Consensus Processes on Complex Networks

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1. Introduction

1. Introduction

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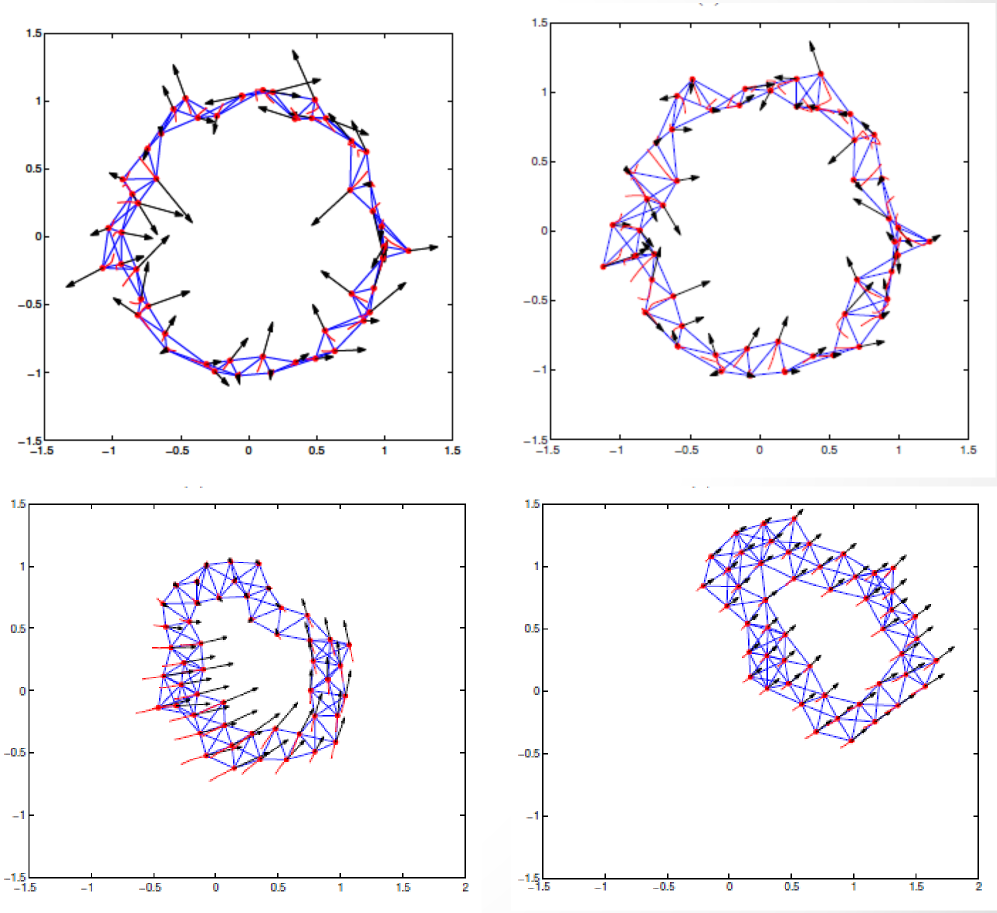
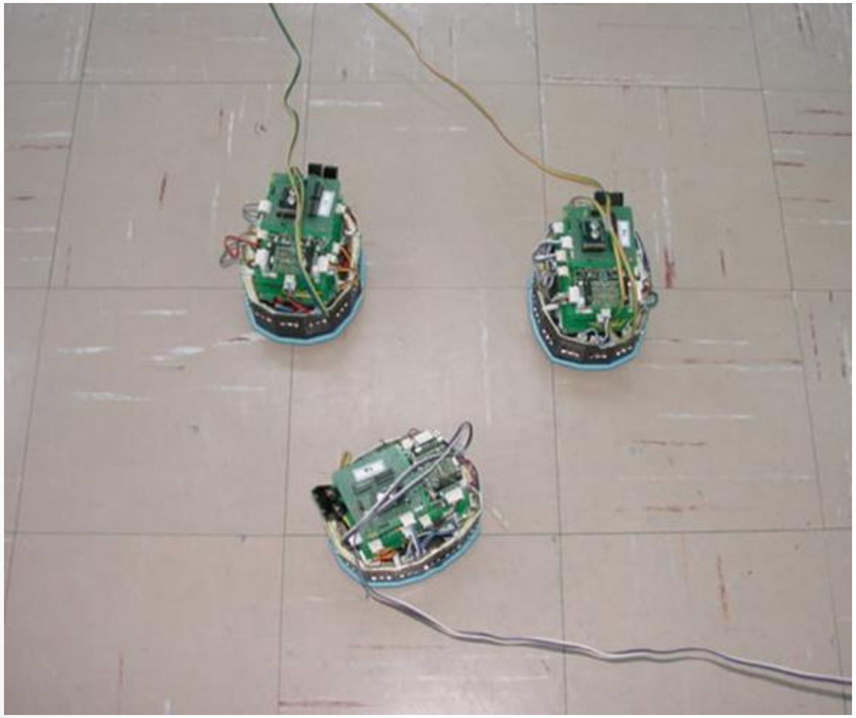
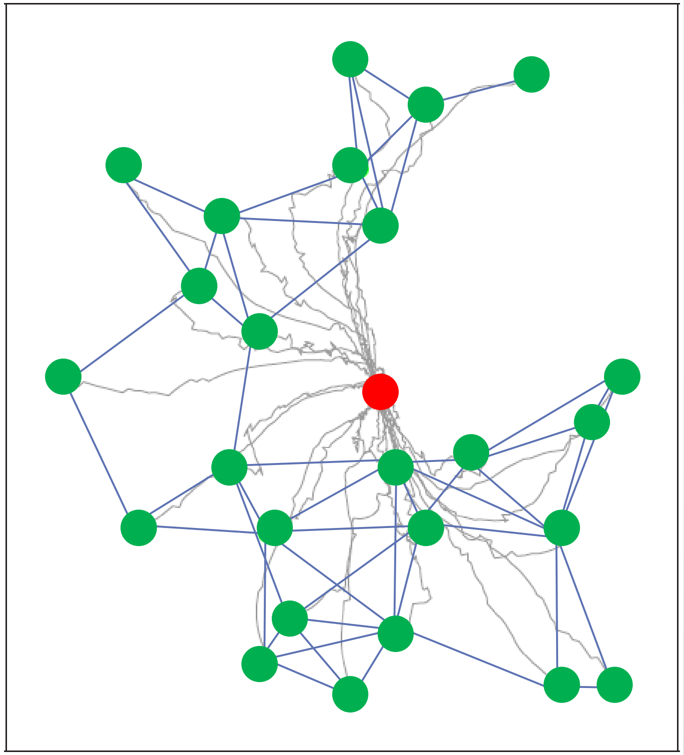


1. Introduction

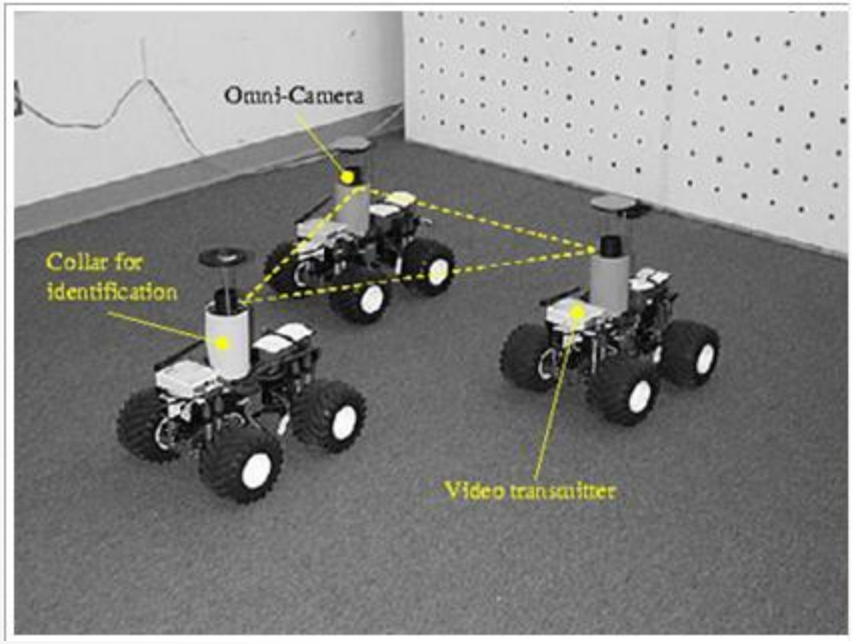
SPATIAL RENDEZVOUS



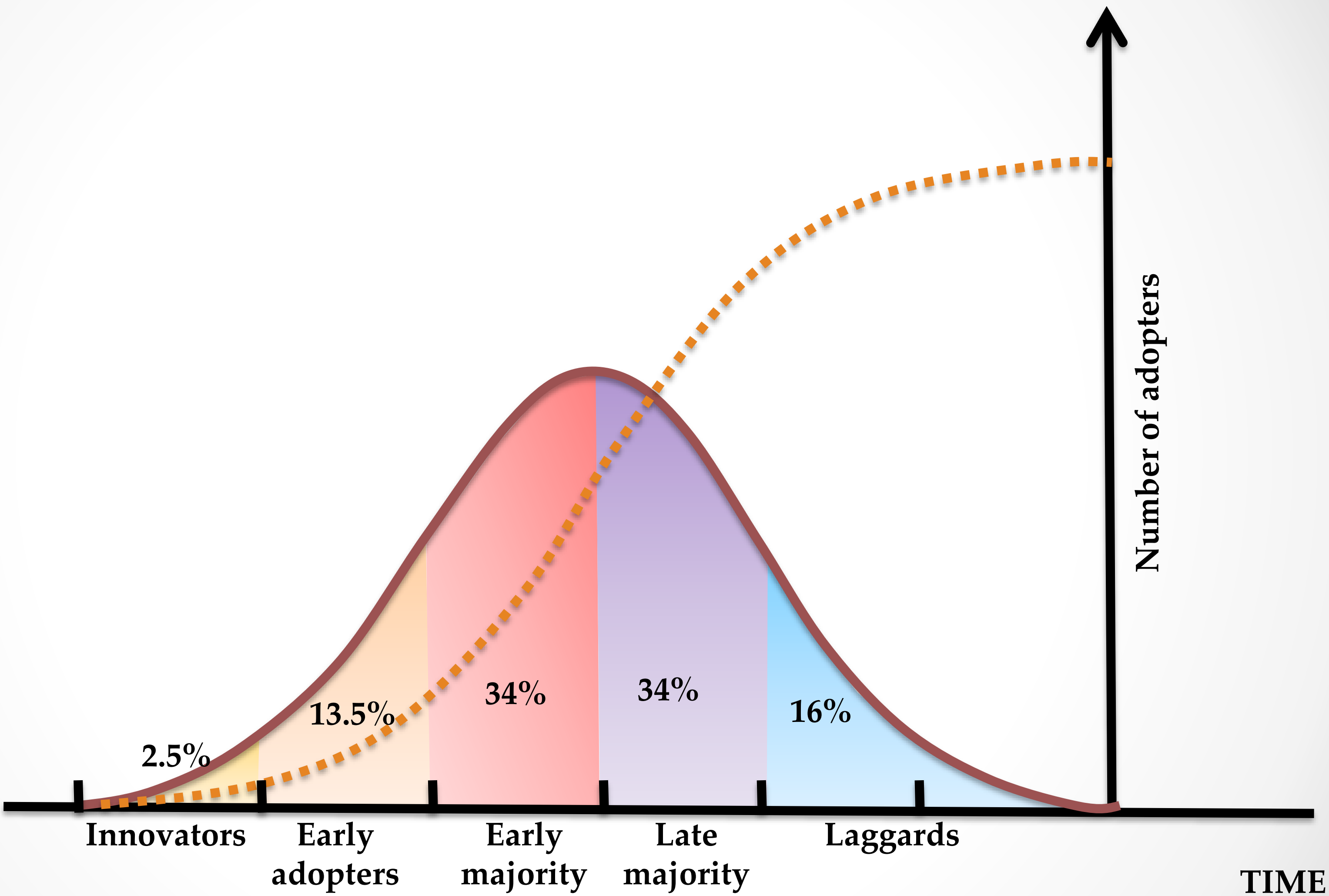


1. Introduction

FORMATION CONTROL

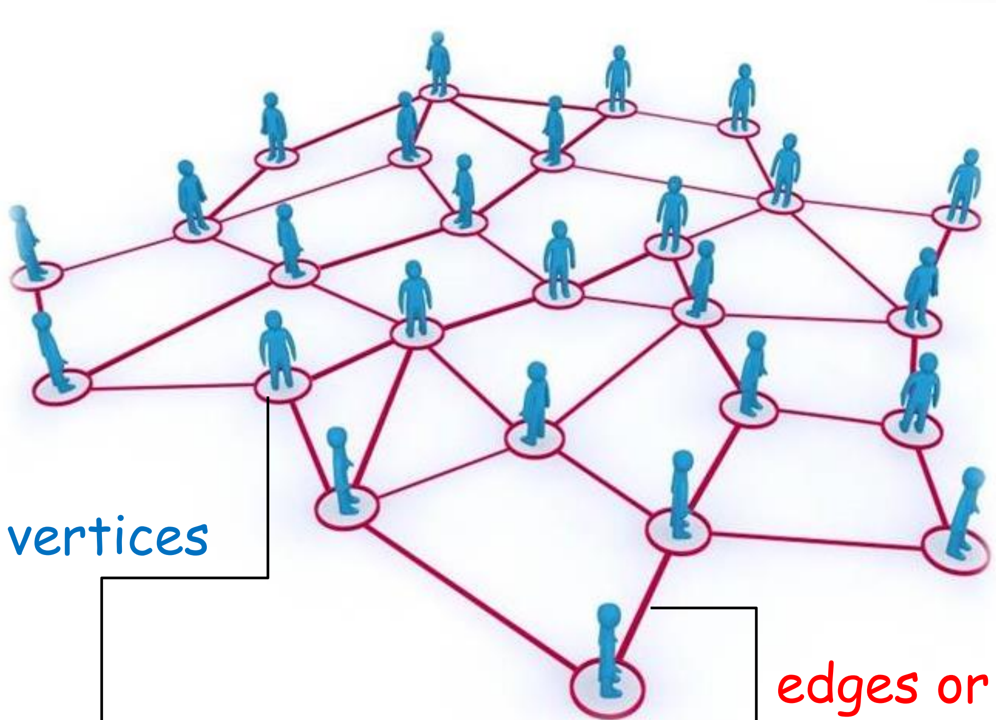


11. Diffusion on networks



2. Networks

$$G = (V, E)$$



nodes or vertices

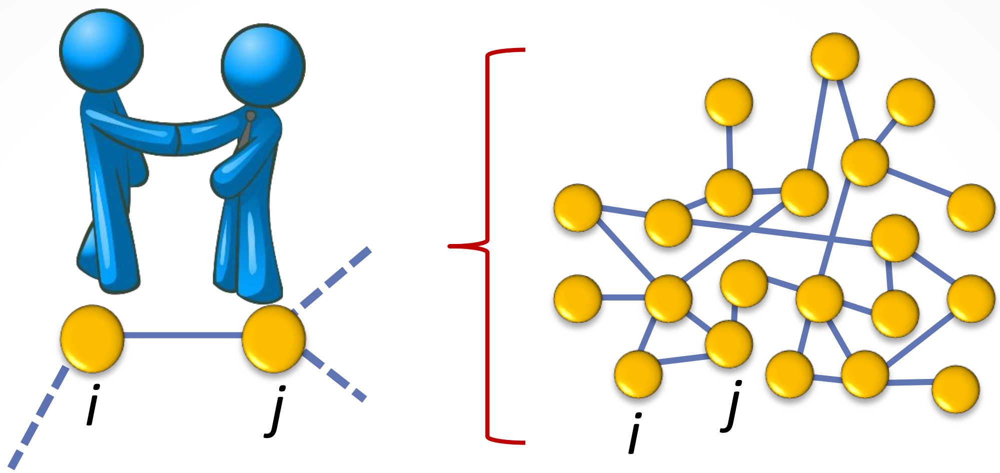
edges or links

$$V = \{v_1, \cdots, v_n\}$$

$$E \subseteq V \times V$$

Estrada & Knight: *A First Course on Network Theory,* Oxford Univ. Press, 2015

3. The consensus model



opinion of *j* at time *t*

opinion of *i* at time *t*

$$\frac{du_i}{dt} = \dot{u}_i(t) = \sum_{j \sim i} \left[u_j(t) - u_i(t) \right], \qquad i = 1, \cdots, n$$

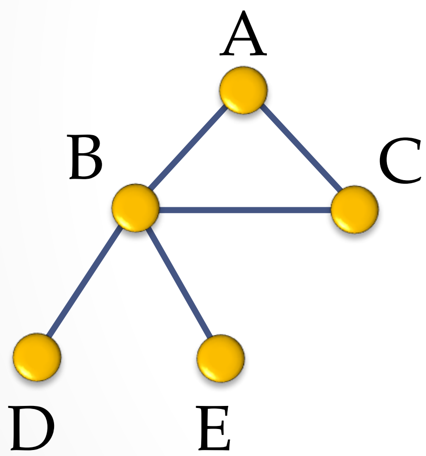
$$u_i(0) = z_i, \qquad z_i \in \Re$$

Olfati-Saber, et al., *Proc. IEEE* 95, **2007** 1.

