# Approximability of the $d$-dimensional Euclidean Capacitated Vehicle Routing Problem

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# Abstract

- We consider the classic single-product Capacitated Vehicle Routing Problem (CVRP) within unit customer demand.
- CVRP in strongly NP-hard even being formulated in Euclidean spaces of fixed dimension.
- Nevertheless, in such a special case the CVRP can be approximated well.
- For instance, in the Euclidean plane, for the problem (and it's various versions) there exist polynomial time approximation schemes (PTAS).
- We propose polynomial time approximation schemes for $\mathbb{R}^d$ (for any fixed $d > 1$.

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# Contents

# Introduction and related word

- Vehicle Routing Problem (VRP) is introduced in [Danzig and Ramser, 1959] for a fleet of gasoline trucks. Curiously, they were sure that this problem can be solved efficiently (in polynomial time).
- The VRP can be defined as the problem of designing the least cost delivery routes from a given depot to a set of spatially distributed customers s.t. some additional constraints (capacity, time-windows, etc.)
- VRP is a strongly NP-hard problem (having TSP as a special case). The problem remains NP-hard even in any fixed dimension Euclidean space [Papadimitriou (1977)], [Lenstra, Rinnooy Kan (1981)].

# Related work

- Metric CVRP is Apx-hard [Asano et al. (1996)].
- PTAS for Euclidean $q$-CVRP in the plane for $q = O(\log\log n)$ [Haimovich, R. Kan (1985)]
- In [Asano et al. (1996)] and [Arora (1998)], is extended for $q = O(\log n / \log\log n)$ and $q = \Omega(n)$.
- PTAS for the plane for $q \leq 2^{\log^{\delta} n}$, where $\delta = \delta(\varepsilon)$ [Adamaszek (2009)].
- $O(n^{(\log n)^{O(1/\varepsilon)}})$ (for any value of $q$) time-complexity QPTAS [Das and Mathieu (2010), (2014)].
- All these results are valid for the plane.
- We extend the results obtained in [Haimovich, Kan (1985)] and [Asano (1996)] to the case of $\mathbb{R}^d$ for any fixed $d$ and any fixed number $m$ of depots
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# Capacitated Vehicle Routing Problem (CVRP)

- **Input:** A complete weighted graph $G^0 = (X \cup Y, E, w)$ and a capacity $q \in \mathbb{N}$. Here $X = \{x_1, \dots, x_n\}$ is a set of clients, $Y = \{y_1, \dots, y_m\}$ is a set of depots, $w : E \to \mathbb{R}_+$ defines inter-node transportation costs.
  - $$x_i \mapsto r_i = \min\{w(y_j, x_i) \colon j = 1, \dots, m\},$$
    $$X_1 \cup \dots \cup X_m = X,\ \ X_j = \{x_i \in X \colon r_i = w(x_i, y_j)\},$$
    such that any client $x_i$ is assigned to the nearest depot $y_j$.
  - any feasible route $R$ has a form $y_{j_s}, x_{i_1}, \dots, x_{i_t}, y_{j_f}$, where $x_{i_1}, \dots, x_{i_t}$ are distinct clients and $t \leq q$. $w(R) = w(\{y_{j_s}, x_{i_1}\}) + w(\{x_{i_1}, x_{i_2}\}) + \dots + w(\{x_{i_t}, y_{j_f}\})$.
- **The problem is**, for a given graph $G$ and capacity $q$, to find a cheapest set of tours visiting each client once.
- If $m = 1$, we have the Single Depot Capacitated Vehicle Routing Problem (SDCVRP), otherwise MDCVRP.
- MDCVRP1: routes can start and terminate at different depots. MDCVRP2: for any route $R$, $y_{j_s} = y_{j_f}$.

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# Special settings

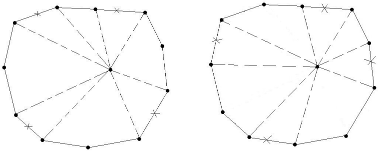
## Metric CVRP

The weight function $w$ meets the triangle inequality. For any vertices $x_{i_1}, x_{i_2}$ and $x_{i_3}$, $w(x_{i_1}, x_{i_2}) \leq w(x_{i_1}, x_{i_3}) + w(x_{i_2}, x_{i_3})$.

