Application of Market Models to Network Equilibrium Problems

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1 Introduction

Investigation of complex systems with active elements having their own interests and sets of actions is usually based on a suitable equilibrium concept. Such a concept should equilibrate different interests and opportunities of the elements (agents, participants) and provide ways of its proper implementation within some accepted basic (information) behavior framework of the system under investigation.

For instance, the classical perfectly (Walrasian) and imperfectly (Cournot - Bertrand) competitive models (see e.g. [Nikaido (1968)], [Okuguchi and Szidarovszky (1990)] and references therein), reflect different equilibration mechanisms and information frameworks. We recall that actions of any separate agent within a perfect competition model can not impact the state of the whole system, hence any agent may utilize some integral system parameters (say, prices), rather than the information about the behavior of other agents. On the contrary, actions of any separate agent in an imperfectly competitive model can change the state of the whole system. Therefore, the model is formulated as a (non-cooperative) game problem and is usually based on the well-known Nash equilibrium concept [Nash (1951)]. Nevertheless, real systems (markets) may give wide variety of these features and different information frameworks. Hence, flexible equilibrium models could also be very useful for derivation of efficient decisions in complex systems.

In this talk, we consider a general two-side market model with divisible commodities and price functions of participants. It is based on the auction market models proposed in [Konnov (2006)], [Konnov (2007a)], where the equivalence result with a variational inequality problem was established. Afterwards, some extensions to the multi-commodity case and applications to resource allocation in telecommunication networks were suggested in [Konnov (2007b)], [Konnov (2013a)]. The alternative equilibrium concept related to this model was proposed in [Konnov (2015a)], where it was also shown that the same equilibrium state can be attained within different mechanisms and information exchange schemes, including the completely decentralized competitive mechanism.

2 A general multi-commodity market equilibrium model

We start our considerations from a general market model with n divisible commodities, which somewhat extends those in [Konnov (2007b)], [Konnov (2015a)]; see also [Konnov (2016a)] for the vector model. For each commodity s, each trader i chooses some offer value x_{is} in his/her capacity segment $[\alpha'_{is}, \alpha''_{is}]$ and has a price function g_{is}. Similarly, each buyer j chooses some bid value y_{js} in his/her capacity segment $[\beta'_{js}, \beta''_{js}]$ and has a price function h_{js}. We denote by I_s and J_s the finite index sets of traders and buyers attributed to commodity s and set $N = \{1, \dots, n\}$. Clearly, each trader/buyer can be attributed to many commodities. We suppose that the prices may in principle depend on all the bid/offer volumes of all the commodities. That is, if we set $x_{(s)} = (x_{is})_{i \in I_s}$, $x = (x_{(s)})_{s \in N}$, $y_{(s)} = (y_{js})_{j \in J_s}$, $y = (y_{(s)})_{s \in N}$, and $w = (x, y)$, then $g_{is} = g_{is}(w)$ and $h_{js} = h_{js}(w)$. Let b_s denote the value of the external excess demand for commodity s, then $b = (b_s)_{s \in N}$. If it equals zero, the market is closed.

Any market solution must satisfy the balance equation, hence we obtain the feasible set of offer/bid values

$$W = \prod_{s \in N} W_s, \text{ where}$$

$$W_s = \left\{ w_{(s)} = (x_{(s)}, y_{(s)}) \;\middle|\; \begin{array}{l} \sum\limits_{i \in I_s} x_{is} - \sum\limits_{j \in J_s} y_{js} = b_s; \\ x_{is} \in [\alpha'_{is}, \alpha''_{is}], i \in I_s, \\ y_{js} \in [\beta'_{js}, \beta''_{js}], j \in J_s \end{array} \right\};$$

for $s \in N$.

