Derivation of the Consumption Euler Equation

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About Euler Equation

- First-order condition (FOC) for the optimal consumption dynamics
- Shows how household choose current consumption \( c_t \), when explicit consumption function is non available
- The most popular version

\[
E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1} \right] = 1. \tag{1}
\]

- Here
  - \( E_t[\cdot] \equiv E[\cdot|I_t] \) — all the information available in period \( t \)
  - \( c_t \) — real consumption
  - \( R_{t,t+1} \) — real interest rate between \( t \) and \( t+1 \)
  - \( \beta \) — subjective discount
  - \( \gamma \) — relative risk aversion coefficient
  - \( 1/\gamma \) — elasticity of intertemporal substitution
Household Optimization Problem

- Expected lifetime utility

\[ U_t = E_t \left[ u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \ldots \right] \rightarrow \max_{c_t, c_{t+1}, c_{t+2}, \ldots} \] (2)

- Preferences (time-separable, labor-separable, constant relative risk aversion)

\[ u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} + \ldots \] (3)

- Budget constraint

\[ A_{t+1} = (A_t + y_t - c_t)R_{t,t+1}. \] (4)

- Here, \( A_t \) is the stock of assets and \( y_t \) denotes labor income
Derivation. Calculus of Variations Approach

- Full differential of $U_t$
  \[ dU_t = E_t \left[ u'(c_t)dc_t + \beta u'(c_{t+1})dc_{t+1} + \beta^2 u'(c_{t+2})dc_{t+2} + \ldots \right]. \] (5)

- Assume we are in optimum so that
  \[ dU_t/dc_t = 0. \] (6)

- Consider the case when household transfer money from $t + 1$ to $t$ only so that
  \[ dc_{t+1} = -R_{t,t+1}dc_t. \] (7)

- From equations 5, 6, 7
  \[ E_t \left[ u'(c_t) - \beta u'(c_{t+1})R_{t,t+1} \right] = 0, \] (8)
  \[ E_t \left[ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t,t+1} \right] = 1, \] (9)
  \[ E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1} \right] = 1. \] (10)
Bellman Equation

- Let’s rewrite household optimization problem in a recursive form (Bellman equation)

\[ V_t(A_t) = \max_{c_t} \left\{ u(c_t) + \beta E_t \left[ V_{t+1}(A_{t+1}) \right] \right\}. \]  \hspace{1cm} (11)

- Budget constraint

\[ A_{t+1} = (A_t + y_t - c_t)R_{t,t+1}. \]  \hspace{1cm} (12)

- FOC

\[ u'(c_t) + \beta E_t \left[ V'_{t+1}(A_{t+1}) \frac{\partial A_{t+1}}{\partial c_t} \right] = 0, \] \hspace{1cm} (13)

- Easy to show that in optimum \( V'_{t+1}(A_{t+1}) = u'(c_{t+1}) \) (Envelope theorem).

- From the budget constraint \( \frac{\partial A_{t+1}}{\partial c_t} = -R_{t,t+1} \).

- Substituting this into equation 13, we get

\[ u'(c_t) = \beta E_t \left[ u'(c_{t+1})R_{t,t+1} \right], \] \hspace{1cm} (14)

\[ E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)}R_{t,t+1} \right] = 1, \] \hspace{1cm} (15)

\[ E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1} \right] = 1. \] \hspace{1cm} (16)
Liquidity constraint

- Borrowing constraint: \( A_t + y_t - c_t \geq 0 \)
- Calculus of variations
  - Benefit from transferring money: \( E_t [u'(c_t) - \beta u'(c_{t+1})R_{t,t+1}] \geq 0 \)
  - Euler equation for those who at the corner
    \[
    E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1} \right] \leq 1. \tag{17}
    \]
- Bellman equation
  \[
  V_t(A_t) = \max_{c_t} \left\{ u(c_t) + \beta E_t [V_{t+1}(A_{t+1})] - \lambda_t (A_t + y_t - c_t) \right\} \tag{18}
  \]
- \( \lambda_t \) shows the increase of utility from relaxing constraint by $1
- Euler equation
  \[
  E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1} \right] = 1 - \frac{\lambda_t}{u'(c_t)}, \tag{19}
  \]
  \[\lambda_t = 0 \text{ if } A_t + y_t - c_t > 0.\] \tag{20}
Liquidity constraint (2)

- Different borrowing and lending rates: \( R_{t,t+1}^C > R_{t,t+1}^D \)
- Budget constraint

\[
A_{t+1}^D - A_{t+1}^C = (A_t^D - A_t^C + y_t - c_t)(\alpha_t R_{t,t+1}^D + (1 - \alpha_t)R_{t,t+1}^C)
\] (21)

- Euler equations

\[
E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1}^C \right] \geq 1, \quad (22)
\]

\[
E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t,t+1}^D \right] \leq 1. \quad (23)
\]