## Euclidean CVRP

In this case, the depot and all the customers locations are points in $d$-dimensional Euclidean space $X \cup \{x_0\} \subset \mathbb{R}^d$ and $w(x_i, x_j) = \|x_i - x_j\|_2$.

# ITP heuristic: the scheme



- Relate the initial SDCVRP problem with TSP problem for the graph $G = G^0 \langle X \rangle$.
- Take an arbitrary Hamiltonian cycle $H$ in the graph $G$.
- Starting from $x_1$, break this cycle into $l = \lceil n/q \rceil$ disjoint segments such that each of them contains at most $q$ customers.
- Then, connect the endpoints of any segment with depot $y_1$ to provide a feasible solution for the initial problem.
- Perform the same procedure iteratively for any other starting point $x_i$ and construct $n$ feasible solutions $V_1, \ldots, V_n$ of the initial instance of SDCVRP.
- Output the best (cheapest) one $S_{\text{ITP}} = \arg\min w(V_j)$.

# ITP heuristic: general case

Let $\bar{r} = 1/n \sum_{i=1}^{n} r_i$. Compare $w(S_{\text{ITP}})$ with $w(H)$

**Lemma 1 (upper bound)**

$$w(S_{\text{ITP}}) \leq 2 \left\lceil \frac{n}{q} \right\rceil \bar{r} + \left(1 - \frac{\lceil n/q \rceil}{n}\right) w(H)$$

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## Proof.

- For $l = \lceil n/q \rceil$, each edge $\{x_{i_1}, x_{i_2}\}$ of the tour $H$ is included $n - l$ times to the solutions $V_1, \ldots, V_n$ and $l$ times is replaced with 'radial' edges $\{y_1, x_{i_1}\}$ and $\{y_1, x_{i_2}\}$ of costs $r_{i_1}$ and $r_{i_2}$.
- Therefore, the total cost $C$ of all solutions $V_1, \ldots, V_n$ is equal to $C = 2l \sum_{i=1}^{n} r_i + (n - l)T(X)$.
- Thus,

$$w(S_{\mathrm{ITP}}) \leq \frac{C}{n} = 2l\bar{r} + (1 - l/n)T(X) = 2 \left\lceil \frac{n}{q} \right\rceil \bar{r} + \left(1 - \frac{\lceil n/q \rceil}{n}\right) w(H).$$

□

# ITP heuristic: the metric case

On the other hand, for an arbitrary feasible solution $S$ of the SDCVRP and the corresponding cycle $H$

**Lemma 2 (lower bound)**

$$w(S) \geq \max \left\{ 2\frac{n}{q}\bar{r}, w(H) \right\}.$$

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## Proof.

- The bound $w(S) \geq w(H)$ follows from the triangle inequality.
- Let $X_1, \ldots, X_l$ be client subsets visited by different routes (of $S$). Again, by the triangle inequality,

$$w(S) \geq \sum_{j=1}^{l} 2 \max_{x_i \in X_j} r_i \geq 2 \sum_{j=1}^{l} \frac{\sum_{x_i \in X_j} r_i}{|X_j|} \geq 2 \sum_{j=1}^{l} \frac{\sum_{x_i \in X_j} r_i}{q} = 2\frac{n}{q}\bar{r}.$$

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# ITP heuristic: the metric case

As a consequence, for optimum values of the SDCVRP and TSP defined by graphs $G^0$ and $G$

**Theorem 3 (Haimovich and R.Kan)**

$$\max\left\{2\frac{n}{q}\bar{r}, \mathrm{TSP}^*\right\} \leq \mathrm{SDCVRP}^* \leq 2\left\lceil\frac{n}{q}\right\rceil\bar{r} + \left(1 - \frac{1}{q}\right)\mathrm{TSP}^*.$$

**Corollary 4**

*Suppose, in ITP, we use a Hamiltonian cycle $H$, such that $w(H) \leq (1+\varepsilon)\mathrm{TSP}^*$. Then, the cost $w(S_{\mathrm{ITP}})$ of ITP-based approximate solution will be*

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# Fixed ratio approximation for the SDCVRP