A vector $\bar{w} = (\bar{x}, \bar{y}) \in W$ is said to be a *market equilibrium point* if there exists a price vector $\bar{p} = (\bar{p}_s)_{s \in N}$ such that

$$g_{is}(\bar{w}) \begin{cases} \geq \bar{p}_s & \text{if} \quad \bar{x}_{is} = \alpha'_{is}, \\ = \bar{p}_s & \text{if} \quad \bar{x}_{is} \in (\alpha'_{is}, \alpha''_{is}), \\ \leq \bar{p}_s & \text{if} \quad \bar{x}_{is} = \alpha''_{is}, \end{cases} \quad \text{for } i \in I_s; \tag{1}$$

and

$$h_{js}(\bar{w}) \begin{cases} \leq \bar{p}_s & \text{if} \quad \bar{y}_{js} = \beta'_{js}, \\ = \bar{p}_s & \text{if} \quad \bar{y}_{js} \in (\beta'_{js}, \beta''_{js}), \\ \geq \bar{p}_s & \text{if} \quad \bar{y}_{js} = \beta''_{js}, \end{cases} \quad \text{for } j \in J_s; \tag{2}$$

for $s \in N$.

We now give the basic relation between the market equilibrium problem (1)–(2) and a variational inequality (VI, for short).

Proposition 2.1 *(a) If $(\bar{w}, \bar{p})$ satisfies (1)–(2) for $s \in N$ and $\bar{w} \in W$, then $\bar{w}$ solves VI: Find $\bar{w} \in W$ such that*

$$\sum_{s \in N} \left[\sum_{i \in I_s} g_{is}(\bar{w})(x_{is} - \bar{x}_{is}) - \sum_{j \in J_s} h_{js}(\bar{w})(y_{js} - \bar{y}_{js}) \right] \geq 0 \quad \forall w \in W. \tag{3}$$

(b) If a vector $\bar{w}$ solves VI (3), then there exists $\bar{p} \in \mathbb{R}^n$ such that $(\bar{w}, \bar{p})$ satisfies (1)–(2) for $s \in N$.

The presence of the price functions is invoked by complexity of the whole system, i.e., the price functions may contain participants' intentions or reflect interdependence (mutual influence) of the elements, which need not be known to the participants.

It follows from Proposition 2.1 that we can establish existence results for equilibrium problems of form (1)–(2) by using suitable results from the theory of VIs. For instance, if the feasible set W is bounded and the cost mapping of VI (3) is continuous, then equilibrium problem (1)–(2) has a solution. In the unbounded case, we need certain coercivity assumptions. We follow the approach from [Konnov (2015a)], [Konnov (2016a)] and consider for simplicity the case where all the lower bounds α'_{is} and β'_{js} of capacities are greater than $-\infty$, whereas some upper bounds α''_{is} and β''_{js} can be absent. Then, for each commodity $s \in N$, we define the index sets

$$I^u_s = \{i \in I_s \mid \alpha''_{is} = +\infty\} \text{ and } J^u_s = \{j \in J_s \mid \beta''_{js} = +\infty\},$$

and take the following coercivity condition.

(C) *There exists a number* $r > 0$ *such that for any point* $w = (x, y) \in W$ *and for each* $s \in N$ *it holds that*

$$\forall l \in J_s^u, y_{ls} > \max\{r, \beta'_{js}\} \Longrightarrow \exists k \in I_s^u \text{ such that}$$
$$x_{ks} > \alpha'_{ks} \text{ and } g_{ks}(w) \geq h_{ls}(w).$$

This condition seems rather natural: at any feasible point w and for each fixed commodity s, any large demand value of buyer l invokes existence of a trader k whose price is not less than the price of buyer l.

Proposition 2.2 *Suppose that the set* W *is nonempty, all the functions* g_{is} *and* g_{js} *are continuous for all* $i \in I$, $j \in J$, *and* $s \in N$. *If condition* **(C)** *is fulfilled, then VI (3) has a solution.*

3 Partial linearization methods

Due to Proposition 2.1, we can take various iterative solution methods for optimization and variational inequality problems (see e.g. [Konnov (2007a)], [Konnov (2015a)], [Konnov (2013b)]) for finding solutions of the market equilibrium problems of form (1)–(2). We now intend to consider a special integrable class of these problems that admits efficient iterative solution methods.

