On the number of maximal independent sets in complete q-ary trees

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- *An independent set* in a graph is a subset of its pairwise non-adjacent vertices
- *A maximal independent set* is an independent set which is maximal by inclusion
- Denote by $\mathrm{i}(G)$ (respectively, $\mathrm{mi}(G)$) the number of all (respectively, maximal) independent sets in a graph G.

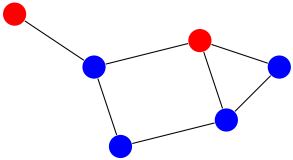

Fig. 1. The independent and the maximal independent sets.

- Denote by $T_{q,n}$ the complete q-ary tree of height n.

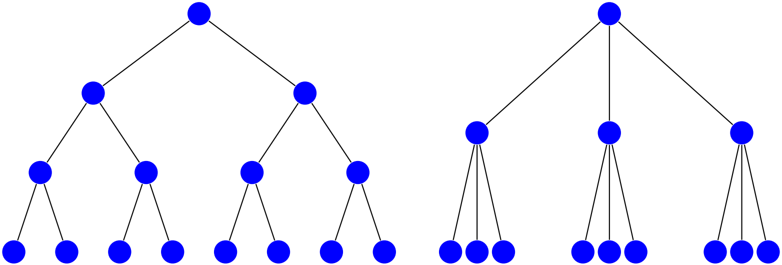

Fig. 2. The trees $T_{2,3}$ and $T_{3,2}$

In 1983 P. Kirschenhofer, H. Prodinger, and R. Tichy proved the following statement:

Lemma 1.

There exist constants $\beta_q, \alpha_q, \alpha_{q,1}, \alpha_{q,2}$ ($\alpha_{q,1} \neq \alpha_{q,2}$), such that for any $q \in \{2,3,4\}$ as $n \longrightarrow \infty$ the asymptotic equality $i(T_{q,n}) \sim \alpha_q \cdot (\beta_q)^{q^n}$ holds and for any $q \geq 5$ as $k \longrightarrow \infty$ the asymptotic equalities $i(T_{q,2k}) \sim \alpha_{q,1} \cdot (\beta_q)^{q^{2k}}$ and $i(T_{q,2k+1}) \sim \alpha_{q,2} \cdot (\beta_q)^{q^{2k+1}}$ hold.

Main result

We investigate the asymptotic behavior of the parameter $\mathrm{mi}(T_{q,n})$. Our main results are the following theorems:

Theorem 1.

There exist constants a_2 и b_2 such that the asymptotic equality $mi(T_{2,n}) \sim a_2 \cdot (b_2)^{2^n}$ holds as $n \longrightarrow \infty$.

Theorem 2.

For any sufficiently large q, there exist pairwise distinct constants $a_q^{(1)}, a_q^{(2)}, a_q^{(3)}$ and a constant b_q such that as $k \longrightarrow \infty$ the asymptotic equalities $mi(T_{q,3k}) \sim a_q^{(1)} \cdot (b_q)^{q^{3k}}, mi(T_{q,3k+1}) \sim a_q^{(2)} \cdot (b_q)^{q^{3k+1}}, mi(T_{q,3k+2}) \sim a_q^{(3)} \cdot (b_q)^{q^{3k+2}}$ hold.

We prove that the following relation holds:

$$\mathrm{mi}(T_{q,n}) = \mathrm{mi}(T_{q,n-2})^{q^2} + \mathrm{mi}(T_{q,n-1})^{q} -$$

$$-(\mathrm{mi}(T_{q,n-1}) - \mathrm{mi}(T_{q,n-3})^{q^2})^{q}$$

Solving this relation, we get the equality

$$\mathrm{mi}(T_{q,n}) = a_{q,n} \cdot (b_q)^{q^n}$$

Here $1 \leq a_{q,n} \leq 2$ and the constant b_q depends only on the parameter q.

Numerical computations

- Our purpose is to determine the value q_0, such that for every $q \geq q_0$ Theorem 2 holds.
- We know that for every $q \geq 2$ the equality $\mathrm{mi}(T_{q,n}) = a_{q,n} \cdot (b_q)^{q^n}$ holds, where $1 \leq a_{q,n} \leq 2$ and the number b_q is a constant.
- For every q we calculate the first elements of the sequence $\{a_{q,n}\}$. If the subsequences $\{a_{q,3k}\}, \{a_{q,3k+1}\}, \{a_{q,3k+2}\}$ have a limit, then Theorem 2 holds.