Further, for any $\rho$-approximation algorithm for TSP, we obtain

$$\frac{w(S_{\mathrm{ITP}})}{\mathrm{SDCVRP}^*} \le \frac{2\lceil \frac{n}{q} \rceil \bar{r} + (1 - l/n)\rho \mathrm{TSP}^*}{\max\left\{2\frac{n}{q}\bar{r}, \mathrm{TSP}^*\right\}}$$

$$\le \frac{q}{n} + 1 + \left(1 - \frac{\lceil n/q \rceil}{n}\right)\rho \le \frac{q}{n} + 1 + \left(1 - \frac{1}{q}\right)\rho,$$

- For $q = o(n)$, any $\rho$-approximation algorithm for the metric TSP induces asymptotically $(1+\rho)$-approximation algorithm for the metric SDCVRP.
- Since the running time of the ITP is at most $O(n^2)$, the overall complexity of any ITP-based approximation algorithm is defined by the running time of the underline approximation algorithm for TSP.
- Well-known Christofides algorithm produces 5/2-approximation algorithm for SDCVRP with time-complexity $O(n^3)$.

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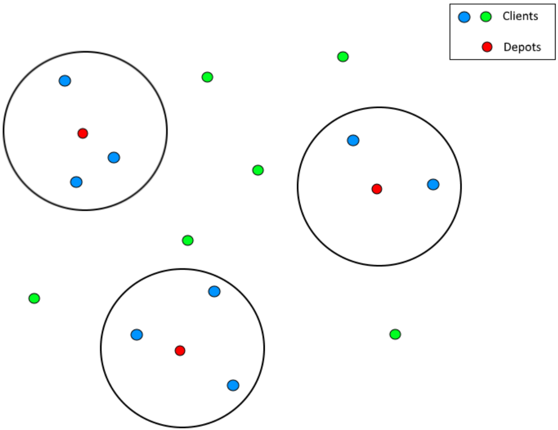
# Combined ITP scheme (CITP)

- **Input:** a complete weighted $n$-graph $G^0(X \cup \{y\}, E, w)$, a capacity $q \in \mathbb{N}$, and and accuracy $\varepsilon > 0$.
- **Output:** $(1+\varepsilon)$-approximate solution $S_{\mathrm{CITP}}$ of the CVRP.
- relabel the clients so that $r_1 \geq r_2 \geq \ldots \geq r_n$;
- take a value $k = k(\varepsilon)$ specifying partition $X$ into subsets $X(k) = \{x_1, \ldots, x_{k-1}\}$ of outer and $X \setminus X(k)$ inner clients;
- find an exact solution $S^*(X(k))$ for MDCVRP specified by $G^0 \langle X(k) \cup Y \rangle$;
- using ITP, find an approximate solution $S_{\mathrm{ITP}}(X_j \setminus X(k))$ for each subgraph $G^0 \langle X_j \setminus X(k) \cup \{y_j\} \rangle$;
- output $S_{\mathrm{CITP}} = S^*(X(k)) \cup S_{\mathrm{ITP}}(X_1 \setminus X(k)) \cup \ldots \cup S_{\mathrm{ITP}}(X_m \setminus X(k))$.

# Combined ITP scheme (CITP)



# Main result

**Theorem 5**

*For any $\rho$-approximation algorithm with the running time of $O(n^c)$ used for the inner TSP, for any fixed $q$, $m \geq 1$, $\rho \geq 1$, and $d \geq 2$, the CITP scheme is an Efficient Polynomial Time Approximation Scheme (EPTAS) for the Euclidean CVRP with time complexity of $O(n^c + n^2 + mK^q 2^K)$, where $K = k(\varepsilon)$.*

**Remark**

The CITP scheme remains a PTAS for the Euclidean CVRP even for slightly relaxed restrictions on its parameters, e.g. for any fixed $d$, $\rho$, and $q = O((\log\log(n))^{1/d})$.

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# Proof sketch: the case of single depot

## The main idea

- for any $k$ obtain an upper bound $e(k)$ of the relative error in terms of $r_k$ and $\mathrm{TSP}^*(X \setminus X(k))$
- use the new upper bound for the optimum value of a TSP instance embedded into an Euclidean sphere of radius $r_k$
- show that for any given accuracy bound $\varepsilon$ there exists $k = k(\varepsilon, \rho, m)$ (does not depending on $n$) such that $e(k) \leq \varepsilon$.