3. The consensus model

$$\dot{u}_i(t) = \sum_{j\sim i}\left[u_j(t) - u_i(t)\right], \qquad i = 1, \cdots, n$$

Example:



- $\dot{u}_A(t) = (u_B(t) - u_A(t)) + (u_C(t) - u_A(t))$
- $\dot{u}_B(t) = (u_A(t) - u_B(t)) + (u_C(t) - u_B(t)) + (u_D(t) - u_B(t)) + (u_E(t) - u_B(t))$
- $\dot{u}_C(t) = (u_A(t) - u_C(t)) + (u_B(t) - u_C(t))$
- $\dot{u}_D(t) = (u_B(t) - u_D(t))$
- $\dot{u}_E(t) = (u_B(t) - u_E(t))$

3. The consensus model

We can rearrange the previous equations as follows:

$$\dot{u}_A(t) = \overbrace{u_B(t) + u_C(t)}^{\text{State of the nearest neighbours of A}} - \overset{\text{Degree of A}}{2}\,u_A(t)$$

$$\dot{u}_B(t) = u_A(t) + u_C(t) + u_D(t) + u_E(t) - 4u_B(t)$$

$$\dot{u}_C(t) = u_A(t) + u_B(t) - 2u_C(t)$$

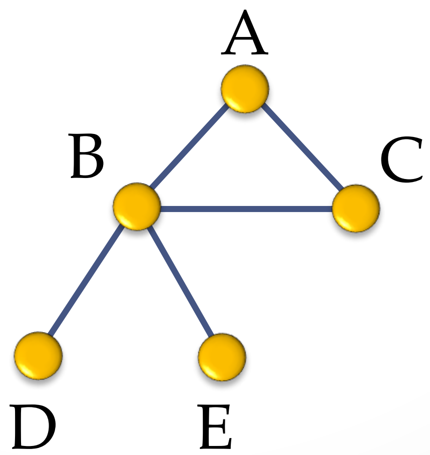
$$\dot{u}_D(t) = u_B(t) - u_D(t)$$

$$\dot{u}_E(t) = u_B(t) - u_E(t)$$

3. The consensus model

In matrix form we get:

$$\begin{bmatrix} \dot{u}_A(t) \\ \dot{u}_B(t) \\ \dot{u}_C(t) \\ \dot{u}_D(t) \\ \dot{u}_E(t) \end{bmatrix} = -\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_A(t) \\ u_B(t) \\ u_C(t) \\ u_D(t) \\ u_E(t) \end{bmatrix}$$



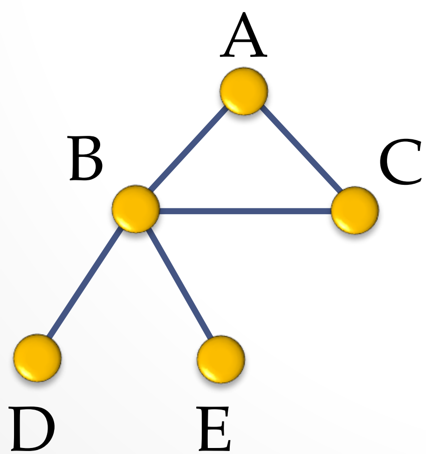
3. The consensus model

The previous equation is now written in the following compact form:

$$\dot{\vec{\mathbf{u}}}(t) = -\mathbf{L}(G)\vec{\mathbf{u}}(t), \qquad \vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0$$

$$L_{uv} = \begin{cases} -1 & \text{if } (uv) \in E, \\ k_u & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

Example:



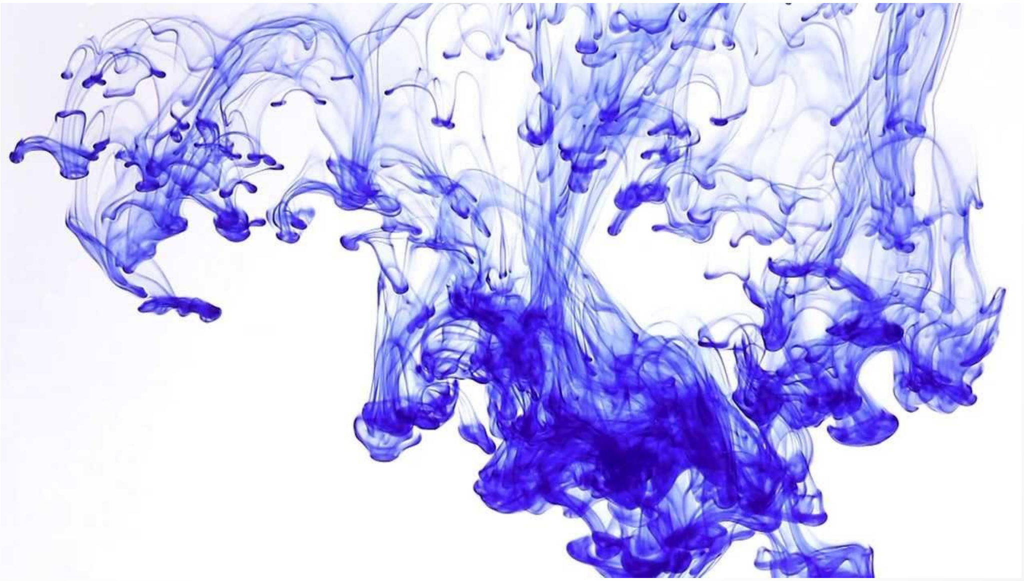
$$\mathbf{L} = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Olfati-Saber, et al., *Proc. IEEE* 95, **2007** 1.

3. The consensus model

$$\dot{\mathbf{u}}(t) = -\mathbf{L}(G)\mathbf{u}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0$$

$$\dot{\mathbf{u}}(t) = D\frac{\partial^2 \mathbf{u}(t)}{\partial t^2}, \qquad \mathbf{u}(0) = \mathbf{u}_0$$



3. The consensus model

Definition 1: The *consensus set* $A \subseteq \Re^n$ is the subspace **span**{1}, that is

$$A = \{u \in \Re^n \mid u_i = u_j, \forall i, j\}$$

Discrete-time Consensus

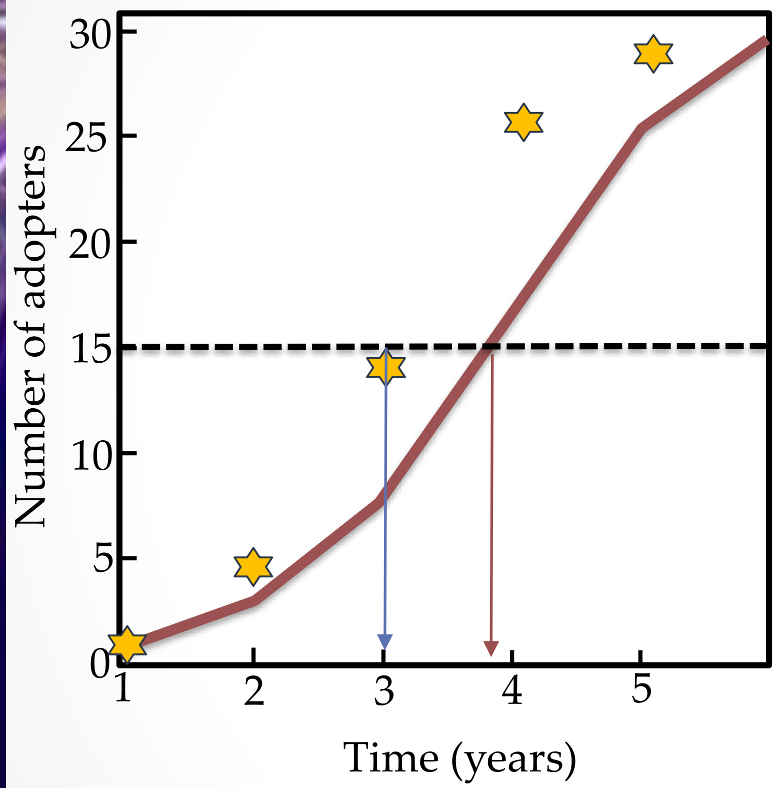
$$\vec{\mathbf{u}}[k+1] = (\mathbf{I} - \varepsilon\mathbf{L})\vec{\mathbf{u}}[k]$$

$$0 < \varepsilon < 1/k_{\max}$$

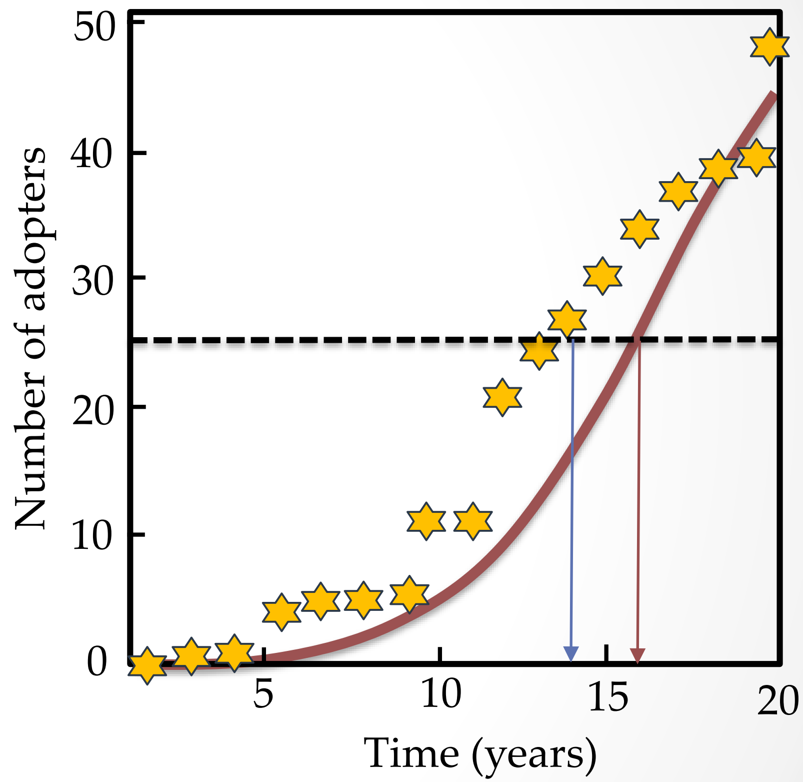
Olfati-Saber, et al., *Proc. IEEE* 95, **2007** 1.

11. Diffusion on networks

Diffusion of a mathematical method among high schools



Diffusion of a biotech product among Brazilian farmers



Observation

Normal diffusion

Estrada, Vargas-Estrada, *Sci. Rep.*, **3**, 2013.

Laplacian Matrix

$$\mathbf{L} = \mathbf{K} - \mathbf{A}$$

Example:

$$\mathbf{K} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Estrada & Knight: *A First Course on Network Theory*, Oxford Univ. Press, 2015

4. The Laplacian matrix

The spectral decomposition of the Laplacian matrix is written as:

$$\mathbf{L} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

$$\mathbf{V} = \begin{pmatrix} \varphi_1(1) & \varphi_2(1) & \cdots & \varphi_n(1) \\ \varphi_1(2) & \varphi_2(2) & \cdots & \varphi_n(2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(n) & \varphi_2(n) & \cdots & \varphi_n(n) \end{pmatrix} \qquad \mathbf{\Lambda} = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{pmatrix}$$

Estrada & Knight: *A First Course on Network Theory,* Oxford Univ. Press, 2015

4. The Laplacian matrix

Lemma 1: The Laplacian matrix is positive semidefinite:

$$0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$$

Lemma 2: Let G be a connected network. Then, the Laplacian matrix has only one zero eigenvalue:

$$0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$$

Estrada & Knight: *A First Course on Network Theory*, Oxford Univ. Press, 2015

5. Consensus in undirected networks

Solution of the consensus dynamic model

$$\mathbf{u}(t) = e^{-t\mathbf{L}}\mathbf{u}_0$$

$$e^{-t\mathbf{L}} = e^{-t\left(\mathbf{U\Lambda}U^T\right)} = \mathbf{U}e^{-t\mathbf{\Lambda}}\mathbf{U}^T$$

$$= e^{-t\mu_1}\varphi_1\varphi_1^{\ T} + e^{-t\mu_2}\varphi_2\varphi_2^{\ T} + \cdots + e^{-t\mu_n}\varphi_n\varphi_n^{\ T}$$

$$\mathbf{u}(t) = e^{-t\mu_1}\left(\varphi_1^{\ T}\mathbf{u}_0\right)\varphi_1 + e^{-t\mu_2}\left(\varphi_2^{\ T}\mathbf{u}_0\right)\varphi_2 + \cdots + e^{-t\mu_n}\left(\varphi_n^{\ T}\mathbf{u}_0\right)\varphi_n$$

Olfati-Saber, et al., *Proc. IEEE* 95, **2007** 1.

5. Consensus in undirected networks

Theorem 3: Let G be a connected network. The undirected consensus model converges to the consensus set with a rate of convergence that is dictated by μ_2 .

Proof: $\mu_1 = 0$ and $\mu_j > 0, \forall j \neq 1$. Thus

$$\mathbf{u}(t) \rightarrow \left(\varphi_1{}^T \mathbf{u}_0\right)\varphi_1 = \frac{\mathbf{1}^T \mathbf{u}_0}{n}\mathbf{1} \quad \text{as} \quad t \rightarrow \infty$$

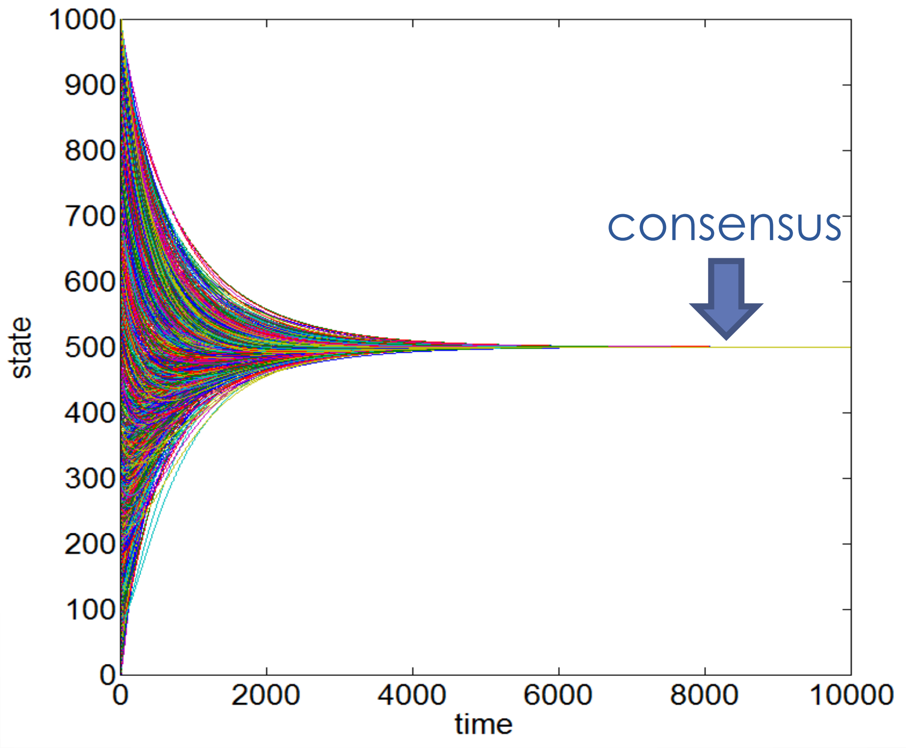
Hence $\mathbf{u}(t) \rightarrow A$ as $t \rightarrow \infty.$

As μ_2 is the smallest positive eigenvalue of the graph Laplacian, it dictates the slowest mode of convergence in the previous equation. □

Mesbahi & Egerstedt., *Graph Theory Methods in Multiagent Networks*, Princeton, 2010 .

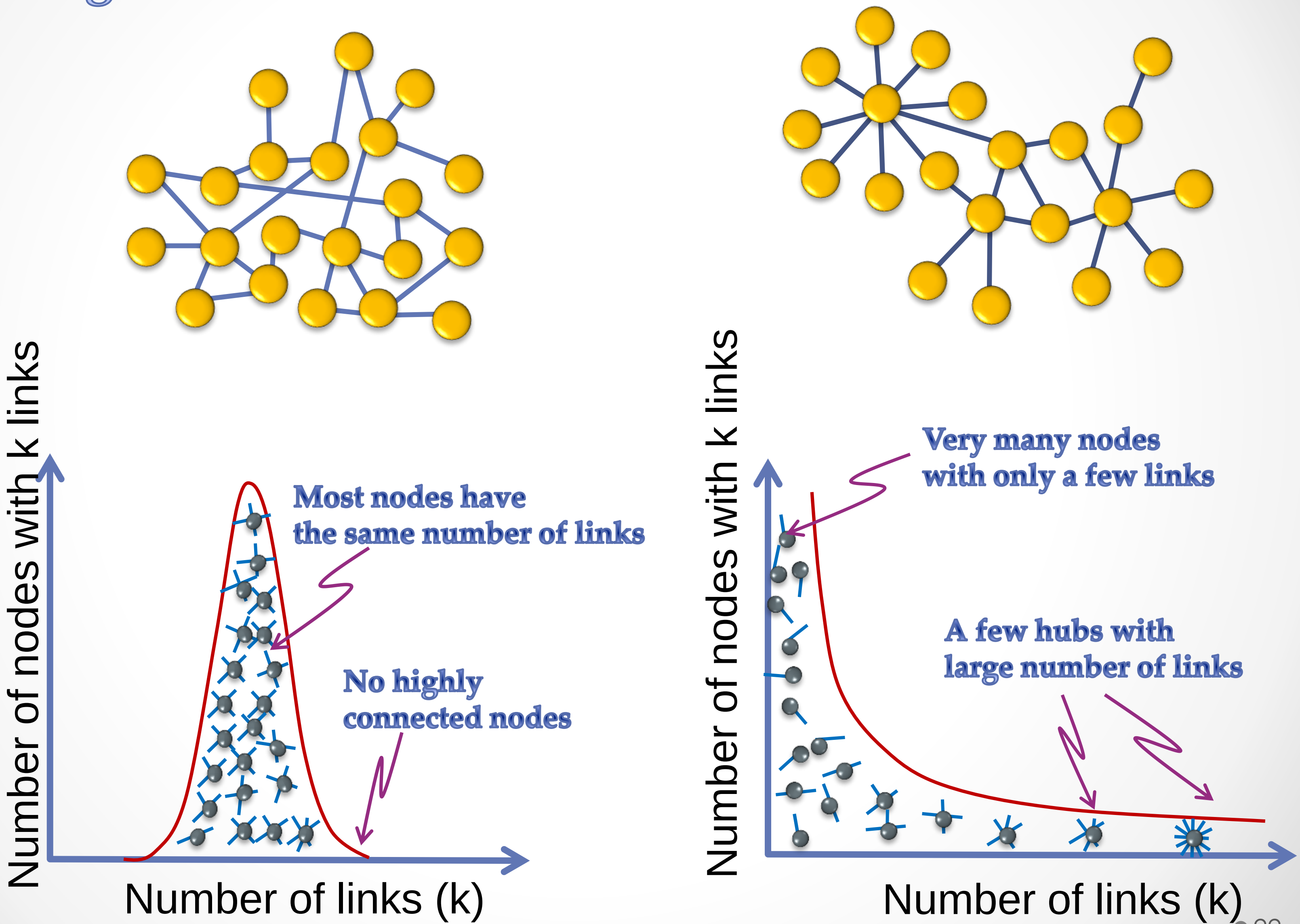
5. Consensus in undirected networks

Proposition 4: A necessary and sufficient condition for the consensus model to converge to the consensus subspace from an arbitrary initial condition is that the network is connected.



