On integer programming with bounded sub-determinants

The Integer Programming in Simplex and Computing of The Simplex Width.

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Some definitions

$\Delta_k(A)$ – the maximum of absolute values of $k \times k$ sub-determinants of A.

$\Delta(A) = \Delta_{rank(A)}(A)$.

We say that $A \in \mathbb{Z}^{m \times n}$ is *k-modular* if $\Delta(A) \leq k$, where $k \in \mathbb{N}$.

We say that $A \in \mathbb{Z}^{m \times n}$ is *strict k-modular* if each $rank(A) \times rank(A)$ minor of A equal to k, $-k$ or 0.

We refer to 2-modular matrices as *bimodular* (Veselov, Chirkov).

We refer to 1-modular matrices as *unimodular*.

We say that $A \in \mathbb{Z}^{m \times n}$ is *almost unimodular* if $\Delta(A) = 2$ and $\Delta_{rank(A)-1}(A) \leq 1$. (Cornuéjols, Zuluaga)

Introduction

Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^n$.
And consider the polyhedron $P(A, b) = \{x \in \mathbb{R}^n : Ax \leq b\}$.

We investigate two problems:

P1: Is the $P(A, b) \cap \mathbb{Z}^n \neq \varnothing$. (NP-complete)

P2: Find the optimal solution of $max\{cx : x \in P(A, b) \cap \mathbb{Z}^n\}$ problem, where $c \in \mathbb{Z}^n$. (NP-hard)

In general, these problems are equivalent in terms of the polynomial reducibility.

Introduction

There are some quasi-polynomial algorithm approaches to solve the problem P2:

m and $\Delta(A\,b)$ are fixed: Dynamic programming. (Papadimitriou, Wolsey, Beckmann)

n is fixed: Lenstra's algorithm. (Lenstra, Grötschel, Lovász, Scarf, Kannan, Chirkov)

A is a square matrix: Group minimization. $O(n\Delta(A))$ time complexity. (Gomory)

A is k-modular: Gomory's (Shevchenko's in russian) conjecture, that there is polynomial algorithm to solve P2.

Integer polyhedron

Let $V(P)$ is the set of vertices of a polyhedron P. A polyhedron $P(A, b)$ is called integral polyhedron, if $V(P) \subseteq \mathbb{Z}^n$, $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^n$.

If the P is the integral polyhedron, than we can use some linear programming polynomial algorithms to solve the problem P2. (Khachiyan, Karmarkar, Novikov, Pardalos, Rao, Mehrotra and etc.)

Some testing criteria of integrality of P:

A is unimodular matrix: We can test unimodularity of the matrix by polinomial time. (Seymour, Hoffman, Kruskal)

Total dual integrality (TDI): It is the property of the system $Ax \leq b$. (Edmonds, Giles).

Any integral polyhedron can be represented by TDI system.

Some new definitions

Let A_{i*} be rows of matrix A.
Let u be a vertex of $P(A, b)$, $I(u) = \{i : A_{i*}u = b_i\}$,
$N(u) = \{x : A_{i*}x \leq b_i, \ where \ i \in I(u)\}$.
So $N(u)$ is the cone spanned by edges, which are adjacent to u.

Integer program with bimodular matrix

Following three theorems were proved by S.I. Veselov and A.J. Chirkov:

Theorem

If A be bimodular matrix, $b \in \mathbb{Z}^n$, and $P(A, b)$ be full-dimensional, then $P(A, b) \cap \mathbb{Z}^n \neq \emptyset$. There is polynomial time algorithm to construct the desired integer point.

Theorem

For every vertex v of $P_{int} = conv(P(A, b) \cap \mathbb{Z}^n)$ exists vertex u of P, such that v lies on a some edge of $N(u)$.

Theorem

If each $n \times n$ minor of A is not a zero, and A is bimodular. Than there is a polynomial time algorithm to solve $max\{cx : Ax \leq b,\ x \in \mathbb{Z}^n\}$ problem.

Integer program with bimodular matrix

From the theorem 1 follows:

Corollary

If A be bimodular matrix, then the feasibility problem for $P(A, b)$ is polynomially solvable.

Corollary

Let the extended matrix $\begin{pmatrix} c \\ A \end{pmatrix}$ is bimodular, than the problem $max\{cx : Ax \le b,\ x \in \mathbb{Z}^n\}$ can be solved using the polynomial time algorithm.

$\{0, 1\}$-case (graph interpretation)

The significant result were obtained if the matrix is a $\{0, 1\}$-matrix. (Alekseev & Zaharova)

Theorem

Let $A \in \{0, 1\}^{m \times n}$ and the extended matrix $\binom{\mathbf{1}}{A}$ is k-modular and rows of the A have at most 2 units. Than $max\{\mathbf{1}^T x : Ax \leq \mathbf{1},\ x \in \mathbb{Z}^n\}$ problem can be solved using the polynomial time.

Integer program with almost unimodular matrix

The next theorem follows from the theorem 1.

Theorem

Let A is almost unimodular. Than there is polynomial time algorithm to solve $max\{cx : Ax \leq b,\ x \in \mathbb{Z}^n\}$ problem.

