money: $\mathrm{E}_t\left[u'(c_t) - \beta u'(c_{t+1}) R_{t,t+1}\right] \geq 0$
  - Euler equation for those who at the corner

$$\mathrm{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t,t+1}\right] \leq 1. \tag{17}$$

- Bellman equation

$$V_t(A_t) = max_{c_t}\left\{u(c_t) + \beta \mathrm{E}_t\left[V_{t+1}(A_{t+1})\right] - \lambda_t(A_t + y_t - c_t)\right\} \tag{18}$$

  - λ_t shows the increase of utility from relaxing constraint by $1
  - Euler equation

$$\mathrm{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t,t+1}\right] = 1 - \frac{\lambda_t}{u'(c_t)}, \tag{19}$$

$$\lambda_t = 0 \quad if \quad A_t + y_t - c_t > 0. \tag{20}$$

Liquidity constraint (2)

- Different borrowing and lending rates: $R^C_{t,t+1} > R^D_{t,t+1}$
- Budget constraint

$$A^D_{t+1} - A^C_{t+1} = (A^D_t - A^C_t + y_t - c_t)(\alpha_t R^D_{t,t+1} + (1-\alpha_t) R^C_{t,t+1}) \qquad (21)$$

- Euler equations

$$\mathrm{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} R^C_{t,t+1} \right] \geq 1, \qquad (22)$$

$$\mathrm{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} R^D_{t,t+1} \right] \leq 1. \qquad (23)$$