5. Consensus in undirected networks

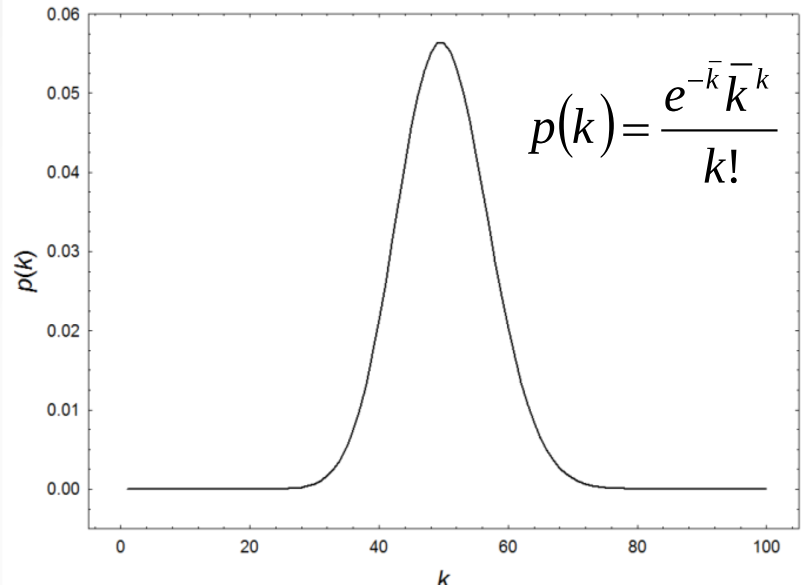
Degree distribution



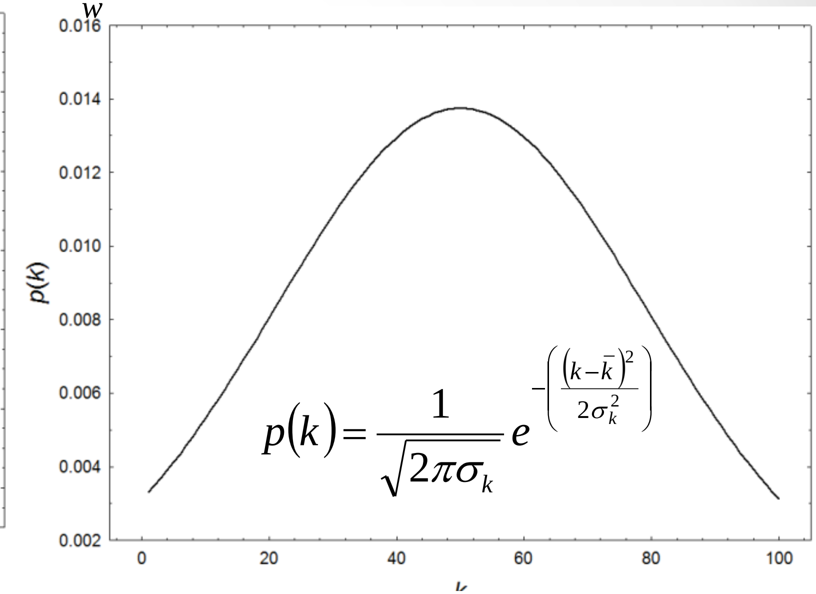
Estrada & Knight: *A First Course on Network Theory*, Oxford Univ. Press, 2015

5. Consensus in undirected networks

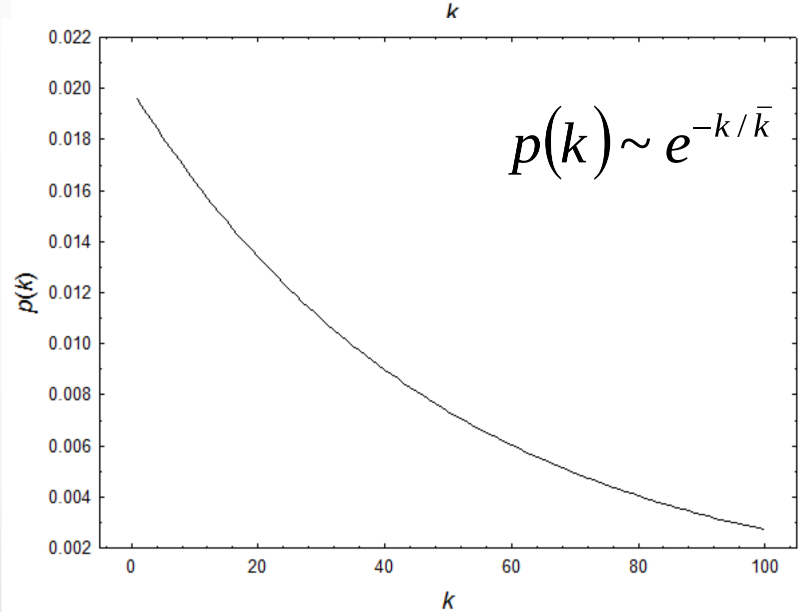
$$p_u = k_v / \sum_w k_w$$



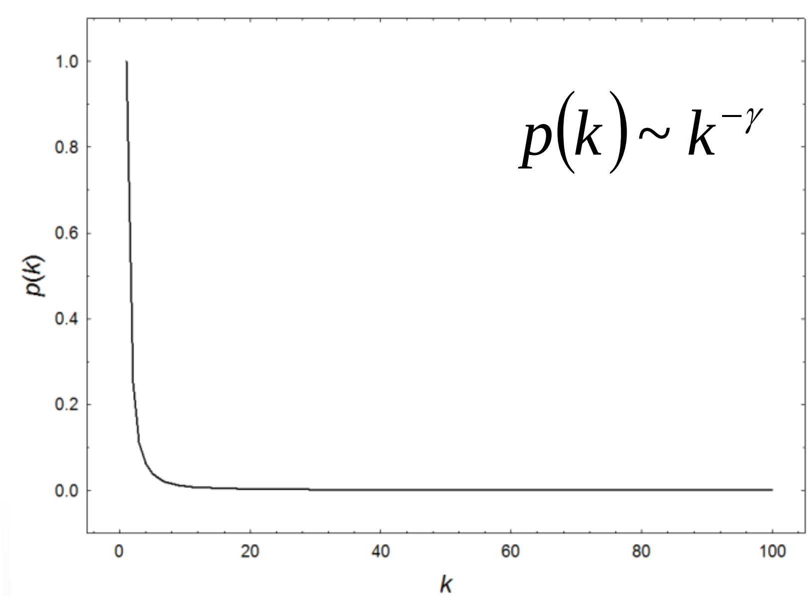
$$p(k) = \frac{e^{-\bar{k}}\bar{k}^{k}}{k!}$$



$$p(k) = \frac{1}{\sqrt{2\pi\sigma_k}} e^{-\left(\frac{(k-\bar{k})^2}{2\sigma_k^2}\right)}$$



