Selective bi-coordinate variations for network equilibrium problems with mixed demand

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For solving large dimensional problems, coordinate descent methods are attractive, and now they are intensively developed and applied. For conditional optimization problems, one can take marginal-based bi-coordinate descent methods originally proposed in [1], [2]. More flexible versions of bi-coordinate descent methods were presented in papers [3], [4]. They describe a method of bi-coordinate variations with special threshold control and tolerances for optimal resource allocation problems with simplex type constraints.

In the present paper, we apply this approach to network equilibrium problems with mixed demand, propose the corresponding modification of the method for this problem, perform numerical calculations, and show its efficiency in comparison with the conditional gradient method.

Network equilibrium problems with fixed demand

Let us consider a network composed of a set of nodes V and a set of directed links A.

W is a set of origin-destination (O/D) pairs (i, j), $i, j \in V$. For each O/D-pair with index $w \in W$ a set of paths P_w is known (each path is a simple chain of links starting at the origin and ending at the destination of O/D-pair) and a demand value $y_w > 0$ is given, which presents a flow outgoing from the origin and ingoing to the destination. Usually it corresponds to the transport or information flow. We denote $I = 1, \dots, n$, where $n = \sum_{w \in W} |P_w|$.

The problem is to distribute the required flows for all O/D pairs among the set of paths by using a certain (equilibrium) criterion.

We denote by x_p a value of flow passing along path p.

Then the feasible set for the path flow vectors is defined as follows:

$$X = \left\{ x \;\middle|\; \sum_{p \in P_w} x_p = y_w, x_p \geq 0,\ p \in P_w, w \in W \right\}. \tag{1}$$

Paths and links are connected with the help of the incidence matrix with elements

$$\alpha_{pa} = \begin{cases} 1, & \text{if link } a \text{ belongs to path } p, \\ 0, & \text{otherwise.} \end{cases}$$

The flow value for each link $a \in A$ is defined as the sum of the corresponding path flows:

$$f_a = \sum_{w \in W} \sum_{p \in P_w} \alpha_{pa} x_p \tag{2}$$

For each link a a continuous cost function c_a is given; it can depend on all link flows in the general case. The summary cost function for path p has the form:

$$g_p(x) = \sum_{a \in A} \alpha_{pa} c_a(f). \tag{3}$$

The equilibrium condition for this network consists in finding an element $x^* \in X$ such that

$$\forall w \in W, q \in P_w, x_q^* > 0 \Longrightarrow g_q(x^*) = \min_{p \in P_w} g_p(x^*), \qquad (4)$$

therefore, only paths with minimal costs have nonzero flows. It is the Nash equilibrium based on the user-optimization principle: a network equilibrium is established when no OD pair can decrease its cost by making a unilateral decision to change its path flows.

It is well known that this problem is equivalent to the variational inequality: Find a point $x^* \in X$ such that

$$\langle G(x^*), x - x^* \rangle \geq 0 \qquad \forall x \in X, \qquad (5)$$

where the vector G is composed of components $g_p, p \in P_w, w \in W$, respectively.

Network equilibrium problems with elastic demand

In contrast to the network equilibrium problem with fixed demand, in the problem with elastic demand, the demand values are variables. Then the feasible set takes the form

$$K = \left\{ (x, y) \,\middle|\, \sum_{p \in P_w} x_p = y_w, x_p \geq 0,\ p \in P_w, w \in W \right\} .$$

Here y is a vector with variable components $y_w,\ w \in W$.

In this problem, for each O/D pair $w \in W$ a continuous so-called disutility function h_w with respect to demand is given. In the general case, it can depend on the whole demand vector y.

Therefore, the network equilibrium problem with *elastic* demand is to find an element $(x^*, y^*) \in K$ such that

$$\langle G(x^*), x - x^* \rangle - \langle H(y^*), y - y^* \rangle \geq 0 \quad \forall (x, y) \in K . \tag{6}$$

Here the vector H is composed of the components h_w, respectively.