Let us first take a problem of minimization of the sum of two functions $\mu(w) + \eta(w)$ over a feasible set $W \subseteq \mathbb{R}^m$, or briefly,

$$\min_{w \in W} \to \{\mu(w) + \eta(w)\}. \tag{4}$$

We suppose that the set $W \subset \mathbb{R}^m$ is non-empty, convex, and compact, both the functions are convex and $\mu : \mathbb{R}^m \to \mathbb{R}$ is smooth. Moreover, the minimization of the function η over the set W is not supposed to be difficult.

In this case one can apply the partial linearization (PL for short) method, which was first proposed in [Mine and Fukushima (1981)].

Method (PL).

Choose a point $w^0 \in W$ and set $k = 0$. At the k-th iteration, $k = 0, 1, \dots$, we have a point $w^k \in W$. Find some solution v^k of the problem

$$\min_{v \in W} \to \left\{ \langle \mu'(w^k), v \rangle + \eta(v) \right\} \tag{5}$$

and define $p^k = v^k - w^k$ as a descent direction at w^k. Take a suitable stepsize $\lambda_k \in (0, 1]$, set $w^{k+1} = w^k + \lambda_k p^k$ and $k = k+1$.

The stepsize can be found either with some one-dimensional minimization procedure as in [Mine and Fukushima (1981)] or with an inexact Armijo type linesearch; see also [Patriksson (1998)], [Bredies et al. (2009)] for substantiation and further development.

The usefulness of this approach becomes clear if problem (4) is (partially) decomposable, which is typical for very large dimensional problems. For instance, let

$$\eta(w) = \sum_{s \in N} \eta_s(w_{(s)}) \text{ and } W = \prod_{s \in N} W_s,$$

where $w_{(s)} \in W_s \subset \mathbb{R}^{m_s}$, so that $m = \sum\limits_{s \in N} m_s$, i.e., there is some concordant partition of the initial space $\mathbb{R}^m$. Then we have the problem

$$\min_{w \in \prod\limits_{s \in N} W_s} \to \left\{ \mu(w) + \sum_{s \in N} \eta_s(w_{(s)}) \right\}, \tag{6}$$

and (5) becomes equivalent to several independent problems of the form

$$\min_{v_{(s)} \in W_s} \to \left\{ \left\langle v_{(s)}, \frac{\partial \mu(w^k)}{\partial w_{(s)}} \right\rangle + \eta_s(v_{(s)}) \right\}; \tag{7}$$

for $s \in N$. The above descent method admits various componentwise iterative schemes; see e.g. [Patriksson (1999)].

Our market equilibrium problem from the previous section written as VI (3) is reduced to problem (4) in the case where the price functions are integrable, i.e.

$$g_{is}(w) = \frac{\partial \mu(w)}{\partial x_i}, \ i \in I_s,$$

and

$$h_{js}(w) = -\frac{\partial \eta_s(w_{(s)})}{\partial y_j}, \ j \in J_s; \ s \in N.$$

This is the case if these functions are separable, i.e. $g_{is}(w) = g_{is}(x_{is})$ for each $i \in I_s$ and $h_{js}(w) = h_{js}(y_{js})$ for each $j \in J_s$, for all $s \in N$. More precisely, VI (3) becomes the necessary optimality condition for (4). The reverse assertion is true if the functions μ and η are convex.

We now describe an adaptive cyclic component-wise PL method for problem (6), which is some implementation of that from [Konnov (2016b)]. For each point $w \in W$ and each $s \in N$, we define by $V_s(w)$ the solution set of the optimization problem:

$$\min_{v_{(s)} \in W_s} \to \left\{ \left\langle v_{(s)}, \frac{\partial \mu(w)}{\partial w_{(s)}} \right\rangle + \eta_s(v_{(s)}) \right\}.$$

As above we suppose that the functions μ and η are convex, μ is smooth, the set $W \subset \mathbb{R}^m$ is non-empty, convex, and compact.