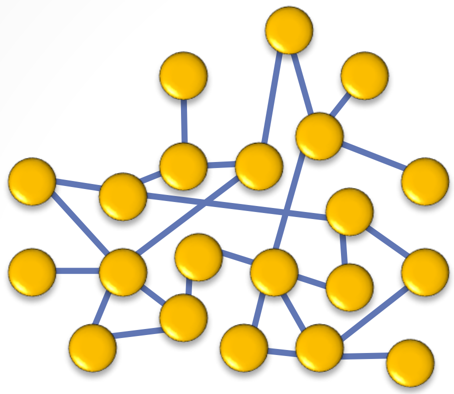
$$p(k) \sim e^{-k/\bar{k}}$$

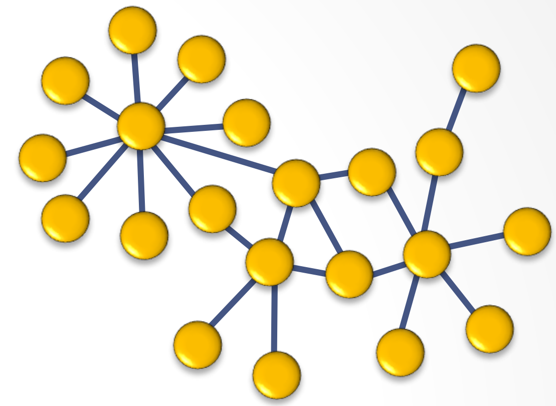


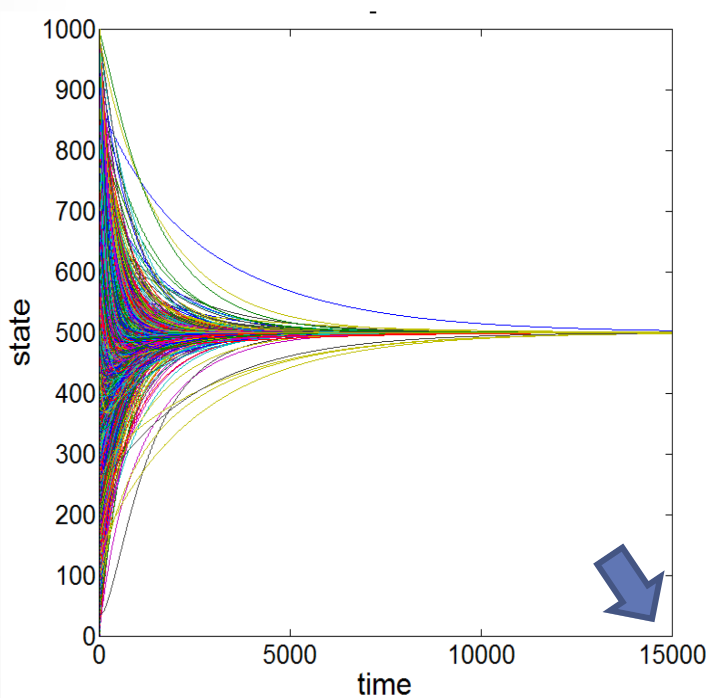
$$p(k) \sim k^{-\gamma}$$

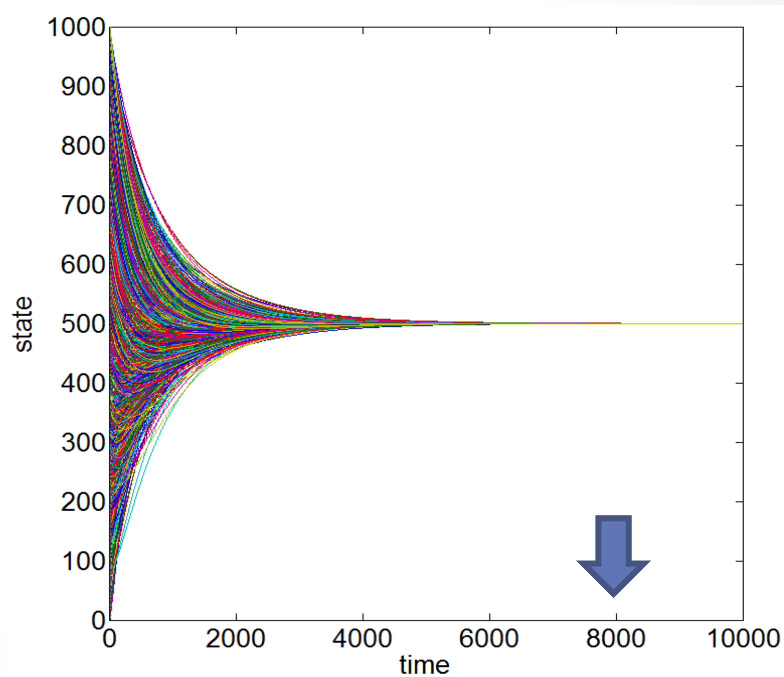
5. Consensus in undirected networks

Degree distribution



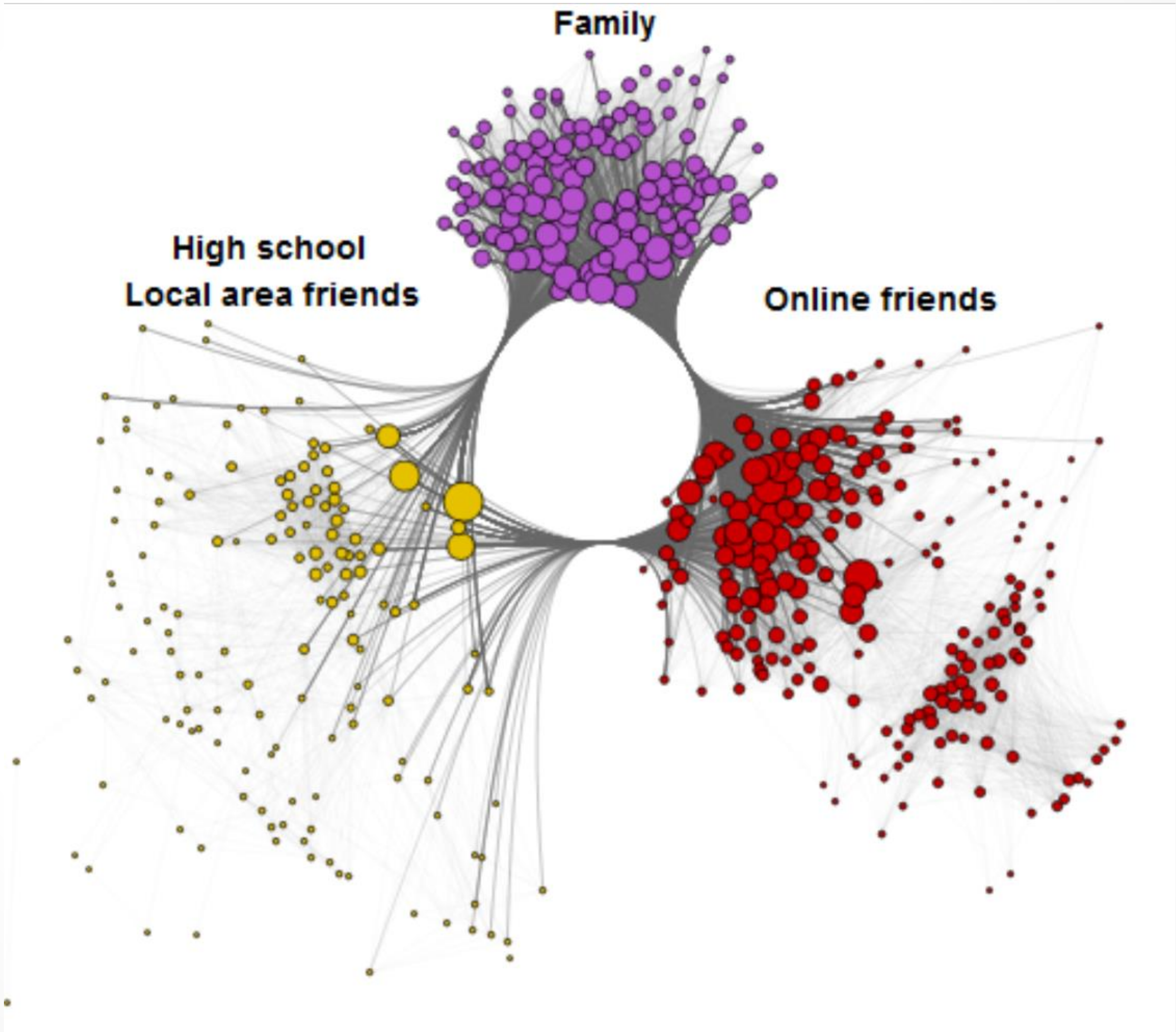






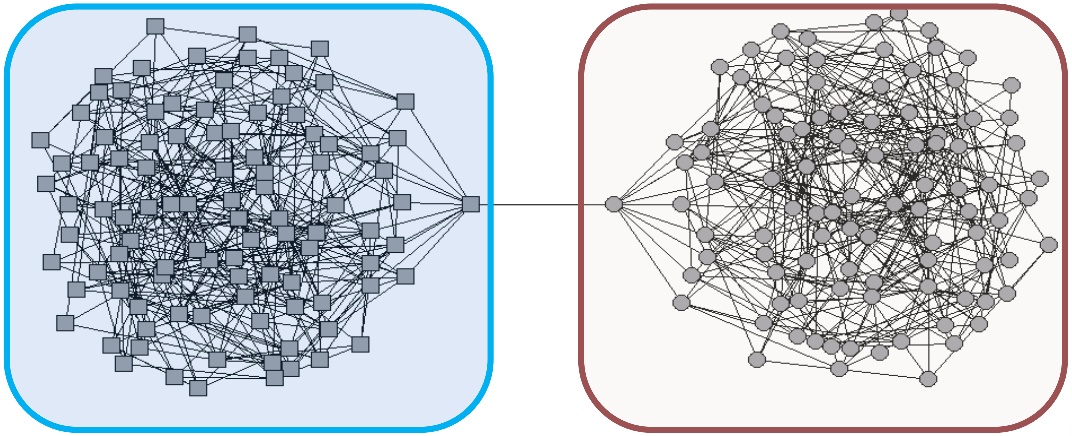
5. Consensus in undirected networks

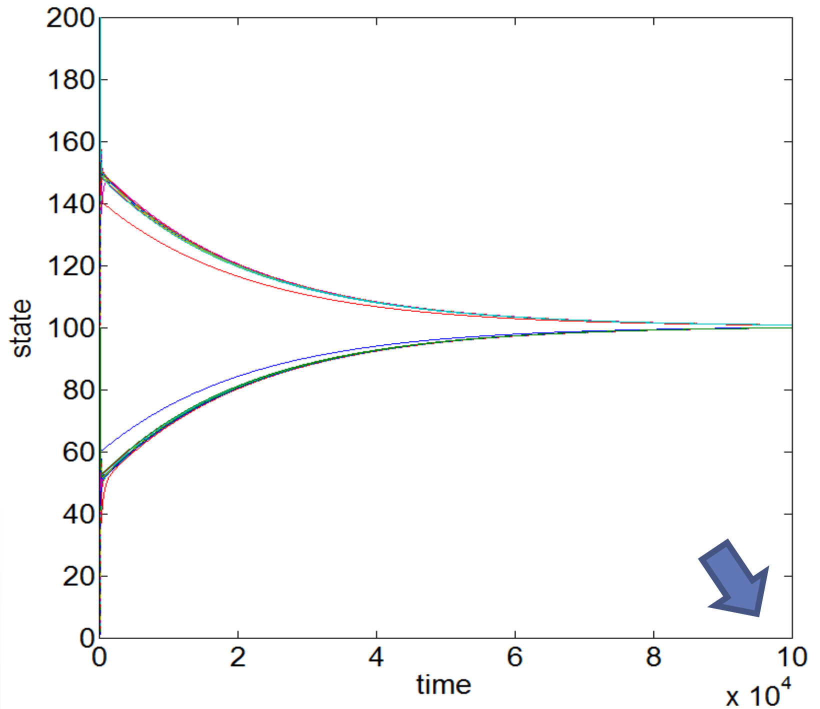
Communities



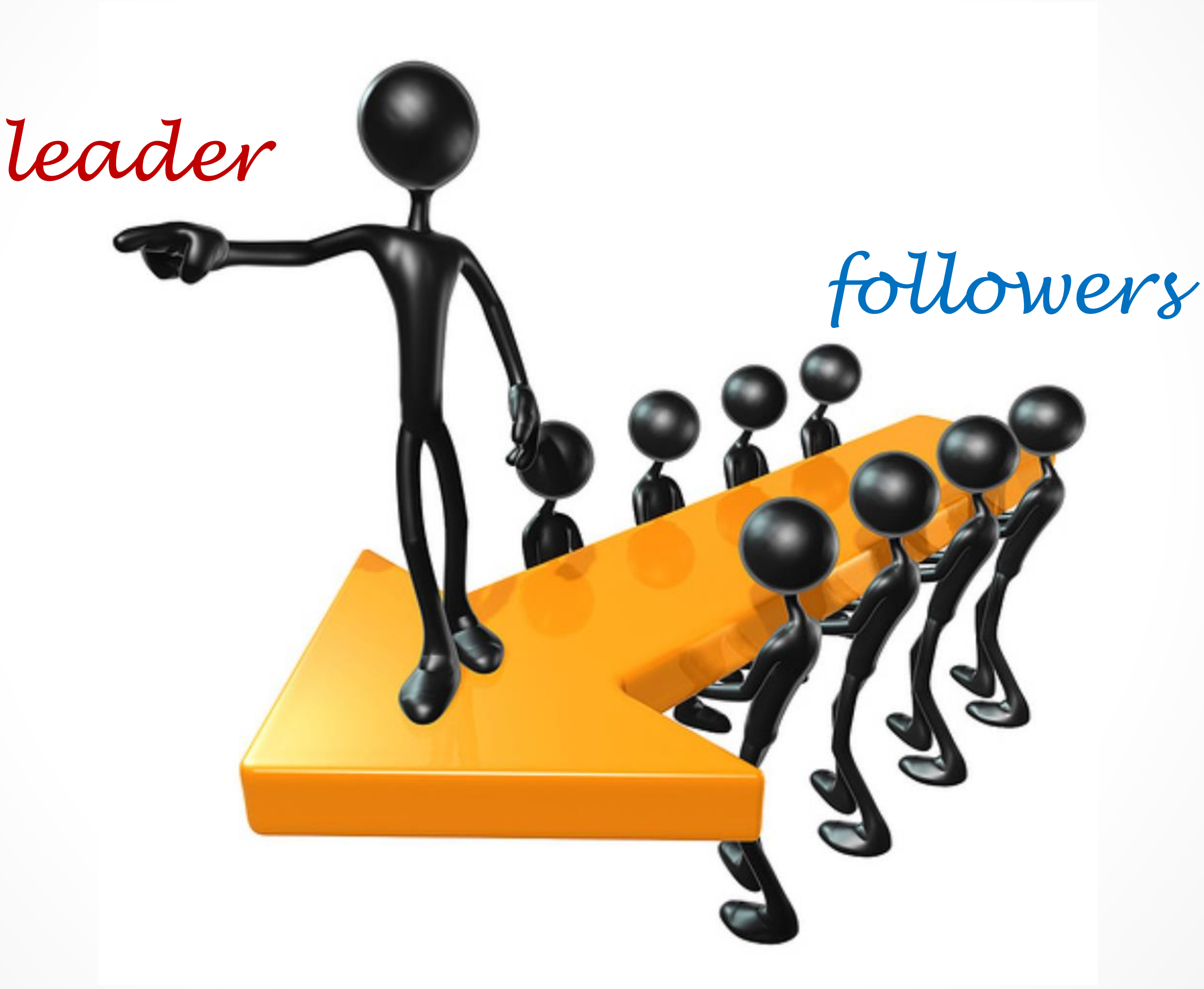
5. Consensus in undirected networks

Communities





6. Leader-followers consensus

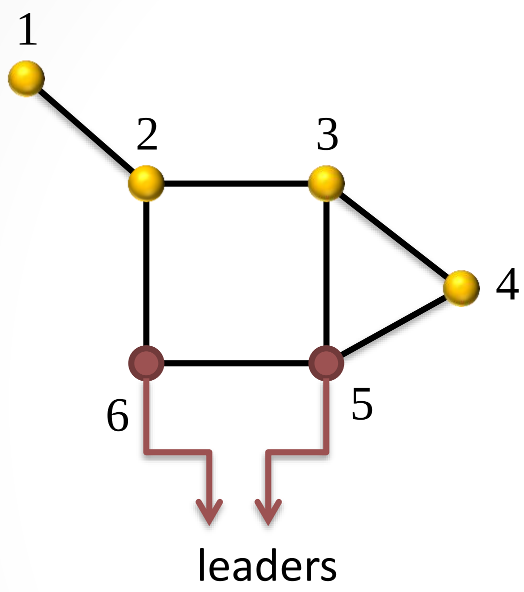


Mesbahi & Egerstedt., *Graph Theory Methods in Multiagent Networks*, Princeton, 2010 .

6. Leader-followers consensus

Consider the partition of the network into n_l leaders and n- n_l followers.

Example:



followers

$$\mathbf{L} = \left(\begin{array}{cccc|cc} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & 0 & -1 & 2 \end{array}\right)$$

leaders

$$\mathbf{L}_l = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \qquad \mathbf{L}_f = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad \mathbf{L}_{fl} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ -1 & 0 \\ -1 & 0 \end{pmatrix}$$

6. Leader-followers consensus

The consensus dynamics of a leader-follower system is described by:

$$\begin{bmatrix} \dot{\mathbf{u}}_f \\ \dot{\mathbf{u}}_l \end{bmatrix} = -\begin{bmatrix} \mathbf{L}_f & \mathbf{L}_{fl} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_l \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}$$

$$\dot{\mathbf{u}}_f = -\mathbf{L}_f \mathbf{u}_f - \mathbf{L}_{fl} \mathbf{u}_l$$

Mesbahi & Egerstedt., *Graph Theory Methods in Multiagent Networks*, Princeton, 2010 .

6. Leader-followers consensus

Theorem 5: If the network G is connected then $\mathbf{L}_f$ is positive definite.

Proof: $\mathbf{L}$ is positive semidefinite. If G is connected

$$N(\mathbf{L}) = \text{span}\{\mathbf{1}\}$$

Since,

$$\mathbf{u}_f^T \mathbf{L}_f \mathbf{u}_f = \begin{bmatrix} \mathbf{u}_f^T & 0 \end{bmatrix} \mathbf{L}_f \begin{bmatrix} \mathbf{u}_f \\ 0 \end{bmatrix}$$

and

$$\begin{pmatrix} \mathbf{u}_f^T & 0 \end{pmatrix} \notin N(\mathbf{L})$$

Then

$$\begin{bmatrix} \mathbf{u}_f^T & 0 \end{bmatrix} \mathbf{L}_f \begin{bmatrix} \mathbf{u}_f \\ 0 \end{bmatrix} > 0, \forall \mathbf{u}_f \in \Re^{n_f} \qquad \blacksquare$$

Rahmani et al., *SIAM J. Control Opt.* **48**, 168, 2009.

6. Leader-followers consensus

Theorem 6: Given fixed leader opinions $\mathbf{u}_l$, the equilibrium point under the leader-follower dynamics is

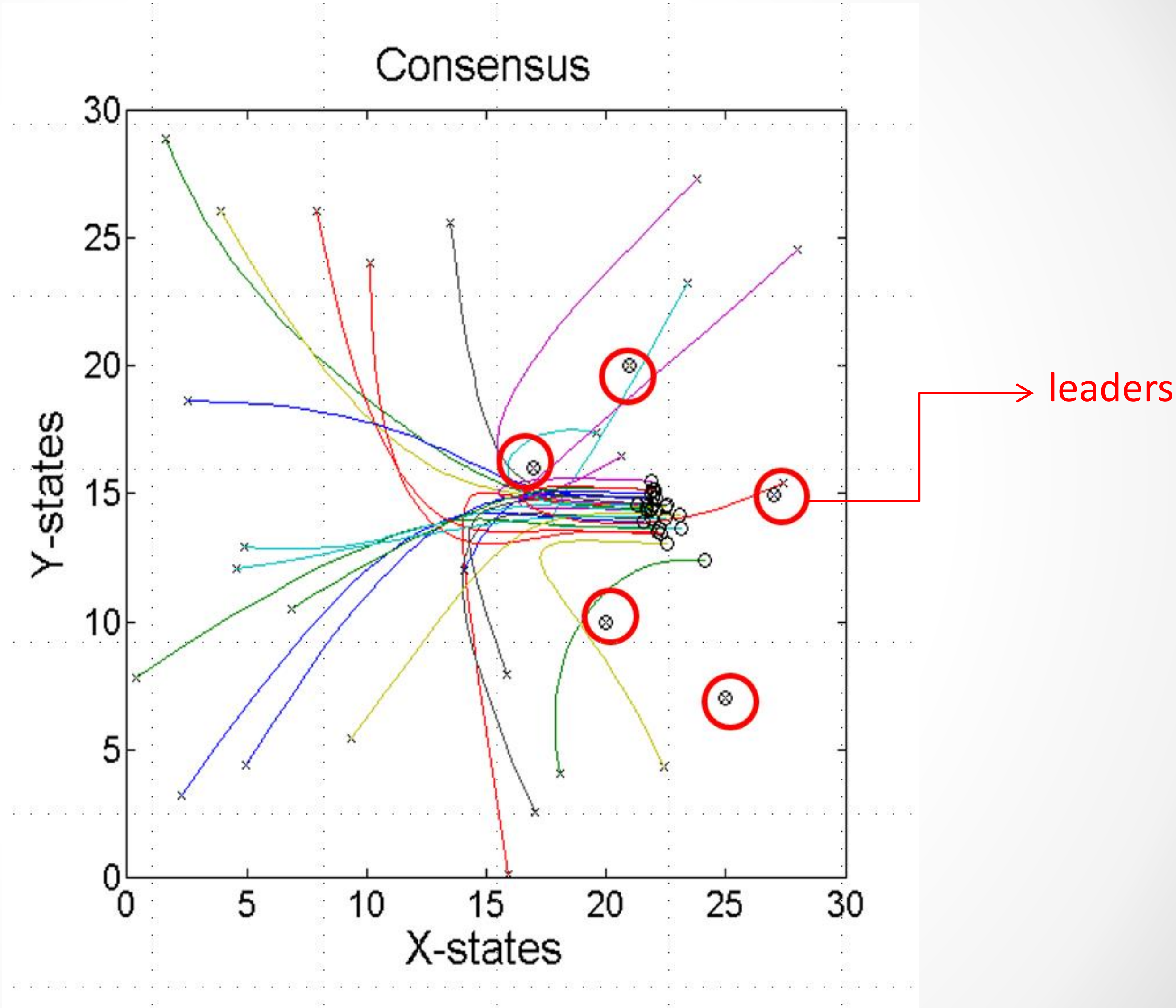
$$\mathbf{u}_f = -\mathbf{L}_f^{-1}\mathbf{L}_{fl}\mathbf{u}_l$$

which is globally asymptotically stable.

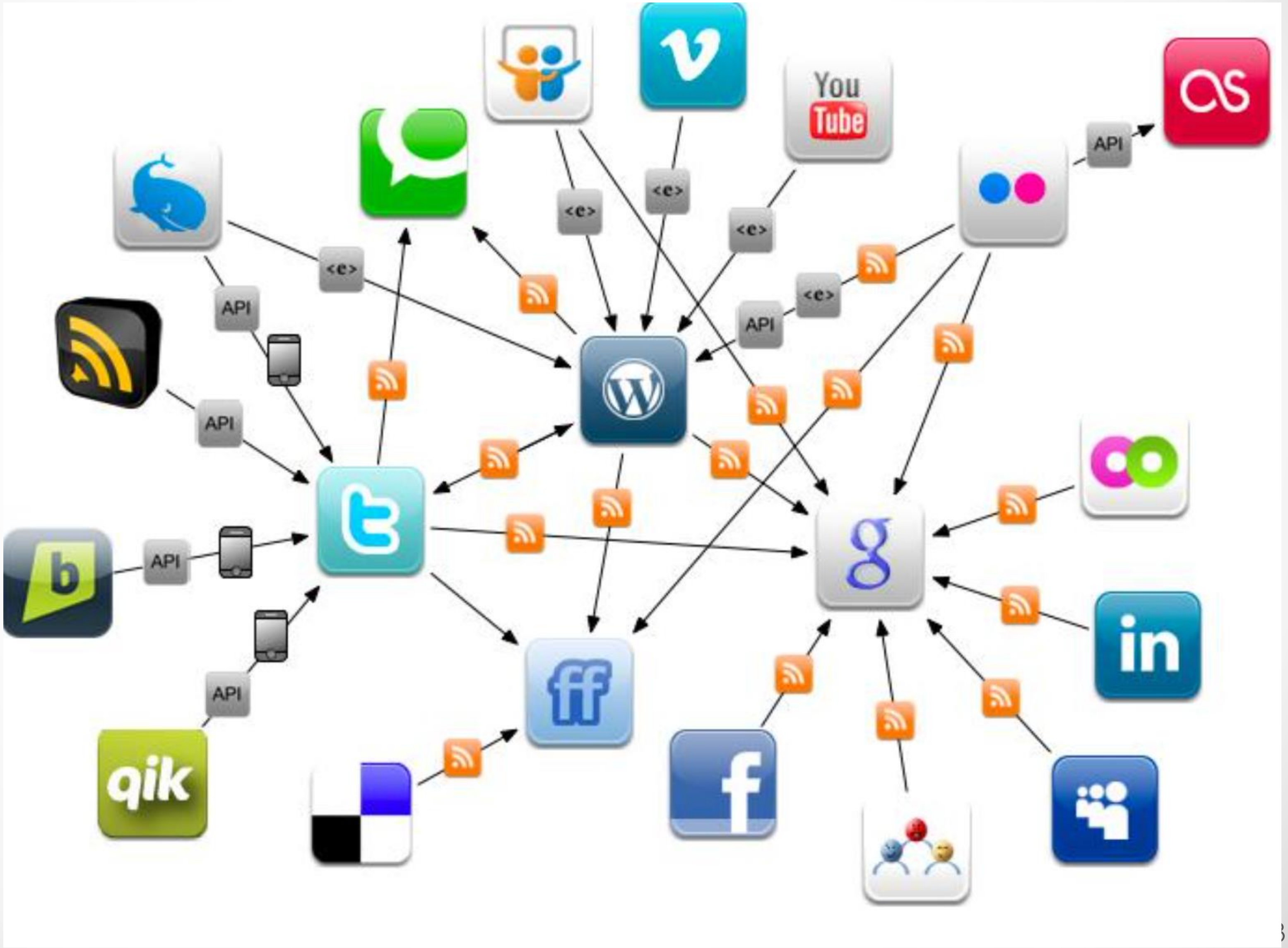
Proof: $\mathbf{L}_f \succ 0$. Thus $\mathbf{L}_f^{-1}$ exists and $\mathbf{u}_f = -\mathbf{L}_f^{-1}\mathbf{L}_{fl}\mathbf{u}_l$ is well defined.

Hence, the equilibrium point is unique. Moreover, because $\mathbf{L}_f \succ 0$, this equilibrium point is globally asymptotically stable.

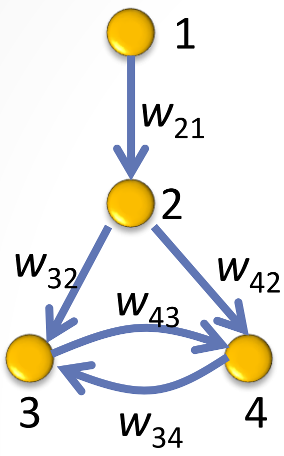
6. Leader-followers consensus



8. Consensus in directed networks



8. Consensus in directed networks



$$\dot{x}_1(t) = 0,$$
$$\dot{x}_2(t) = w_{21}\left(x_1(t) - x_2(t)\right),$$
$$\dot{x}_3(t) = w_{32}\left(x_2(t) - x_3(t)\right) + w_{34}\left(x_4(t) - x_3(t)\right),$$
$$\dot{x}_4(t) = w_{42}\left(x_2(t) - x_4(t)\right) + w_{43}\left(x_3(t) - x_4(t)\right).$$

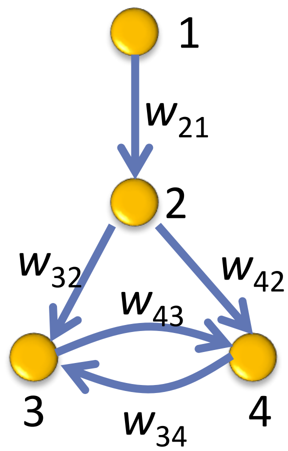
$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -w_{21} & w_{21} & 0 & 0 \\ 0 & -w_{32} & w_{32} + w_{34} & -w_{34} \\ 0 & -w_{42} & -w_{43} & w_{42} + w_{43} \end{pmatrix} \mathbf{x}(t)$$

Mesbahi & Egerstedt., *Graph Theory Methods in Multiagent Networks*, Princeton, 2010 .