It is well known that equilibrium conditions for this problem have the following form: a vector $(x^*, y^*) \in K$ is a solution to problem (6), if for all $p \in P_w, w \in W$ it holds that

$$g_p(x^*) \begin{cases} = h_w(y^*) & \text{if } x_p^* > 0 \,, \\ \geq h_w(y^*) & \text{if } x_p^* = 0 \,. \end{cases}$$

In other words, at each equilibrium point the path costs (for nonzero flows) are equal to the disutility function value for the associated O/D pair.

Network equilibrium problems with mixed demand

At last, we consider the network equilibrium problem with *mixed* demand originally proposed in [5]: Find a vector $(x^*, y^*) \in U$ such that

$$\langle G(x^*), x - x^* \rangle - \langle H(y^*), y - y^* \rangle \geq 0 \quad \forall (x, y) \in U\,, \tag{7}$$

where

$$U = \left\{ (x, y) \,\middle|\, \sum_{p \in P_w} x_p = y_w + y_w^{const}, x_p \geq 0,\ y_w \geq 0,\ p \in P_w, w \in W \right\}.$$

In this problem, for each O/D-pair the variable y_w and fixed y_w^{const} demands are simultaneously presented ($y_w^{const} \geq 0, \forall w \in W$).

It is known that equilibrium conditions for problem (7) are following: a vector $(x^*, y^*) \in U$ is a solution to problem (7) if and only if for all $p \in P_w, w \in W$ it satisfies conditions

(a) if $x_p^* > 0$, then $g_p(x^*) = \min_{q \in P_w} g_q(x^*)$,

(b) if $x_p^* > 0$ and $y_w^* > 0$, then $g_p(x^*) = h_w(y^*)$,

(c) if $x_p^* = 0$ or $y_w^* = 0$, then $g_p(x^*) \geq h_w(y^*)$.

In what follows, we assume that each link cost function c_a depends on f_a only, $\forall a \in A$, each disutility function h_w depends on y_w only, $\forall w \in W$. Then the mappings G and H are potential, and there exist functions

$$\mu_a(f_a) = \int\limits_0^{f_a} c_a(t)dt \quad \forall a \in A\,, \quad \sigma_w(y_w) = \int\limits_0^{y_w} h_w(t)dt \quad \forall w \in W\,.$$

In this case, variational inequality (7) presents the optimality condition for the following optimization problem:

$$\min_{u \in U} \longrightarrow \psi(u), \tag{8}$$

where $u = (x, y)$, $\psi(x, y) = \left\{ \sum\limits_{a \in A} \mu_a(f_a) - \sum\limits_{w \in W} \sigma_w(y_w) \right\}$, $f_a, \forall a \in A$ are defined in (2). Therefore, each solution to problem (8) solves problem (7). The reverse assertion is true, if, for example, the mappings G and $-H$ are monotone.

The method of bi-coordinate variations for network equilibrium problems with mixed demand

In papers [2], [3] a method of bi-coordinate variations with a special threshold control and tolerances has been proposed for solving resource allocation problems with simplex type constraints. Let us explain the idea of this method applied to the network equilibrium problem (8).

At the current iteration we have a point $(x, y) \in U$, which is not a solution to problem (7). At first we define, which coordinates values can decrease. The value is suitable to decreasing, if it exceeds a certain threshold $\varepsilon > 0$. We remind that we have two groups of variables: the path flows x and the variable demands y. We denote the sets of "active" indices by $I_\varepsilon(x) = \{i = 1, 2, \dots, n \mid x_i \geq \varepsilon\}$, $J_\varepsilon(y) = \{j \in W \mid y_j \geq \varepsilon\}$, respectively.

