Computationally efficient PageRank algorithm exploiting graph sparsity

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Purpose of research

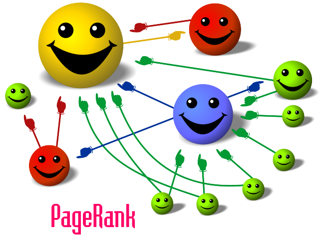
Checking theoretical estimation algorithm based on the Nesterov ideas ranking web-pages on the sparse graphs.

Application

It lets solve a large class ranking problems in logarithmic largest space time.

Notation

- Given a directed graph with n vertices.
- The vertices — sites.
- Oriented edge — links.
- Matrix $\mathbf{P}$ is adjacency matrix for the graph.
- Vector PageRank $\mathbf{x}$ is vector quantity characterizing the importance of each site.
- Sparsity coefficient s is the maximum number of non-zero elements in each column and each row of the matrix $\mathbf{P}$.



Problem Statement

The problem of finding the vector PageRank = The problem of searching left eigenvector $\mathbf{x}$, that

$$\mathbf{x}^T = \mathbf{x}^T \mathbf{P}, \text{ where } \sum_{k=1}^{n} x_k = 1.$$

The problem of searching left eigenvector = The optimization problem

$$f(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x}\|_2^2 + \frac{1}{2}\sum_{k=1}^{n}(-x_k)_+^2 \longrightarrow \min_{\langle \mathbf{x}, \mathbf{e} \rangle = 1},$$

where matrix $\mathbf{A} = \mathbf{P}^T - \mathbf{I}$, $\mathbf{I}$ - the identity matrix, $\mathbf{e} = (1, \dots, 1)$,

Key ideas

- Matrix **A** is sparse with Lipschitz constant L.
- Componentwise descent.

$$x_{k+1} = x_k + \operatorname*{argmin}_{h:\langle h,e\rangle=0} \left\{ f(x_k) + \langle \nabla f(x_k), h\rangle + \frac{L}{2}\|h\|_1^2 \right\},$$

- The number of iterations $O(1/\epsilon^2)$
- The complexity of a single iteration $O(s^2 \ln n)$
- The complexity of the algorithm

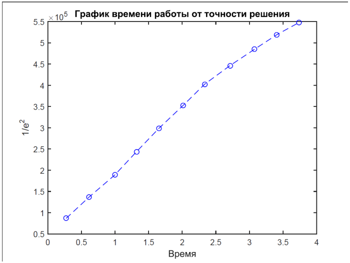
$$O\left(\frac{s^2 \ln n}{\varepsilon^2}\right).$$

- The goal of computational experiment is to check theoretical estimates of the complexity.
- **Input**:the dimension n, the sparsity coefficient s, the generated sparse matrix $\mathbf{A}$ Erdos-Renyi type with parameters n, s, the initial value of the function f_0 and the initial value of the gradient of g_0, precounted $\mathbf{A}^T\mathbf{A}$.
- **Output**: Page Rank $\mathbf{x}$, time.

Results

- Checking dependence $O(1/\epsilon^2)$. Timelines execution of the accuracy of the solution (1-2). Charts have a linear dependence. $\Longrightarrow$ The complexity $O(1/\epsilon^2)$. Theoretical estimates are confirmed.
- The checking of dependence. $O(s^2 \ln n)$. Unable to use the sparsity of the matrix due to the nature of MatLab and memory arrays. Theoretical estimates have not been confirmed.
- Getting Dependence $O(n)$. Timeline of execution of of the dimension (3). Charts have a linear dependence. $\Longrightarrow$ complexity $O(n)$.
- The final complexity: $O(n/\epsilon^2)$

Charts



pic.1 The dependence of the runtime by the number of iterations of the algorithm for $n = 12500$ sites

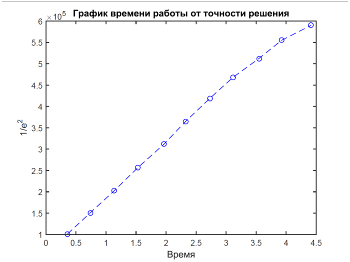


Рис.2 The dependence of the runtime by the number of iterations of the algorithm for $n = 15000$ sites

Графики

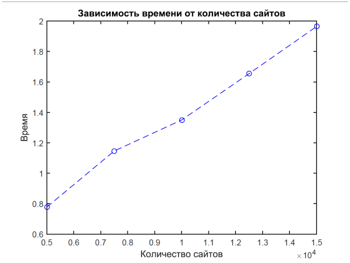


Рис.3 The dependence of the the runtime from the dimension n where 21000 iteration

Comparing with other algorithms

It received the complexity:

- $O\left(\frac{n}{\epsilon^2}\right)$, in practice without sparsity,
- $O\left(\frac{s^2 \ln n}{\varepsilon^2}\right)$ in theory with sparsity.
- The table shows the rate of convergence of the most advanced fast algorithms.
- Our algorithm has the lowest the complexity.

Table

Method	Condition	Complexity
Nazin-Polyak 2008	no	$O\left(\frac{n\ln(\frac{n}{\sigma})}{\epsilon^2}\right)$
Nesterov 2012	S	$O\left(\frac{sn\ln n}{\epsilon^2}\right)$
Juditsky et al 2009	no	$O\left(\frac{n\ln(\frac{n}{\sigma})}{\epsilon^2}\right)$
Grigoriadis-Hachiyan 2009	S	$O\left(\frac{s\ln n\ln(\frac{n}{\sigma})}{\epsilon^2}\right)$
Polyak-Tremba 2012	S	$\frac{2sn}{\epsilon}$
Gasnikov-Dvurechensky 2015	S	$O\left(\frac{s^2\ln n}{\epsilon^2}\right)$

Estimates of the rate of convergence of algorithms PageRank, where S sparsity-condition.

Final

In our work:

- The research method of ranking web pages with sparse graphs.
- It is shown that the theoretical estimate of the number of steps corresponds to the experimental data.
- It is shown that the theoretical estimate of the complexity of the algorithm step does not correspond to the experimental data, due to the nature of programmatic implementation.
- Through the use of 1-norm achieved $O(n)$ arithmetic operations on a step of the algorithm.
- The result is a new, and even with the deterioration estimates of complexity algorithm is one of the fastest algorithms.