8. Consensus in directed networks

$$\dot{\mathbf{u}}(t) = -\mathbf{L}(D)\mathbf{u}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0$$

Example:



$$\mathbf{A} = \begin{pmatrix} 0 & w_{21} & 0 & 0 \\ 0 & 0 & w_{32} & w_{42} \\ 0 & 0 & 0 & w_{43} \\ 0 & 0 & w_{34} & 0 \end{pmatrix}$$

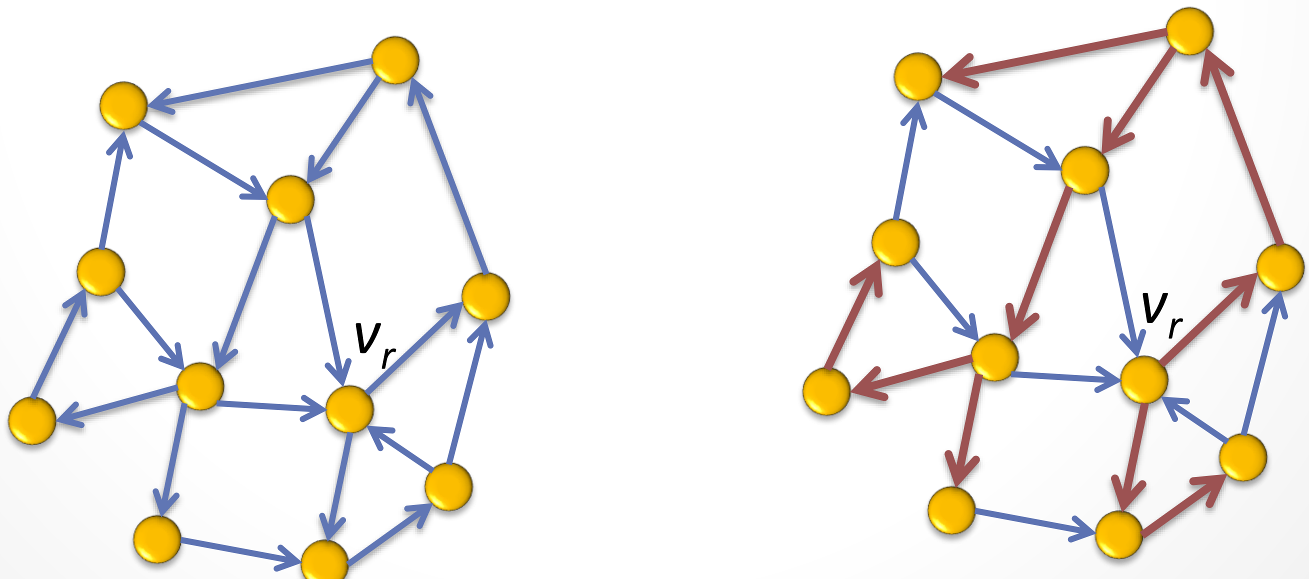
$$\mathbf{L}(D) = \mathbf{Diag}\left(\mathbf{A}^T\mathbf{1}\right) - \mathbf{A}^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -w_{21} & w_{21} & 0 & 0 \\ 0 & -w_{32} & w_{32}+w_{34} & -w_{34} \\ 0 & -w_{42} & -w_{43} & w_{42}+w_{43} \end{pmatrix}$$

8. Consensus in directed networks

Definition 2: A directed graph is a rooted out-branching if:

1. It does not contain a directed cycle and
2. It has a vertex v_r (root) such that for every other vertex v there is a directed path from v_r to v.

Example:



8. Consensus in directed networks

Proposition 13: A directed network contains a rooted out-branching subgraph if and only if $\mathbf{rank}(\mathbf{L}(D)) = n-1$. In that case, $N(\mathbf{L}(D))$ is spanned by the all-ones vector.

Theorem 14: For a directed network *D* containing a rooted out-branching, the state trajectory generated by the consensus dynamic model, initialized from $\mathbf{u}_0$, satisfies

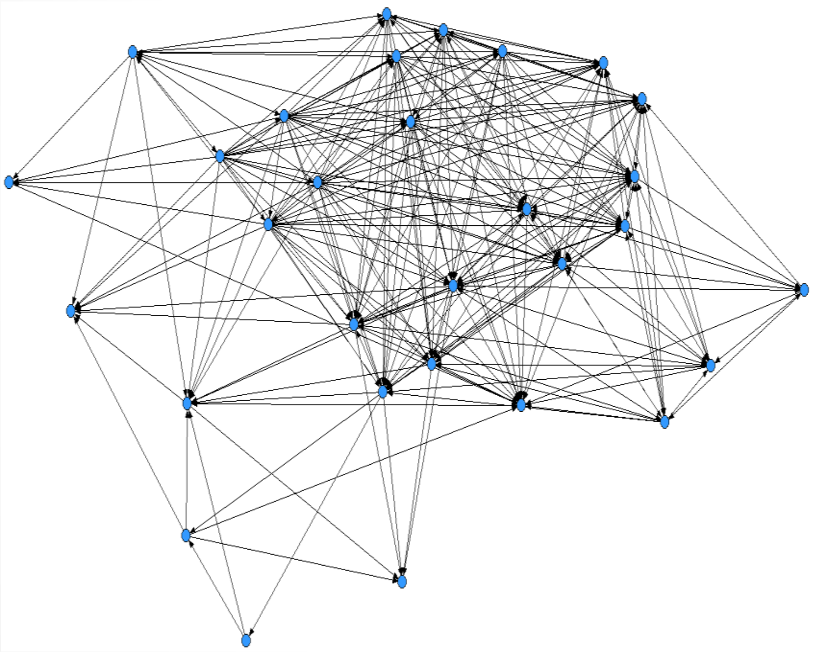
$$\lim_{t\to\infty} \mathbf{u}(t) = \left(\mathbf{p}_1 \mathbf{q}_1^T\right)\mathbf{u}_0$$

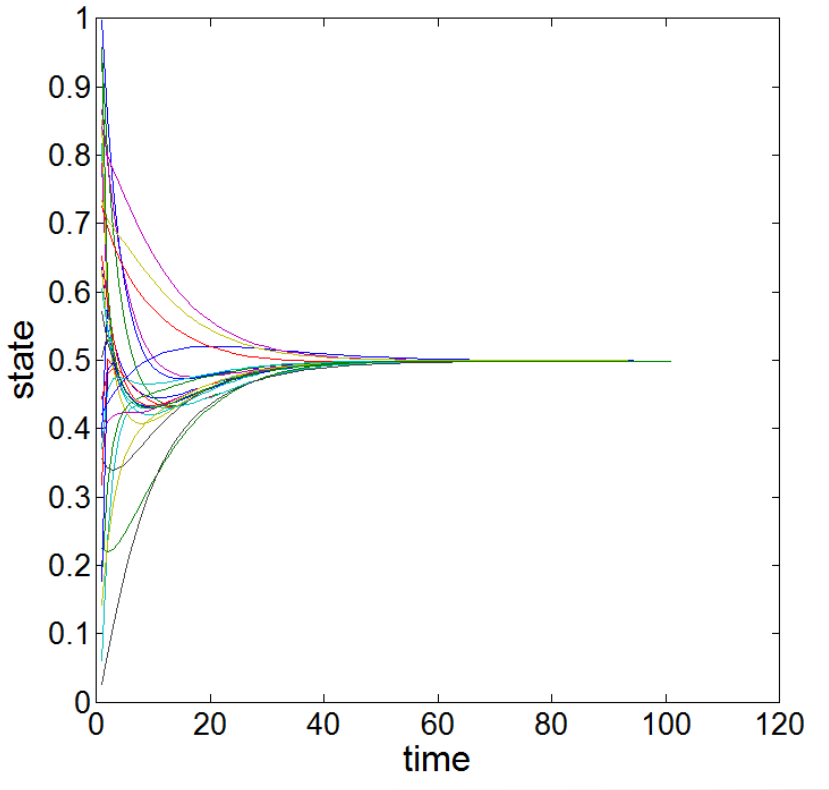
where $\mathbf{p}_1$ and $\mathbf{q}_1^T$, are, respectively, the right and left eigenvectors associated with the zero eigenvalue of L(D), normalized such that $\mathbf{p}_1 \mathbf{q}_1^T = 1$. As a result, one has $\mathbf{u}(t) \to A$ for all initial conditions if and only if *D* contains a rooted out-branching.

Mesbahi & Egerstedt., *Graph Theory Methods in Multiagent Networks*, Princeton, 2010 .

8. Consensus in directed networks

Example:



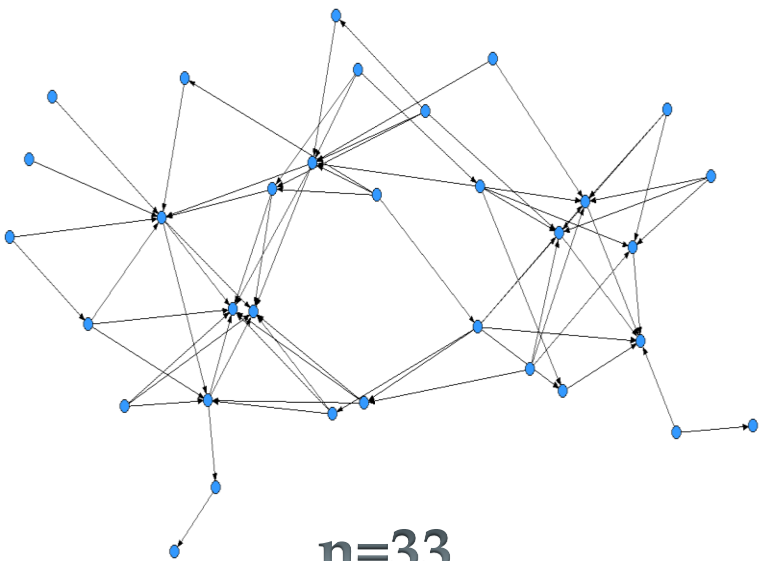


n=30

Rank(L)=29

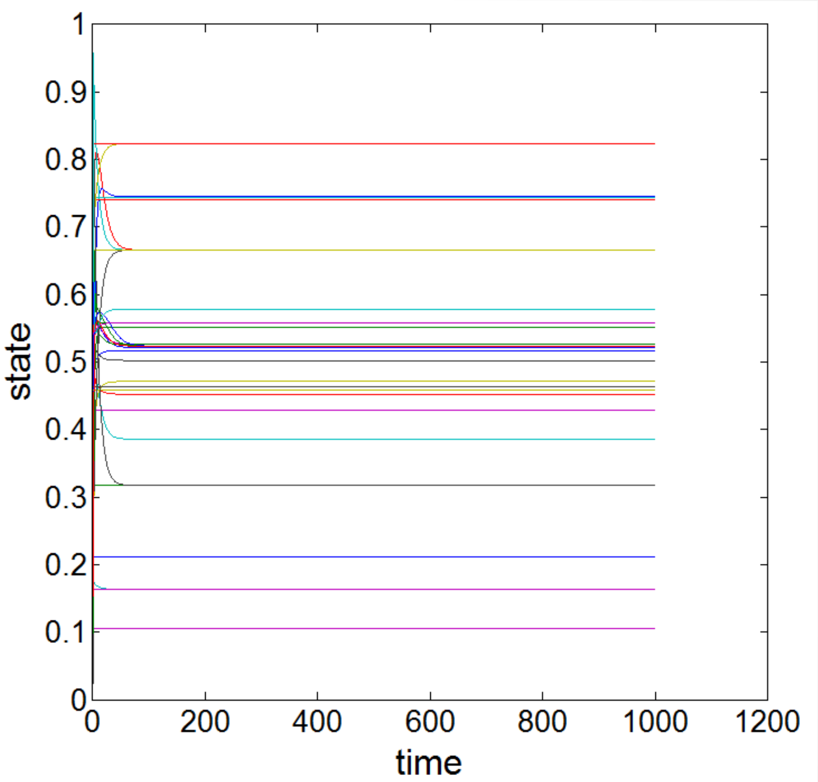
8. Consensus in directed networks

Example:



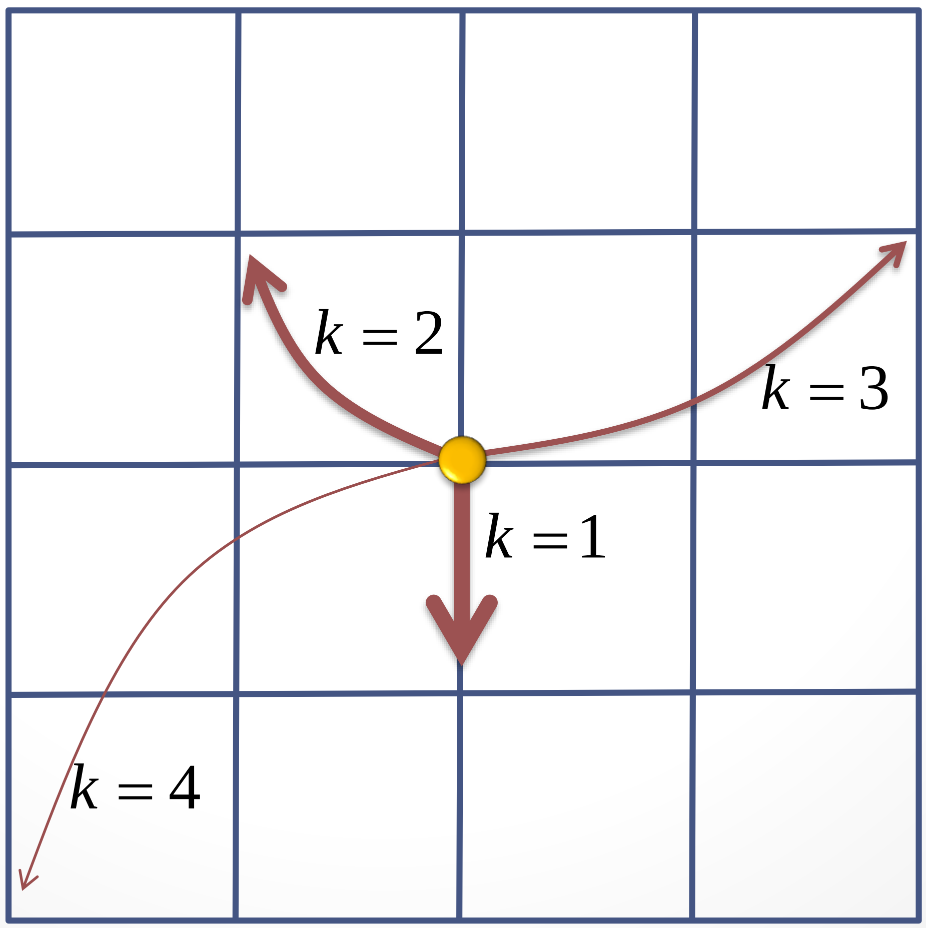
n=33

Rank(L)=21



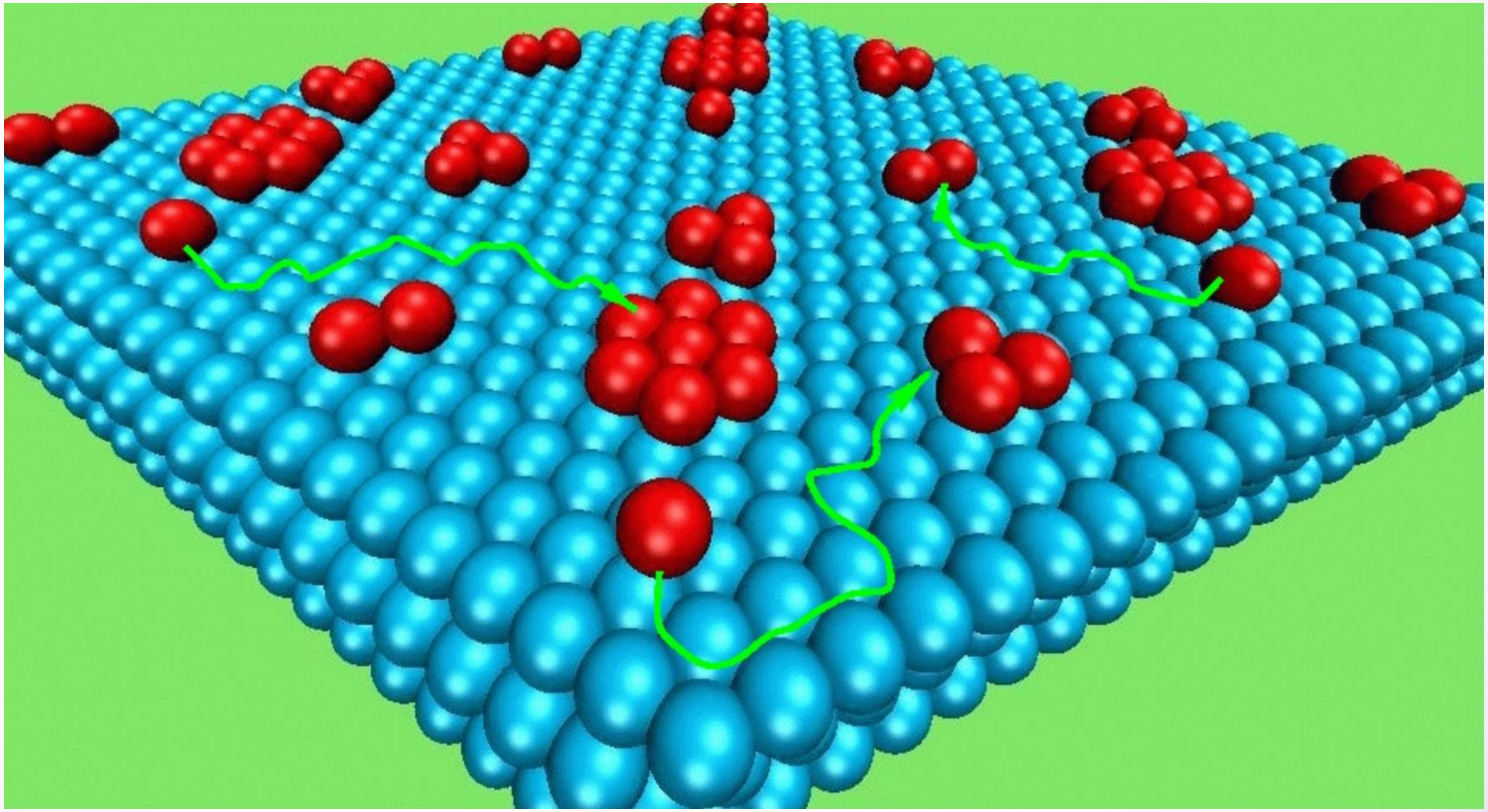
9. Long-range interactions

Let us consider that a particle at a given node can hop not only to its nearest neighbours but to any other node of the network with a probability that decays with the shortest path distance from its current position.



9. Long-range interactions

This situation is frequently observed in the diffusion of atoms and molecules adsorbed on the surface of metals.