Further we note that at any optimal point of problem (7) for any $w \in W$ in view of equilibrium condition (a) the values of corresponding components of vector G with nonzero path flows are equal:

$$\forall w \in W,\ i, j \in P_w,\ x_i^* > 0,\ x_j^* > 0 \Longrightarrow g_i(x^*) = g_j(x^*).$$

At the same time, due to equilibrium condition (b) for each nonzero variable demand the cost values for paths $p \in P_w$ with nonzero flows are equal to the value of disutility function for this O/D-pair $w \in W$:

$$\text{if } x_p^* > 0 \text{ and } d_w^* > 0, \text{ then } g_p(x^*) = \lambda_w(d^*).$$

Therefore, it is reasonable to "adjust" deviating values of path cost functions and disutility functions, i.e. to decrease the great ones and increase the small ones. Hence we can choose a pair of indices (i, j) satisfying any of the following three rules.

$$g_i(x) - h_j(y_j) \le -\delta_k,\ i \in P_j,\ j \in W,$$

$$g_i(x) - h_j(y_j) \ge \delta_k,\ i \in I_{\varepsilon_k}(x),\ j \in J_{\varepsilon_k}(y),\ i \in P_j,$$

$$g_i(x) - g_j(x) \ge \delta_k,\ i \in I_{\varepsilon_k}(x),\ i, j \in P_w.$$

We note that in the bi-coordinate method it is sufficient to choose *one* pair of indices, but we can choose more pairs, for example, one pair for each $w \in W$.

Hence, we formulated the principle of constructing a descent direction. We will use the inexact Armijo type line-search as the rule of step choice.

The proposed method has the two-level scheme. On the inner level, we minimize the objective function with fixed values of parameters ε and δ, and on the upper level we decrease values of these parameters.

Method 1

Step 0. Choose a stop criterion and an accuracy value, an initial point $u^0 \in U$, sequences $\{\varepsilon_k\} \searrow 0$, $\{\delta_k\} \searrow 0$, $k = 1, 2, \ldots$, parameters $\beta \in (0,1)$, $\theta \in (0,1)$. Set $k = 1$.

Step 1. Set $l = 0$, $v^l = u^{k-1}$.

Step 2. If for the point v^l the stop criterion is fulfilled, then we obtained the given accuracy, the iterative process stops. Otherwise set $(x^l, y^l) = v^l$.

Step 3. Choose at least one (or more) pair of indices, *no more than one pair for each* $w \in W$ such that either

$$(i, n+j) : g_i(x^l) - h_j(y_j^l) \leq -\delta_k,\ i \in P_j,\ j \in W, \tag{9}$$

(denote the sets of chosen indices by I_l^+ and J_l^+, respectively) or

$$(i, n+j) : g_i(x^l) - h_j(y_j^l) \geq \delta_k,\ i \in I_{\varepsilon_k}(x^l),\ j \in J_{\varepsilon_k}(y^l),\ i \in P_j, \quad (10)$$

(denote the sets of chosen indices by I_l^- and J_l^-, respectively) or

$$(i, j) : g_i(x^l) - g_j(x^l) \geq \delta_k,\ i \in I_{\varepsilon_k}(x^l),\ i, j \in P_w, \quad (11)$$

(denote the sets of chosen indices by I_l and J_l, respectively). If no such pairs exist, set $u^k = (x^l, y^l)$, $k = k + 1$ and go to Step 1.

Step 4. Construct the descent direction d^l with components

$$d_s^l = \begin{cases} 1, & \text{if } s \in I_l^+ \cup J_l^+ \cup J_l, \\ -1, & \text{if } s \in I_l^- \cup J_l^- \cup I_l, \\ 0, & \text{in all other cases.} \end{cases}$$

Step 5. Find the smallest nonnegative integer b such that the condition is fulfilled (the Armijo type inexact line-search)

$$\psi(v^l + \theta^b \varepsilon_k d^l) - \psi(v^l) \leq \beta \theta^b \varepsilon_k \langle \psi'(v^l), d^l \rangle .$$

Set $\lambda_l = \varepsilon_k \theta^b$, $v^{l+1} = v^l + \lambda_l d^l$, $l = l + 1$ and go to Step 2.

We note that for the network equilibrium problem with *elastic* demand in Step 3 only conditions (9) and (10) should be applied. In the network equilibrium problem with *fixed* demand, there are variables $x \in X$ only, and Step 3 is reduced to condition (11).

Now we establish the convergence properties of the proposed method.