Furthermore we can prove more general theorem by analogy:

Theorem

Let A is bimodular, and $\Delta_{r-s}(A) \leq 1$, where $r = rank(A)$ and s is parameter. Than there is a polynomial time algorithm to solve $max\{cx : Ax \leq b,\ x \in \mathbb{Z}^n\}$ problem. Polynomial has the power that proportional to s.

Flatness theorem

Let $width_c(P) = max\{cx : x \in P\} - min\{cx : x \in P\}$. We use the notion of the width(integer width) of a polyhedron (Minkowski): $width(P) = min\{width_c(P) : c \in \mathbb{Z}^n \setminus \{0\}\}$.

Theorem

(Khinchine) Let $P(A, b)$ is a polytope, such that $P(A, b) \cap \mathbb{Z}^n = \emptyset$. Then $width(P) \leq f(n)$, since $f(n)$ is value, that depends only on n.

Flatness theorem. History.

	f(n) upper bound
Khinchine'48	$(n+1)!$
Babai'86	$2^{O(n)}$
Lenstra-Lagarias-Schnorr'87	$n^{5/2}$
Kannan-Lovasz'88	n^2
Banaszczyk et al'99*	$n^{3/2}$
Rudelson'00	$n^{4/3} log^c n$

*[Banaszczyk-Litvak-Pajor-Szarek'99] Best known bound of $f(n)$

for simplexes is $O(n\, log\, n)$. (Banaszczyk-Litvak-Pajor-Szarek'99)

Bound conjectured to be is $\Theta(n)$. (Best possible)

Flatness theorem for symmetric convex bodies. History.

	f(n) upper bound
Khinchine'48	$n!$
Babai'86	$2^{O(n)}$
Kannan-Lovasz'88	n^2
Banaszczyk'93	$n^{3/2}$
Banaszczyk'96	$n \log n$

$f(n) = \Theta(n)$ for spheres.(Banaszczyk'93)

Bound conjectured to be is $\Theta(n)$. (Best possible)

Empty lattice simplexes. Lower bounds on $f(n)$.

The simplex S with **integral** vertexes called empty lattice, if $S \setminus V(S) \cap \mathbb{Z}^n = \emptyset$, where $V(S)$ is the set of vertexes of S.

Kantor'99: for any $0 < \epsilon \leq 1/e$, $\exists S$ - empty lattice n-simplex, such that $width(S) > \epsilon n$.

András Sebö'00:
Let's define $S_n(k) = conv(s_0, s1, \dots, s_n)$, $s_0 = 0$,
$s_1 = (1, k, 0, \dots, 0)$, $s_2 = (0, 1, k, 0, \dots, 0)$, $\dots$,
$s_{n-1} = (0, \dots, 0, 1, k)$, $s_n = (k, 0, \dots, 0, 1)$.
The width of $S_n(k)$ is k, unless both $k = 1$ and n is even.
If $k + 1 < n$, then $S_n(k)$ is an empty lattice simplex with integral vertexes.

So, the simplex $S_n(n-2)$ is empty lattice simplex and $width(S_n(n-2)) = n - 2$ for any $n \geq 3$.

Flatness theorem and strict Δ-modular matrices.

Theorem

Let A be a strict Δ-modular matrix, $P(A, b)$ be polytope. If width$(P(A, b)) > (n+1)(\Delta - 1)$, then $|\mathscr{P}(A, b) \cap \mathbb{Z}^n| \geq n + 1$. One can find these integer points using a polynomial time algorithm.

Flatness theorem for simplices.

Theorem

(Veselov, Gribanov) Let $A \in \mathbb{Z}^{(n+1)\times n}$, $b \in \mathbb{Z}^{n+1}$, $P = P(A, b)$ *be a simplex and* Δ_{min} *be the minimal absolute value of basis sub-determinants of the matrix A. If* $width(P(A, b)) \geq \Delta_{min} - 1$, *then* $P(A, b) \cap \mathbb{Z}^n \neq \emptyset$. *There is a polynomial time algorithm to find some integer point in P.*

Flatness theorem for simplices.

Theorem

(Veselov, Gribanov) Let $A \in \mathbb{Z}^{(n+1)\times n}$ *be* Δ*-modular,* $b \in \mathbb{Z}^{n+1}$ *and* $P = P(A, b)$ *be a simplex. If* $width(P(A, b)) \geq \Delta - 1$*, then* $P(A, b) \cap \mathbb{Z}^n \neq \emptyset$ *and there is an algorithm with the time complexity* $O(n\Delta)$ *that solves an integer problem* $max\{c^\top x : \; x \in P \cap \mathbb{Z}^n\}$*.*

Integer programming in a simplex.

We develop an quasi-polynomial algorithm of an integer programming for the special class of polytopes. The complexity of the algorithm is $O((n^2 + m)n^{2log^2(\Delta)})$ of elementary arithmetic operations. This algorithm can be applied to simplicies, one gives complexity $O(n^{2log^2(\Delta)+2})$.

Computing of the simplex width.

Computing width of a simplex is NP-hard problem due to András Sebö (2000).

Theorem

Let A be Δ-modular matrix from $\mathbb{Z}^{(n+1)\times n}$, $b \in Z^{n+1}$ *and* $P(A, b)$ *be a simplex. Then the width of P and the flat direction of P can be computed using* $O(\Delta n^{2 log^2(\Delta)+3})$ *elementary arithmetic operations.*

Thank you!