Ala-Nissila, et al., *Adv Phys*. **51**, 949, 2002.

9. Long-range interactions

Let us extend the definition of the Laplacian matrix to account for such long-range hops.

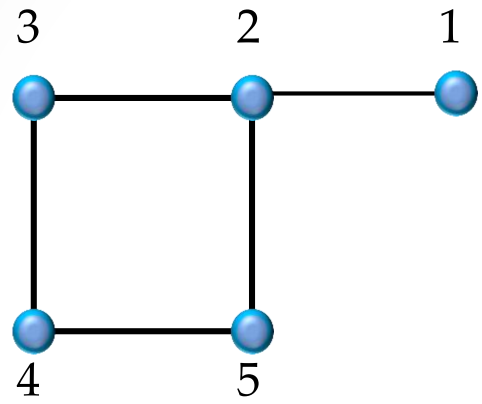
Definition 3: The k-path Laplacian matrix of a connected, undirected graph is a symmetric *nxn* matrix whose entries are given by:

$$\mathbf{L}_k(i,j)=\begin{cases} -1 & d_{i,j}=k, \\ \delta_k(i) & i=j, \\ 0 & \text{otherwise.} \end{cases}$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

9. Long-range interactions

Example:



$$\mathbf{L}_1(G) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{L}_2(G) = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\mathbf{L}_3(G) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

9. Long-range interactions

Proposition 15: The *k*-path Laplacian matrix is positive semidefinite:

$$0 = \mu_1(\mathbf{L}_k) \leq \mu_2(\mathbf{L}_k) \leq \cdots \leq \mu_n(\mathbf{L}_k)$$

Estrada, *Lin. Algebra* Appl., **436**, 3373, 2012.

9. Long-range interactions

We can now generalise the consensus dynamics equation to account for such long-range hops:

$$\dot{\vec{\mathbf{u}}}(t) = -\tilde{\mathbf{L}}\vec{\mathbf{u}}(t)$$

$$\tilde{\mathbf{L}} = \sum_{k=1}^{\Delta} c_k \mathbf{L}_k$$

Mellin transform

$$c_k = k^{-s},$$

$$s \in \Re^{+}$$

Laplace transform

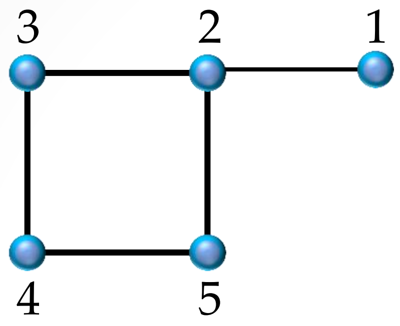
$$c_k = \exp(-l \cdot k),$$

$$l \in \Re^{+}$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

9. Long-range interactions

Example:



$$\tilde{\mathbf{L}} = \sum_{k=1}^{\Delta} c_k \mathbf{L}_k$$

Mellin transformed Laplacian

$$\tilde{\mathbf{L}} = \begin{bmatrix} 1+2\cdot 2^{-s}+3^{-s} & -1 & -2^{-s} & -3^{-s} & -2^{-s} \\ -1 & 3+2^{-s} & -1 & -2^{-s} & -1 \\ -2^{-s} & -1 & 2+2\cdot 2^{-s} & -1 & -2^{-s} \\ -3^{-s} & -2^{-s} & -1 & 2+2^{-s}+3^{-s} & -1 \\ -2^{-s} & -1 & -2^{-s} & -1 & 2+2\cdot 2^{-s} \end{bmatrix}$$

Estrada, *Lin. Algebra* Appl., **436**, 3373, 2012.

9. Long-range interactions

Example:

$$\tilde{\mathbf{L}} = \begin{bmatrix} 1+2\cdot 2^{-s}+3^{-s} & -1 & 2^{-s} & 3^{-s} & 2^{-s} \\ -1 & 3+2^{-s} & -1 & 2^{-s} & -1 \\ 2^{-s} & -1 & 2+2\cdot 2^{-s} & -1 & 2^{-s} \\ 3^{-s} & 2^{-s} & -1 & 2+2^{-s}+3^{-s} & -1 \\ 2^{-s} & -1 & 2^{-s} & -1 & 2+2\cdot 2^{-s} \end{bmatrix}$$

$s \to \infty$ $\qquad\qquad$ $s \to 0$

$$\mathbf{L}(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix} \qquad \mathbf{L}(K_n) = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

9. Long-range interactions

In the discrete-time version of the equation we have:

$$\vec{\mathbf{u}}(t+1) = \tilde{\mathbf{P}}\vec{\mathbf{u}}(t)$$

where:

$$\tilde{\mathbf{P}} = \mathbf{I} - \varepsilon \sum_{k=1}^{\Delta} c_k \mathbf{L}_k ,$$

and Δ is the diameter of the network.

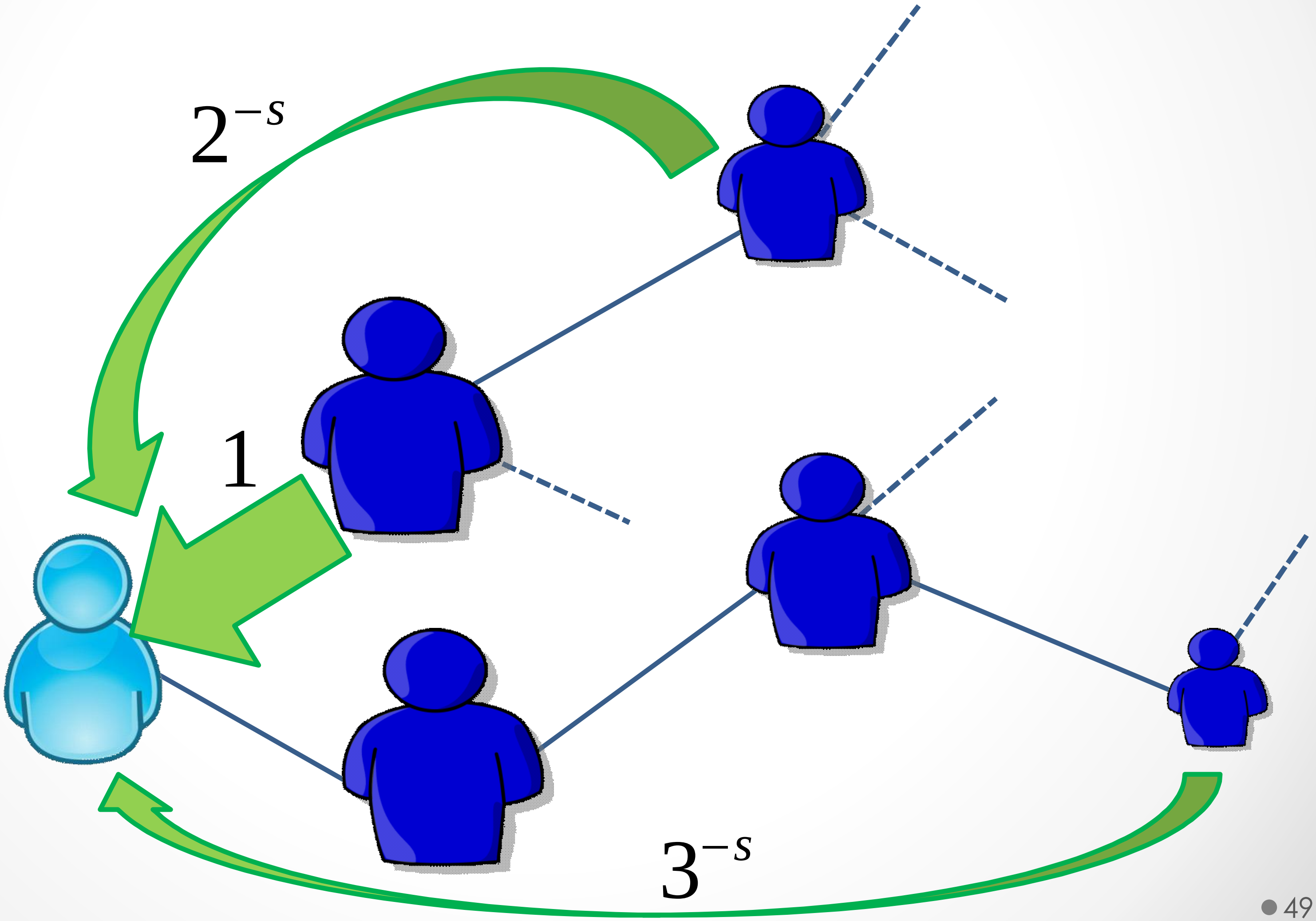
The time step ε is bounded as follows

$$0 < \varepsilon < \left[\sum_{k=1}^{\Delta} \delta_{\max}(k) \right]^{-1}$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

9. Long-range interactions

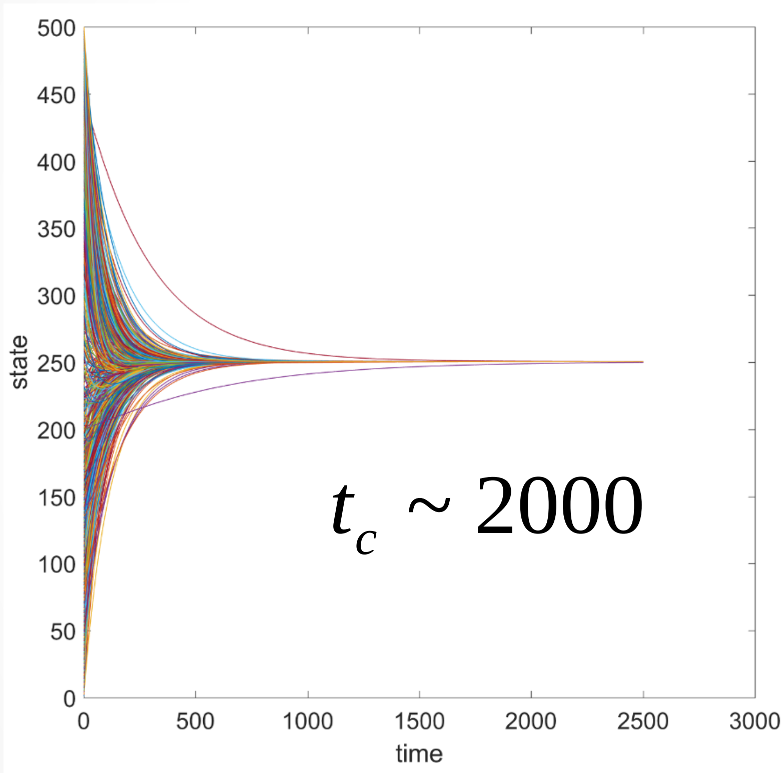
The long-range interaction may account for the indirect peers pressure in a social network.