# Upper bound for $e(k)$

**Lemma 6**

*For an arbitrary $k$,*

$$\begin{aligned}\mathrm{SDCVRP}^*(X, \{y\}) &\\ &\le \mathrm{SDCVRP}^*(X(k), \{y\}) + \mathrm{SDCVRP}^*(X \setminus X(k), \{y\}) \\ &\le \mathrm{SDCVRP}^*(X, \{y\}) + 4(k-1)r_k.\end{aligned}$$

Applying Lemmas 1,2, and 6, obtain

$$\begin{aligned} e(k) &= \frac{w(S_{\mathrm{CITP}}(X)) - \mathrm{SDCVRP}^*(X)}{\mathrm{SDCVRP}^*(X)} \\ &= \frac{\mathrm{SDCVRP}^*(X(k)) + w(S_{\mathrm{ITP}}(X \setminus X(k))) - \mathrm{SDCVRP}^*(X)}{\mathrm{SDCVRP}^*(X)} \\ &\le q(2k-1)\frac{r_k}{\sum_{i=1}^n r_i} + \frac{q\rho}{2\sum_{i=1}^n r_i}\mathrm{TSP}^*(X \setminus X(k)).\end{aligned}$$

# Upper bound for TSP*

- Condider the angular distance defined on the unit Euclidean sphere

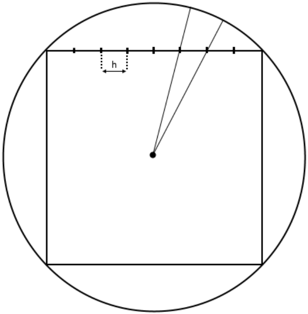
$$dist(x_1, x_2) = \arccos(x_1, x_2),\ (x_1, x_2 \in S^{d-1}).$$

- We need a finite $\varepsilon$-net $N$ for this metric, i.e. a finite subset $N \subset S^{d-1}$ such that, for any $x \in S^{d-1}$, there exists $\xi \in N$ such that $dist(\xi, x) \leq \varepsilon$.

# Upper bound for TSP*

**Lemma 7 (see e.g. Hubbert et al. (2015))**

*For an arbitrary* $h \in (0, h_0)$, $h_0 = \pi/(6\sqrt{d-1})$, *on the sphere* $S^{d-1}$ *there exists an* $h\sqrt{d-1}$*-net* $N = N(d,h)$ *such that* $|N| = Ch^{-(d-1)}$ *for some constant* $C = C(d)$.



# Upper bound for TSP*

## Lemma 8

*For an arbitrary $d > 1$ and a finite $X \subset B(y, R)$ the following bounds*

$$\mathrm{TSP}^*(X) \le \begin{cases} C_1 R^{1/d} (\sum_{i=1}^n r_i)^{(d-1)/d}, & \text{if } \sum_{i=1}^n r_i > RC \frac{(d-1)^{(d+1)/2}}{(\pi/6)^d}, \\ C_2 R, & \text{otherwise}, \end{cases}$$

*are valid, where*

$$C_1 = 2dC^{1/d}(d-1)^{(d-1)/2d} \text{ and } C_2 = 2dC(\pi/6)^{-(d-1)}(d-1)^{(d-1)/2}.$$

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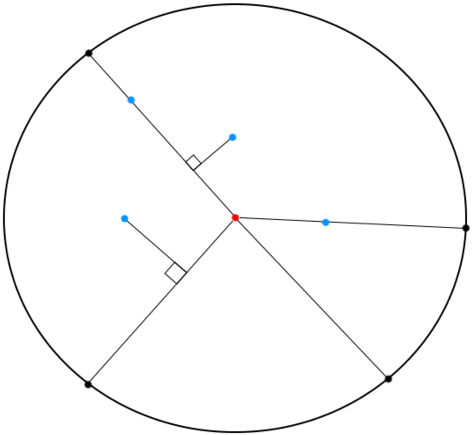