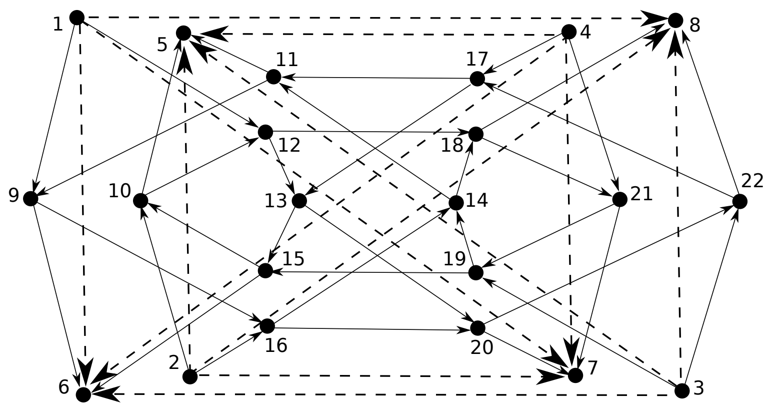
Proposition 1 *The line-search procedure at Step 5 of Method 1 is finite.*

Proposition 2 *Let the function ψ be coercive on U. Then the inner iterative process (Steps 2–5) of Method 1 is finite.*

Theorem 1 *Let the function ψ be coercive on U. Then the sequence $\{u^k\}$ generated by Method 1 has limit points, all of them are solutions to VI (7). Provided that the function ψ is convex, they are also solutions to optimization problem (8).*

Numerical experiments

We compare the method of bi-coordinate variations (BCM) and the ordinary conditional gradient method (CGM) and present the results of preliminary numerical experiments for the network equilibrium problem with fixed demand. In Example 1 we consider the following network.



Example 1, 22 nodes, 12 O/D pairs

We set link cost functions $c_a(f_a) = 1 + 0.5f_a$ for all $a \in A$, the fixed demand $d_w = 5$ for all $w \in W$. We use the stop criterion of the conditional gradient method:

$$\langle \psi'(x), x - \bar{x} \rangle < \Delta, \tag{12}$$

where $\langle \psi'(x), \bar{x} \rangle = \min_{z \in X} \langle \psi'(x), z \rangle$.

Δ	*BCM*	*CGM*
0.1	116 it., 16 ms	885 it., 93 ms
0.01	137 it., 31 ms	11228 it., 141 ms
0.001	147 it., 47 ms	114961 it., 1076 ms

Table 1. Example 1, numbers of iterations and calculation time

In the following examples we used random data. We generated N nodes and K O/D pairs.

The calculation results for several problems with a given error $\Delta = 0.1$ are presented in Table 2. Beside numbers of nodes and O/D-pairs we adduce approximate dimensions of solutions, i.e., the value $\sum_{w \in W} |\bar{P_w}|$.

N	K	$\sum_{w \in W} \lvert\bar{P_w}\rvert$	*BCM*	*CGM*
50	20	200	2715 it., 0.09 s	6302 it., 3.04 s
80	26	600	6143 it., 0.218 s	8254 it., 13.1 s
100	30	850	11029 it., 0.39 s	9516 it., 22.8 s
100	40	1000	13608 it., 0.515 s	11953 it., 32.34 s
200	50	2700	37719 it., 3.5 s	9896 it., 130.2 s

Table 2. Examples with different dimensions.
Numbers of iterations and calculation time

The dimension of the network equilibrium problem (the number of feasible paths for all O/D pairs) is usually great, but the solution often contains many zero values. Therefore in practice we use the following "trick". Instead of sets $P_w,\ w \in W$ we use their approximations $\bar{P}_w$. On the initial stage, we choose some nonempty subset $\bar{P}_w \subset P_w$ for all $w \in W$ and at each iteration they can increase including new shortest paths. At some moment, the subsets $\bar{P}_w$ stop to increase.

Presented results of preliminary numerical calculations show that the proposed method of bi-coordinate variations for network equilibrium problems has essential advantages in comparison with the conditional gradient method and is promising for further investigations.

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