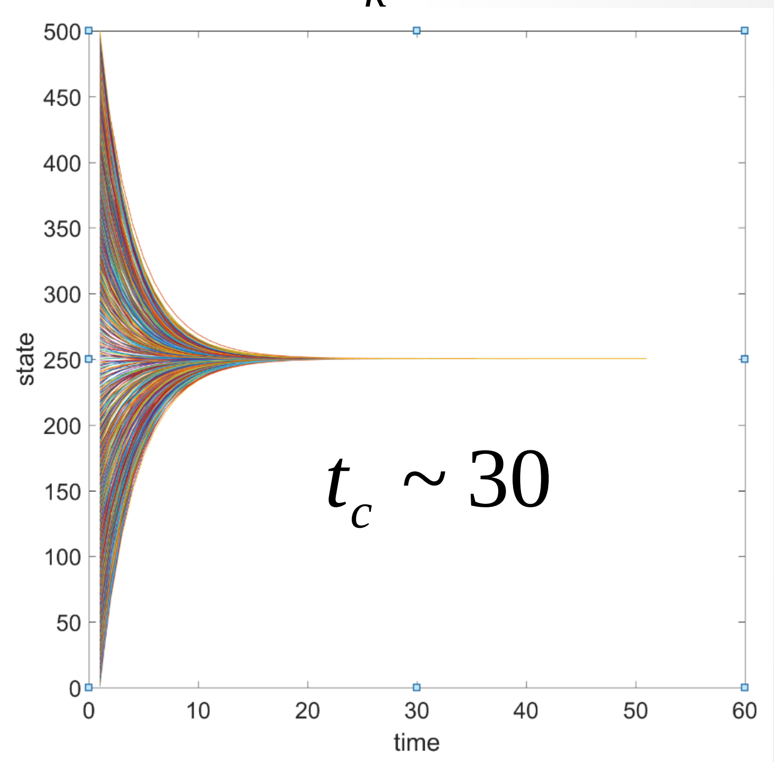


9. Long-range interactions

Erdös-Rényi Random Graph

$$c_k = k^{-1}$$



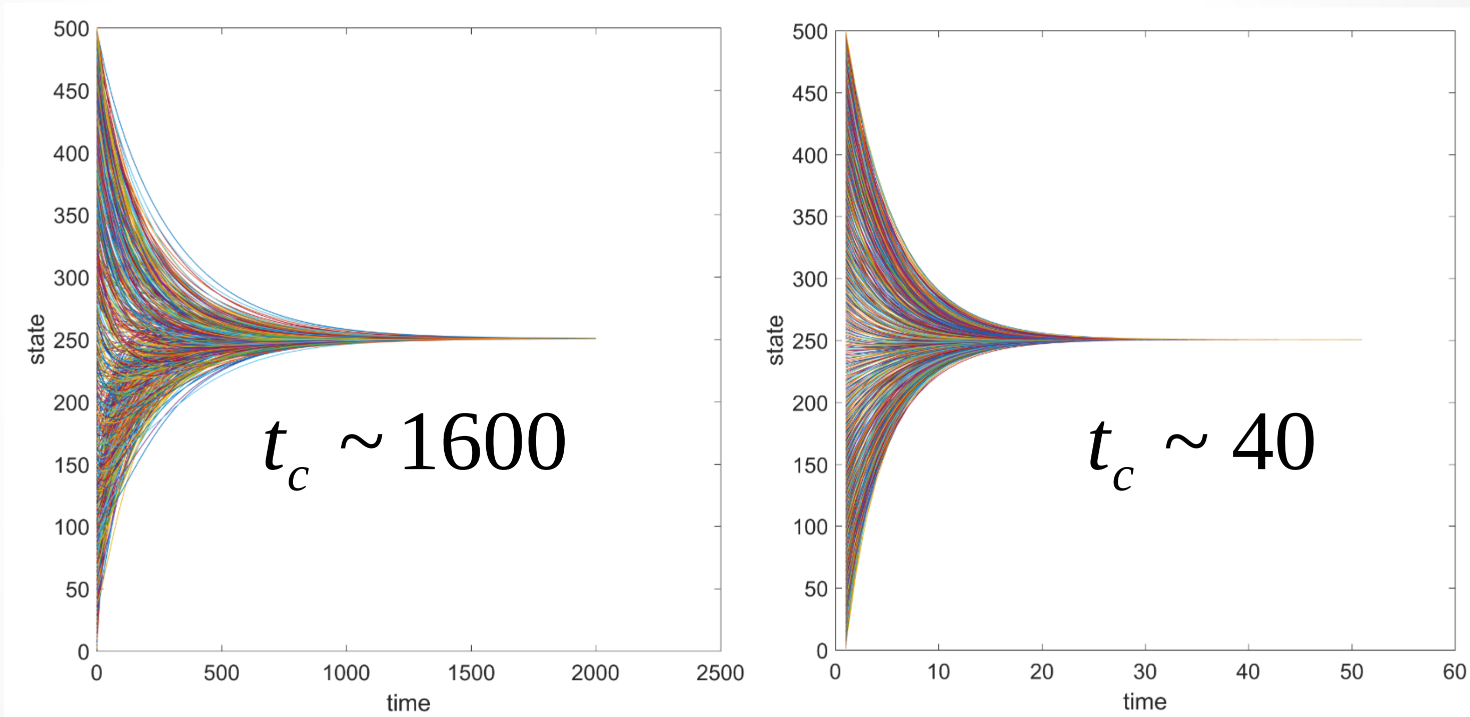


$$G(500,4000)$$

9. Long-range interactions

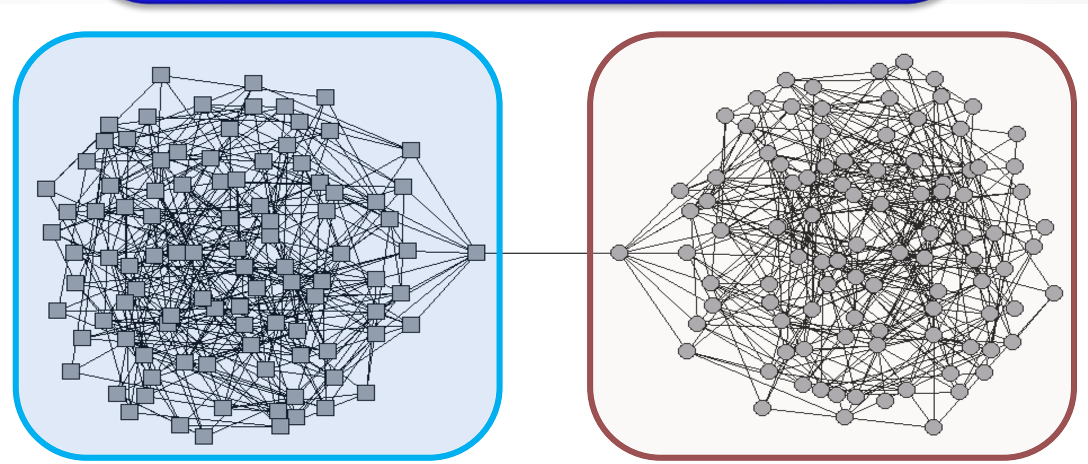
Barabási-Albert Random Graph

$$c_k = k^{-1}$$

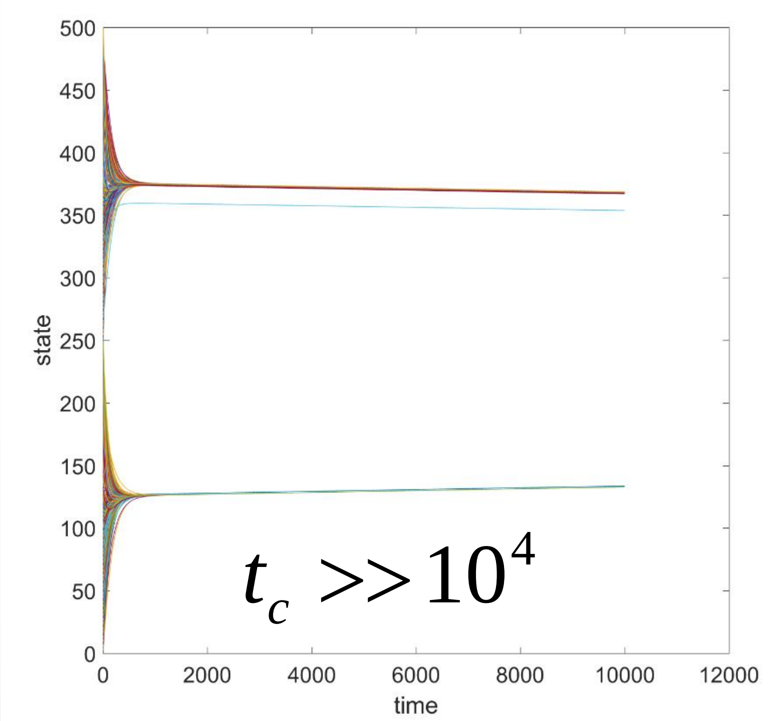


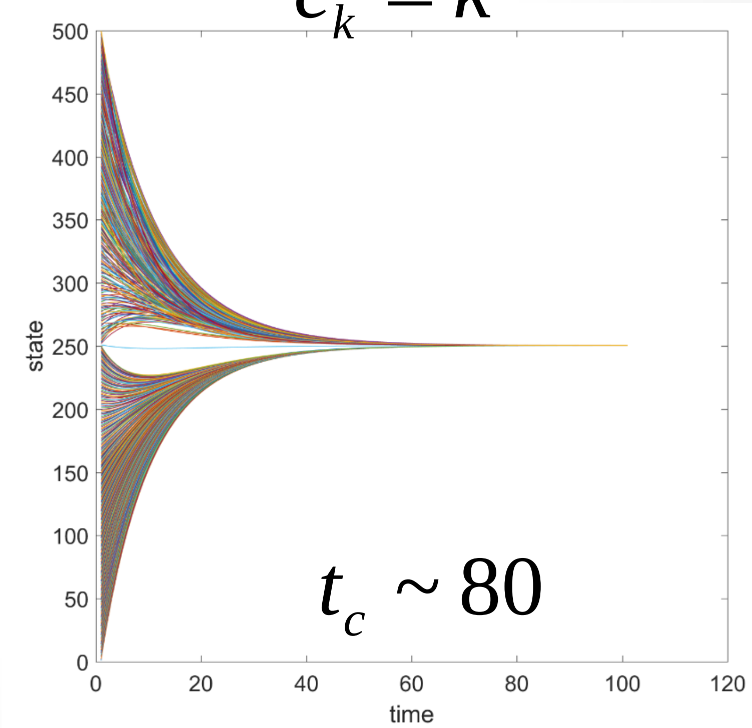
$$G(500,6)$$

9. Long-range interactions



$c_k = k^{-1}$





10. Leader selection

What are the 'best' leaders in a network to reduce the time for consensus of the followers?

Estrada, Vargas-Estrada, *Sci*. *Rep*., **3**, 2013.

10. Leader selection

Theorem 16: The time of consensus averaged over all the nodes in the network is bounded as follow:

Fiedler vector

$$\langle t_c \rangle \geq \frac{1}{n\mu_2} \sum_{p=1}^{n} \ln \left| \frac{\vec{\psi}_2(p)\left(\vec{\psi}_2 \cdot \vec{u}_0\right)}{\delta} \right| .$$

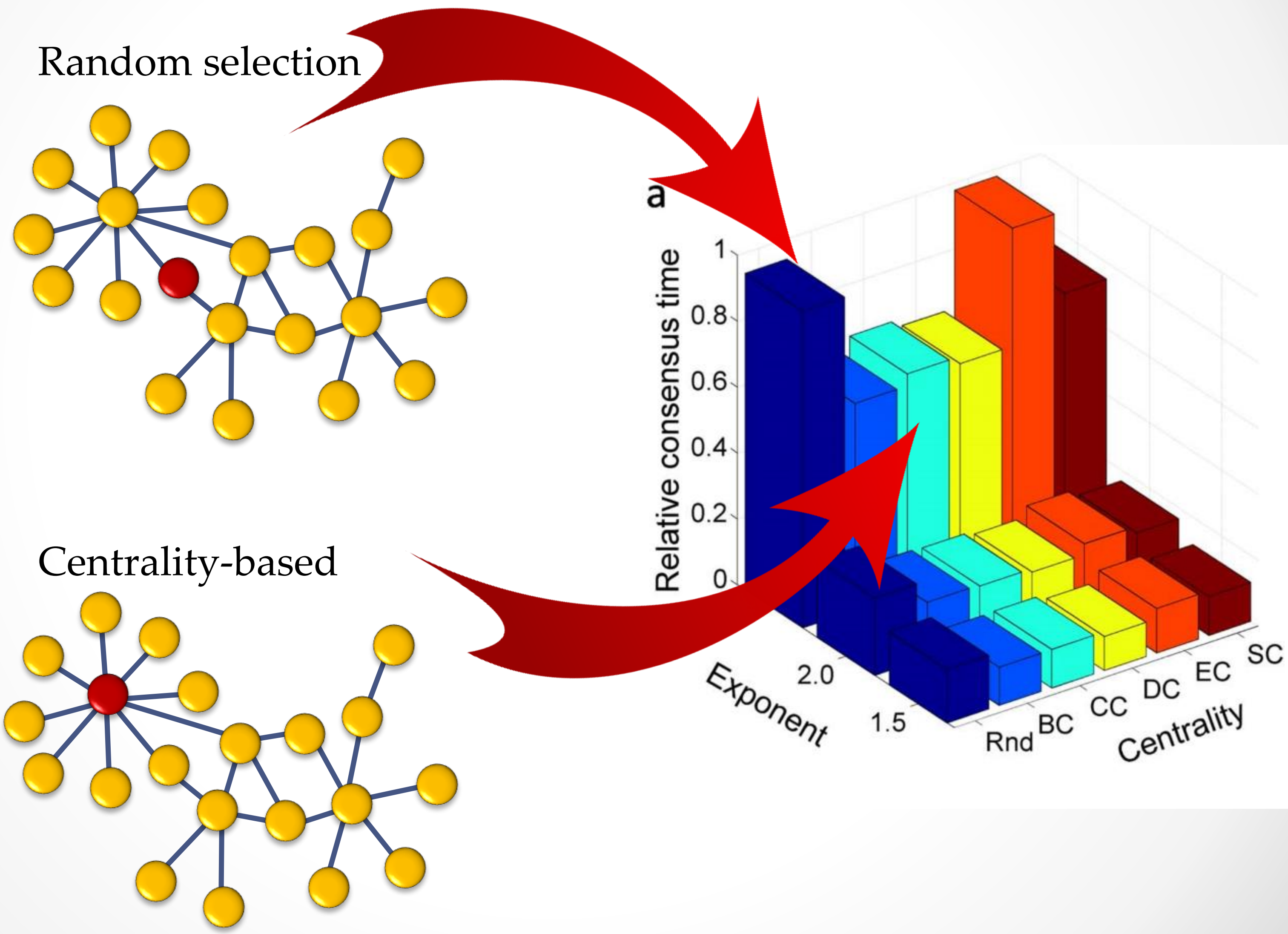
Estrada, Sheerin, *Physica D, Nonlinearity* **323-324**, 2016, 20-26.

5. Consensus in undirected networks

Miroslav Fiedler
1926-2015

10. Leader selection

No long-range interactions

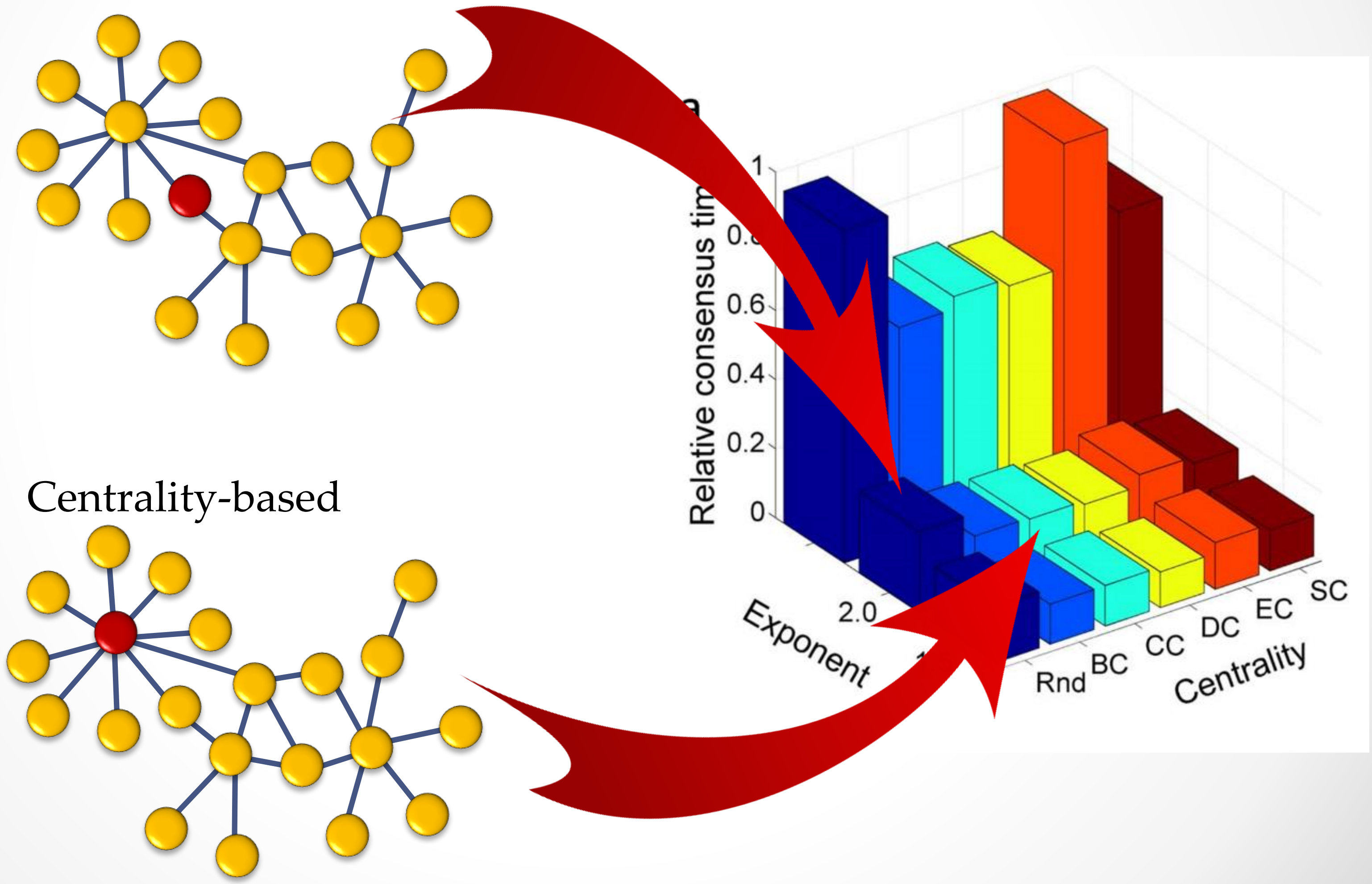


Estrada, Vargas-Estrada, *Sci*. *Rep*., **3**, 2013.

10. Leader selection

Long-range interactions

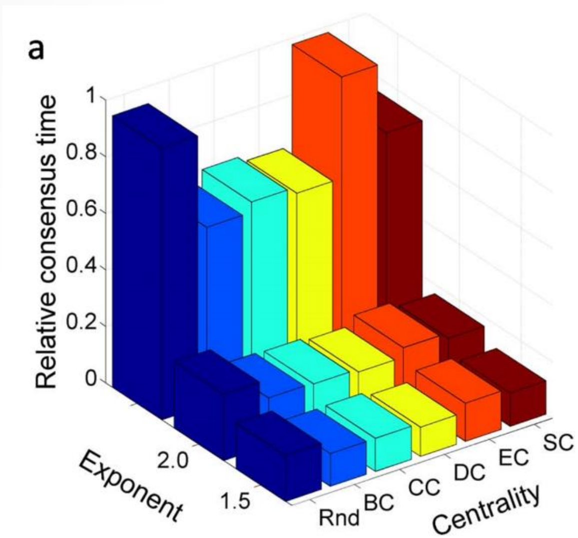
Random selection



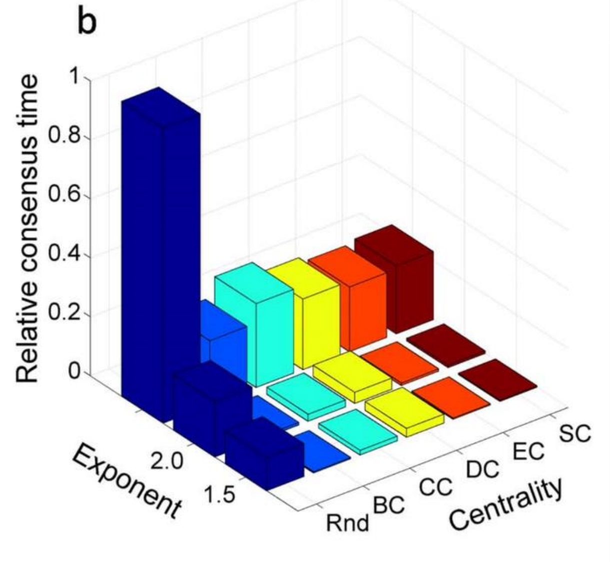
Estrada, Vargas-Estrada, *Sci*. *Rep*., **3**, 2013.