# Show that for any $\varepsilon$ there exist $k$: $e(k) \le \varepsilon$

By Lemma 8,

$$e(k) \le q(2k-1)\frac{r_k}{\sum_{i=1}^n r_i} + \frac{q\rho}{2}\max\left\{C_1\left(\frac{r_k}{\sum_{i=1}^n r_i}\right)^{1/d}, C_2\frac{r_k}{\sum_{i=1}^n r_i}\right\}$$
$$\le q(2k-1)\frac{r_k}{\sum_{i=1}^n r_i} + \frac{q\rho}{2}\max\{C_1, C_2\}\left(\frac{r_k}{\sum_{i=1}^n r_i}\right)^{1/d},$$

since $r_k \le \sum_{i=1}^n r_i$.
Denote $(r_k/\sum_{i=1}^n r_i)^{1/d}$ by $s_k$. Suppose that, for any $t \in \{1,\dots,k\}$,

$$q(2t-1)s_t^d + \frac{q\rho}{2}C^* s_t > \varepsilon$$

is valid, where $C^* = \max\{C_1, C_2\}$ depends on $d$ ultimately.

# Show that for any $\varepsilon$ there exist $k$: $e(k) \leq \varepsilon$

There exist two options.

- $s_t \geq \varepsilon/(q\rho C^*)$ for each $t$. Then,

$$1 \geq \sum_{t=1}^{k} s_t^d \geq k\left(\frac{\varepsilon}{q\rho C^*}\right)^d \text{and } k \leq \left(\frac{q\rho C^*}{\varepsilon}\right)^d.$$

- Let $t_0$ be the smallest number, for which

$$s_{t_0} < \varepsilon/(q\rho C^*).$$

  The same inequality is valid also for each $t_0 \leq t \leq k$, and, by assumption $s_t^d > \varepsilon/(2q(2t-1))$. Therefore

$$1 \geq \sum_{t=1}^{k} s_t^d \geq (t_0 - 1)\left(\frac{\varepsilon}{q\rho C^*}\right)^d + \frac{\varepsilon}{2q}\sum_{t=t_0}^{k}\frac{1}{2t-1}$$

$$\geq (t_0 - 1)\left(\frac{\varepsilon}{q\rho C^*}\right)^d + \frac{\varepsilon}{2q}\int_{t_0}^{k+1}\frac{dt}{2t-1}$$

$$= (t_0 - 1)\left(\frac{\varepsilon}{q\rho C^*}\right)^d + \frac{\varepsilon}{4q}(\ln(2k+1) - \ln(2t_0 - 1)).$$

# Show that for any $\varepsilon$ there exist $k$: $e(k) \leq \varepsilon$

Without loss of generality suppose that $\varepsilon \leq 4q\rho$. This equation together with the obvious (for $d > 1$) condition $C^* \geq 4$ implies

$$\left(\frac{\varepsilon}{q\rho C^*}\right)^d \leq \frac{\varepsilon}{4q},$$

and

$$\left(\frac{\varepsilon}{q\rho C^*}\right)^{-d} \geq t_0 - 1 + \ln(2k+1) - \ln(2t_0 - 1). \tag{1}$$

Minimizing the RHS of (1) subject to $t_0 \in \{1, \ldots, k\}$, we obtain

$$k \leq \frac{1}{2} e^{\left(\frac{q\rho C^*}{\varepsilon}\right)^d}.$$

And, finally, we come to the decision that the segment

$$\left[1, \frac{1}{2} e^{\left(\frac{q\rho C^*}{\varepsilon}\right)^d} + 1\right]$$

definitely contains the required number $k = k(\varepsilon)$ such that $e(k) \leq \varepsilon$.

# Summary

- Extending the approach to construction approximation algorithms for CVRP on the basis of well-known ITP heuristic, we propose new efficient polynomial time approximation schemas for the Euclidean CVRP for any dimension fixed $d > 1$, number $m$ of depots, approximation ratio $\rho$, and capacity $q$.
- The proposed scheme remains polynomial even for $q = O((\log\log n)^{1/d})$.
- Approximation algorithm used for solution of the inner TSP should not has a fixed approximation ratio. This can be useful for tackling *Big Data*.
- In future work, it would be interesting to extend the results obtained to some special cases of the Euclidean CVRP (time windows, uniform demand, pickup and delivery, etc.)

*Thank you for your attention!*