- The sequence $\{a_{2,n}\}$ has a limit $l = 1.298..$
  This is exactly what Theorem 1 claims.
- It seems that if $3 \le q \le 10$, the sequence $\{a_{q,n}\}$ does not have a limit, and it does not have three partial limits.

q	n								
	10	20	30	40	50	600	700	800	900
2	1.178	1.284	1.303	1.300	1.298	1.298	1.298	1.298	1.298
3	1.045	1.194	1.445	1.510	1.290	1.106	1.329	1.411	1.118
4	1.008	1.226	1.805	1.374	1.028	1.037	1.038	1.040	1.042
5	1.004	1.466	1.566	1.021	1.001	1.765	1.000	1.790	1.019
6	1.008	1.790	1.108	1.000	1.309	1.039	1.000	1.000	1.001
7	1.036	1.691	1.000	1.193	1.410	1.213	1.000	1.694	1.000
8	1.222	1.113	1.000	1.896	1.000	1.000	1.000	1.000	1.000
9	1.593	1.018	1.333	1.034	1.025	1.000	1.018	1.996	1.025
10	1.818	1.000	1.995	1.000	1.053	1.000	1.087	1.000	1.038

Table 1: Some elements of the sequence $\{a(q,n)\}$

Numerical computations. Case $q > 10$

- If $q > 10$, then the sequence $\{a_{q,n}\}$ has three partial limits.
- The limit of the subsequence $\{a_{q,3k+1}\}$ is close to 2.
- It tends to 2 as q tends to infinity.

q	k								
	1	2	3	4	5	60	70	80	90
11	1.942	1.922	1.913	1.909	1.906	1.904	1.904	1.904	1.904
12	1.966	1.958	1.956	1.955	1.955	1.955	1.955	1.955	1.955
13	1.979	1.976	1.976	1.976	1.976	1.976	1.976	1.976	1.976

Table 2: Some elements of the sequence $\{a(q, 3k+1)\}$

Numerical computations. Case $q > 10$

- The limits of the subsequences $\{a_{q,3k+2}\}$ and $\{a_{q,3k}\}$ are close to 1.
- They tend to 1 as q tends to infitity.

q	k							
	1	2	3	4	5	60	70	80
11	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.0008
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
13	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 3: Some elements of the sequence $\{a_{q,3k+2}\}$

q	k							
	1	2	3	4	5	60	70	80
11	1.005	1.007	1.008	1.008	1.008	1.009	1.009	1.009
12	1.003	1.004	1.004	1.004	1.004	1.004	1.004	1.004
13	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001

Table 4: Some elements of the sequence $\{a_{q,3k}\}$

We have 3 cases:

- If $q = 2$, then the sequence $\{a_{2,n}\}$ has a limit and Theorem 1 holds.
- If $3 \leq q \leq 10$, then the sequence $\{a_{q,n}\}$ does not have a limit and it does not have three partial limits.
- If $q > 10$, then the sequence $\{a_{q,n}\}$ has three convergent subsequences and Theorem 2 holds.

Overall conclusion

We investigate the asymptotic behavior of the number of the maximal independent sets in complete q-ary trees as the height of the trees tends to infinity.

For $q = 2$ and for any sufficiently large q we obtain the following asymptotic equalities:

- $\mathrm{mi}(T_{2,n}) \sim a_2 \cdot (b_2)^{2^n}$,
- $\mathrm{mi}(T_{q,3k}) \sim a_q^{(1)} \cdot (b_q)^{q^{3k}}$, $\mathrm{mi}(T_{q,3k+1}) \sim a_q^{(2)} \cdot (b_q)^{q^{3k+1}}$, $\mathrm{mi}(T_{q,3k+2}) \sim a_q^{(3)} \cdot (b_q)^{q^{3k+2}}$ if $q > q_0$.

Numerical computations show that $q_0 = 10$.

We also prove a weaker statement for all $q \geq 2$:

- $\mathrm{mi}(T_{q,n}) = a_{q,n} \cdot (b_q)^{q^n}$, $1 \leq a_{q,n} \leq 2$.

Thank you for your attention!