10. Leader selection

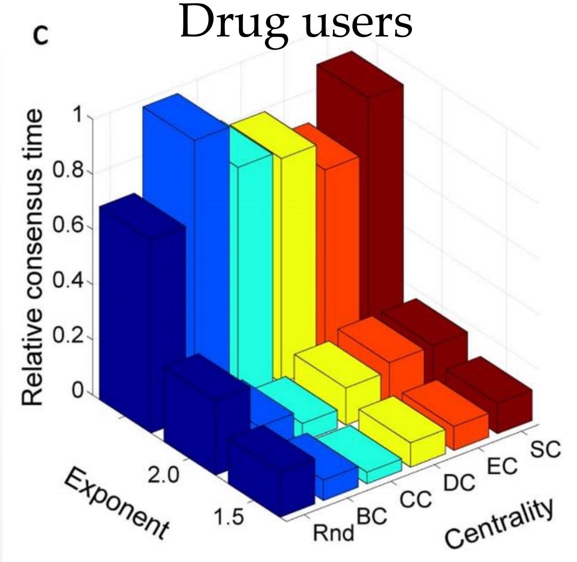
Sawmill



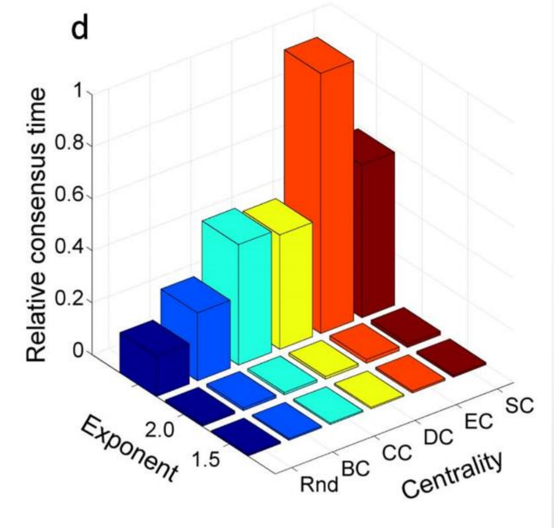
Corporate directors



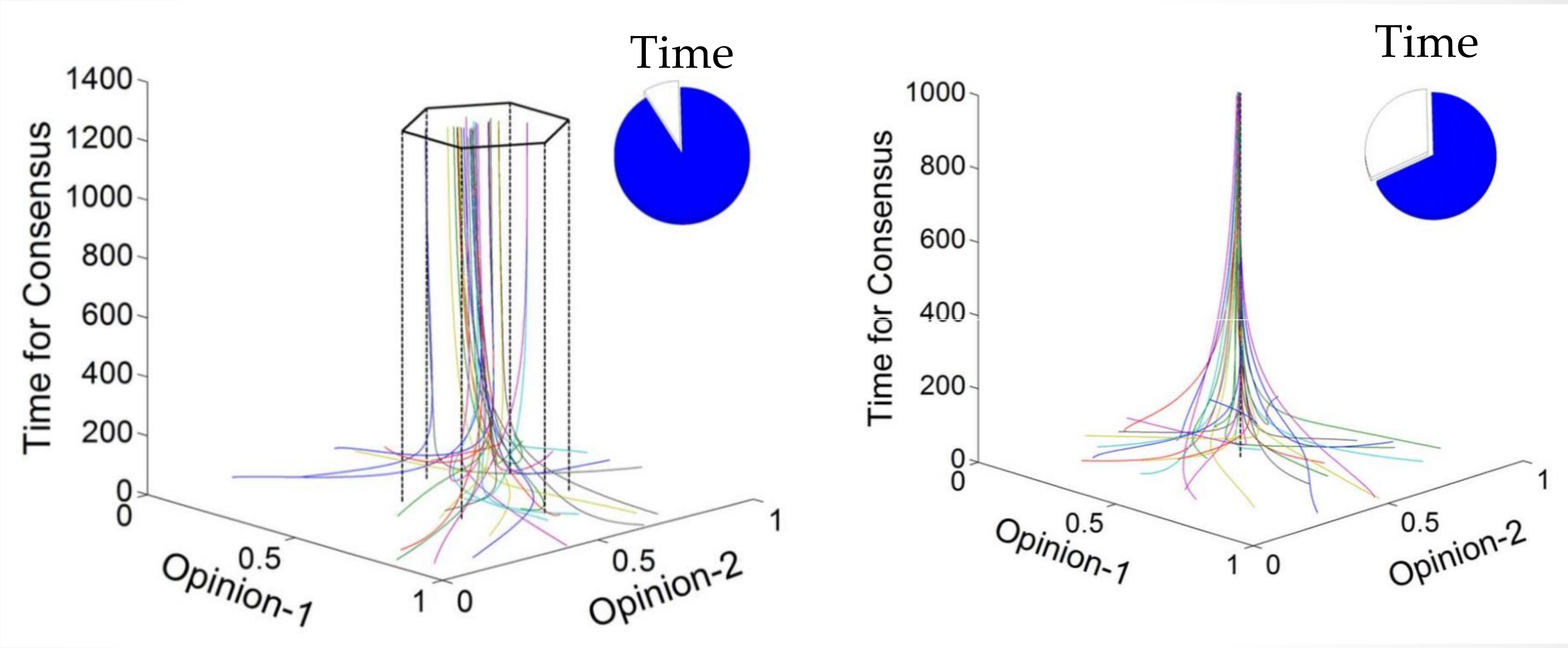
Drug users



Random with communities



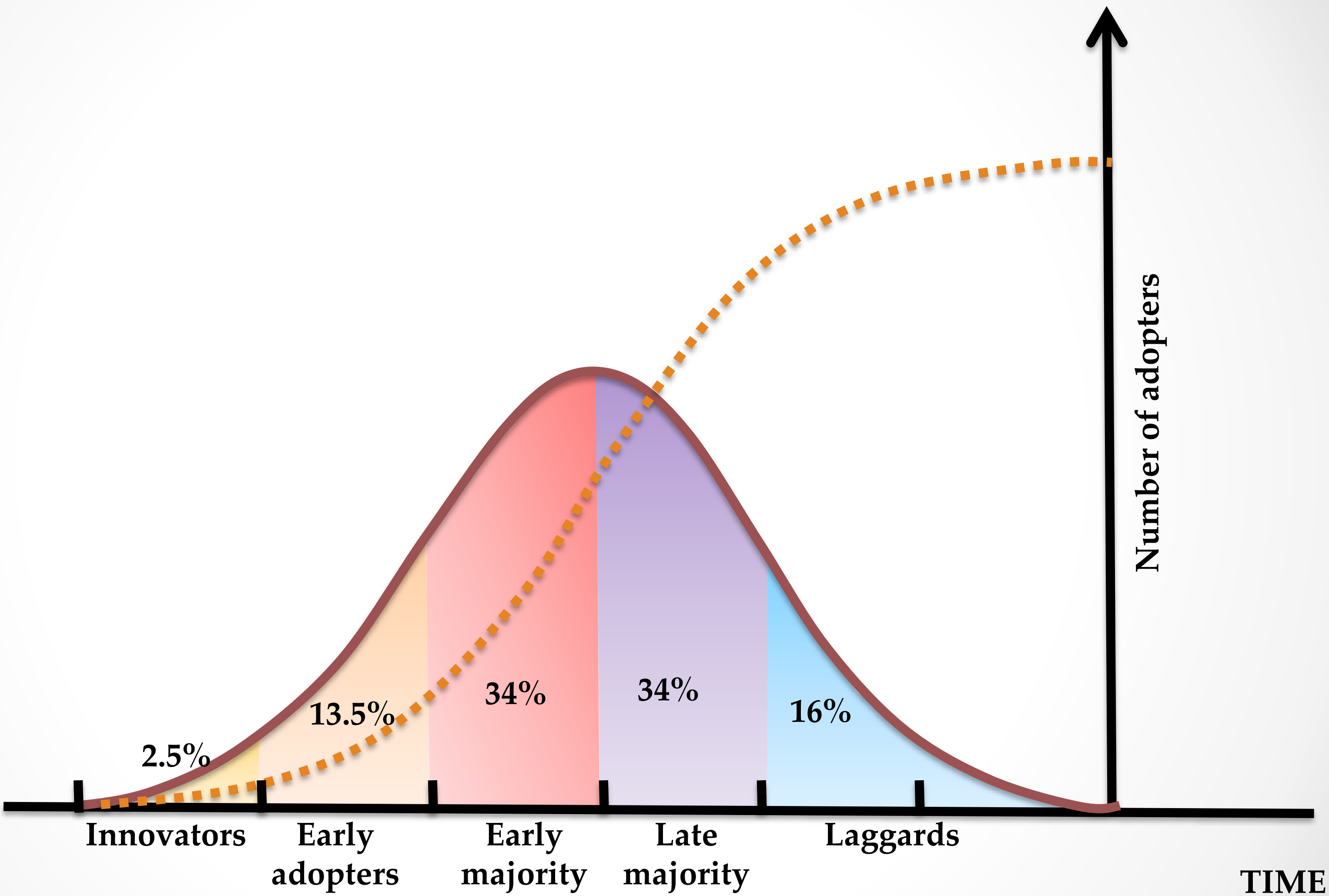
10. Leader selection



LEADERS' COHESIVENESS

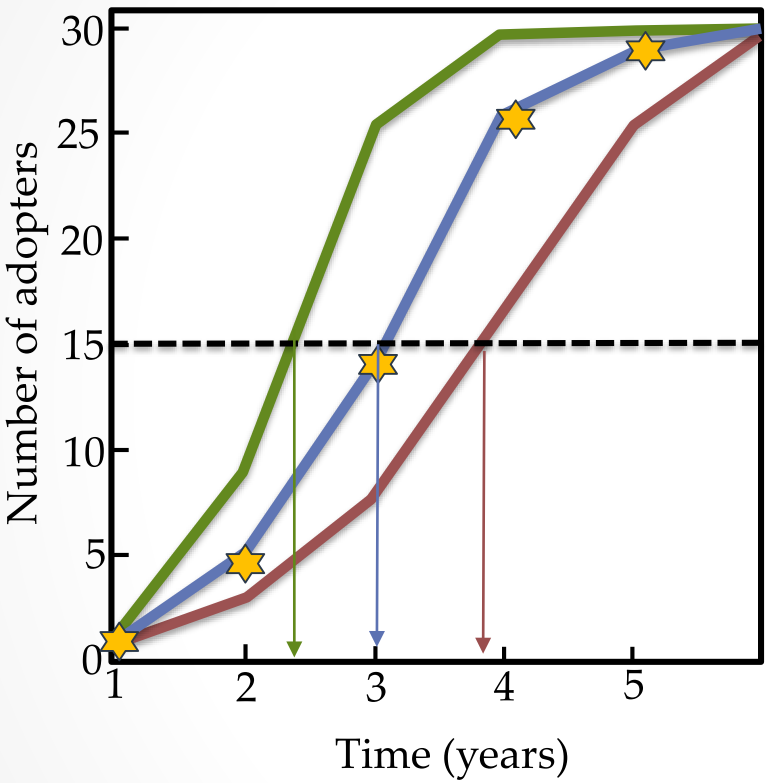
Estrada, Vargas-Estrada, *Sci*. *Rep*., **3**, 2013.

11. Diffusion on networks

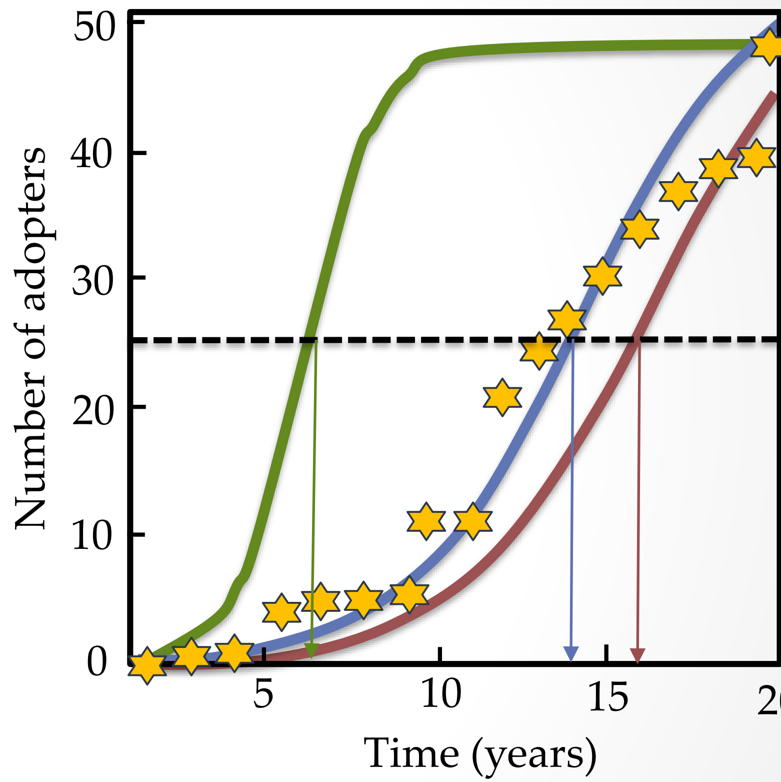


11. Diffusion on networks

Diffusion of a mathematical method among high schools



Diffusion of a biotech product among Brazilian farmers



Observation

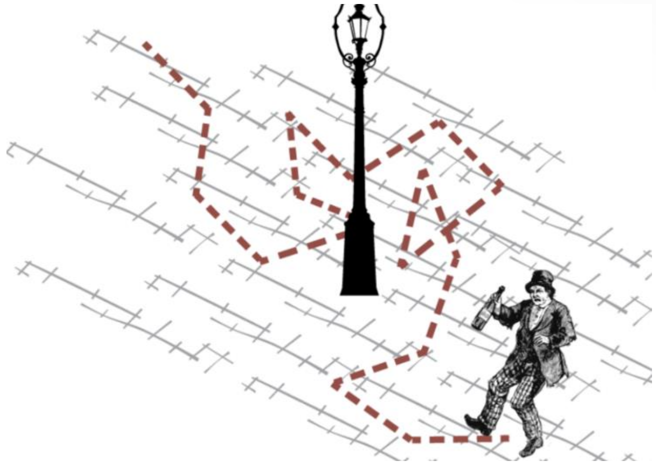
No indirect peers pressure

Moderate indirect peers pressure

High indirect peers pressure

Estrada, Vargas-Estrada, *Sci. Rep.*, **3**, 2013.

12. Random walks on networks



Definition 4: A random walk on a network G is a sequence of nodes $v_0, v_1, \cdots, v_k$ where each v_{t+1} is chosen to be a random neighbour of v_t, $\{v_t, v_{t+1}\} \in E$ and the probability of the transition is given by

$$P_{ij} = \Pr\left(x_{t+1} = v_j \middle| x_t = v_i\right)$$

where

$$\sum_i P_{ij} = 1$$

12. Random walks on networks

- Transition matrix $\mathbf{P}$ has entries

$$P_{ij} = \begin{cases} (k_i)^{-1} & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- In terms of the degree and adjacency matrices

$$P_{ij} = \frac{A_{ij}}{k_i^{out}} = K_{ii}^{-1} A_{ij}$$

- The probability at time $t+1$

$$p_j(t+1) = \sum_i P_{ij} p_i(t) = \sum_i \frac{p_i(t)}{k_i^{out}} A_{ij}$$

- In matrix form

$$\vec{\mathbf{p}}(t+1) = \vec{\mathbf{p}}(t)\mathbf{P} = \vec{\mathbf{p}}(t)(\mathbf{K}^{-1}\mathbf{A})$$

12. Random walks on networks

- The distribution at time t, $\vec{\mathbf{p}}(t)$ can be obtained from the initial distribution $\vec{\mathbf{p}}(0)$

$$\vec{\mathbf{p}}(t) = \vec{\mathbf{p}}(0)\mathbf{P}^t$$

- On nonbipartite networks the random walk converges to the limiting distribution

$$\lim_{t\to\infty} \vec{\mathbf{p}}(t) = \lim_{t\to\infty} \vec{\mathbf{p}}(0)\mathbf{P}^t = \vec{\boldsymbol{\pi}}$$

- The left eigenvalue of the matrix $\mathbf{P}$ is $\lambda = 1$

$$\vec{\boldsymbol{\pi}} = \vec{\boldsymbol{\pi}}\mathbf{P}$$

12. Random walks on networks

- Consider the Laplacian matrix

$$\mathbf{L} = \mathbf{K} - \mathbf{A}$$

and multiply both members by $\mathbf{K}^{-1}$

$$\mathbf{K}^{-1}\mathbf{L} = \mathbf{I} - \mathbf{K}^{-1}\mathbf{A}$$

Then, because

$$\mathbf{P} = \mathbf{K}^{-1}\mathbf{A}$$

we get

$$\mathbf{P} = \mathbf{I} - \mathbf{K}^{-1}\mathbf{L}$$

12. Random walks on networks

- Consider the transition matrix

$$\mathbf{P} = \mathbf{K}^{-1}\mathbf{A}$$

The graph Laplacian can be expressed as

$$\mathbf{L} = \mathbf{K} - \mathbf{A} = \mathbf{K}(\mathbf{I} - \mathbf{P})$$

Thus, the diffusion equation can be expressed in terms of the transition matrix of the random walk on the network

$$\dot{\vec{\mathbf{u}}}(t) = -\mathbf{K}(\mathbf{I} - \mathbf{P})\vec{\mathbf{u}}(t), \qquad \vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0$$

13. Multi-hopper walks on networks

- Replace the Laplacian by the transformed k-path Laplacian

$$\tilde{\mathbf{P}} = \mathbf{I} - \tilde{\mathbf{K}}^{-1}\tilde{\mathbf{L}}$$

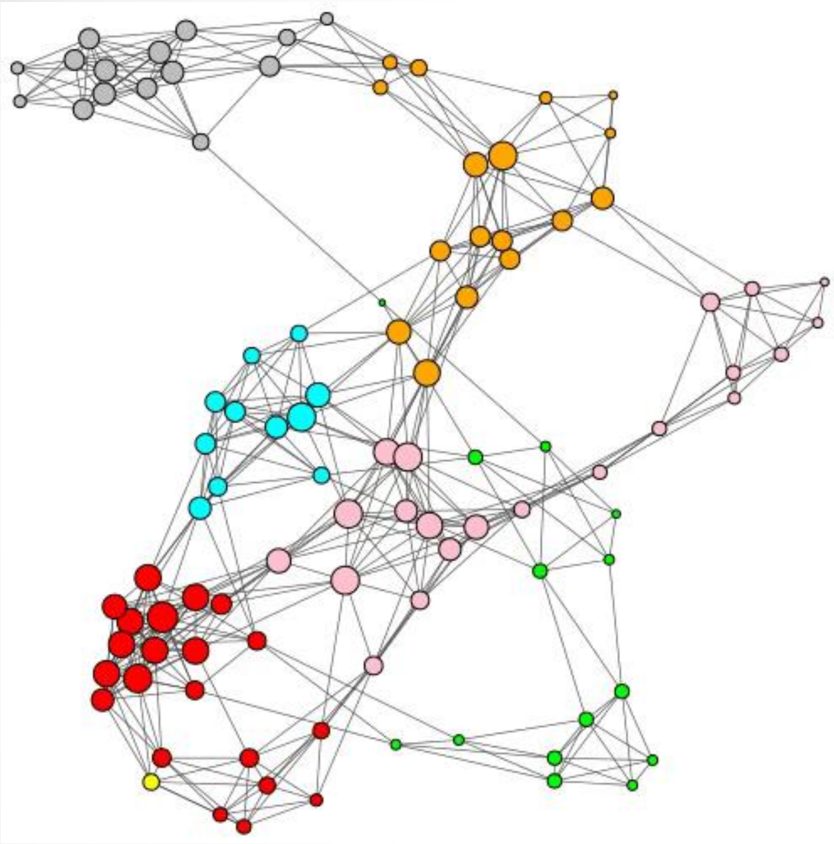
$$\tilde{\mathbf{L}} = \sum_{k=1}^{\Delta} c_k \mathbf{L}_k \qquad \tilde{\mathbf{K}} = diag\left(diag\left(\mathbf{L}_k\right)\right)$$

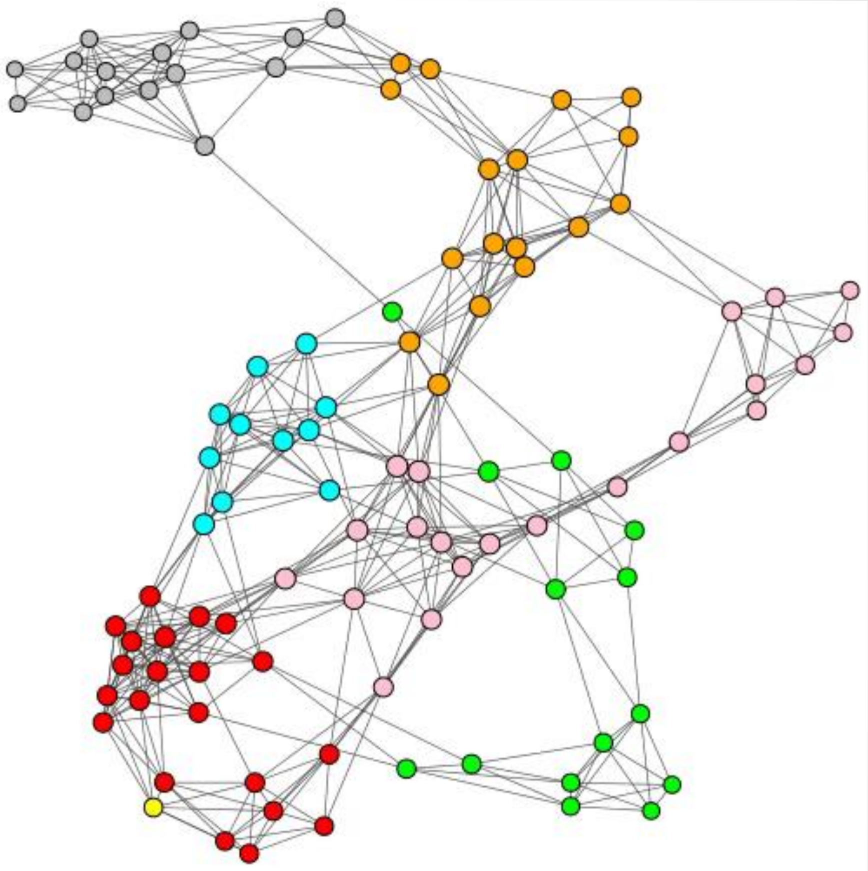
- A multi-hopper random walk evolves as

$$\vec{\mathbf{p}}(t+1) = \vec{\mathbf{p}}(t)\tilde{\mathbf{P}}$$

Estrada et al., Work in progress, 2016.

12. Random walks on networks





13. Codes

Matlab® Codes

University of Strathclyde

Thank you!