Under these assumptions $V_s(w)$ is also non-empty, convex, and compact. We define the gap function

$$\varphi_s(w) = \max_{v_{(s)} \in W_s} \left\{ \left\langle w_{(s)} - v_{(s)}, \frac{\partial \mu(w)}{\partial w_{(s)}} \right\rangle + \eta_s(w_{(s)}) - \eta_s(v_{(s)}) \right\}$$

for each $s \in N$. For brevity, set $f(w) = \mu(w) + \eta(w)$ and denote by $\mathbb{Z}_+$ the set of non-negative integers. The optimal value of the function f in (4) (or (6)) will be denoted by f^*. The adaptive cyclic PL method is described as follows.

Method (CPL).

Initialization: Choose a point $z^0 \in W$, numbers $\beta \in (0,1)$, $\theta \in (0,1)$, and a sequence $\{\delta_l\} \searrow 0$. Set $l = 1$.

Step 0: Set $k = 0$, $d = 0$, $s = 1$, $w^0 = z^{l-1}$.

Step 1: Solve problem (7), find $v_{(s)} \in V_s(w^k)$ and calculate $\varphi_s(w^k)$. If $\varphi_s(w^k) \geq \delta_l$, take

$$p^k_{(i)} = \begin{cases} v_{(s)} - w^k_{(s)} & \text{if } i = s, \\ \mathbf{0} & \text{if } i \neq s; \end{cases}$$

and go to Step 4.

Step 2: Set $d = d + 1$. If $d = n$, set $z^l = w^k$, $l = l + 1$ and go to Step 0. *(Restart)*

Step 3: If $s = n$, set $s = 1$, otherwise $s = s + 1$. Afterwards go to Step 1.

Step 4: Determine j as the smallest number in $\mathbb{Z}_+$ such that

$$f(w^k + \theta^j p^k) \leq f(w^k) - \beta\theta^j \varphi_s(w^k),$$

set $\lambda_k = \theta^j$, $w^{k+1} = w^k + \lambda_k p^k$, $k = k + 1$, $d = 0$. If $s = n$, set $s = 1$, otherwise $s = s + 1$. Afterwards go to Step 1.

Thus, the method has two levels. Each its outer iteration l contains some number of inner iterations in k with the sequential verification of descent value for each component with the fixed tolerance δ_l. Completing each stage, which is marked as restart, leads to decreasing the tolerance value.

Proposition 3.1 *Suppose in addition that the gradient map the function μ is uniformly continuous on W. Then:*
(i) the number of inner iterations at each outer iteration l is finite;
(ii) the sequence $\{z^l\}$ generated by Method (CPL) has limit points, all these limit points are solutions of problem (6), besides,

$$\lim_{l\to\infty} f(z^l) = f^*.$$

The line-search procedure in the method admits various modifications. For instance, we can take the exact one-dimensional minimization rule instead of the current Armijo rule. If the gradient of the function μ is Lipschitz continuous, we can take fixed stepsize values and remove the line-search procedure at all; see [Konnov (2016b)].

Remark 3.1 *Due to the presence of the control sequence* $\{\delta_l\}$*, CPL differs essentially from the usual decomposition methods; see e.g. [Patriksson (1999)], [Migdalas (2004)]. At the same time, this technique is rather usual for non-differentiable optimization methods; see e.g. [Balinski and Wolfe (1975)]. It was also applied in iterative methods for linear inequalities [McCormick (1977)] and for decomposable variational inequalities [Konnov (2002)].*

4 A generalization of network equilibrium problems with elastic demands

We now consider network flow equilibrium problems with elastic (inverse) demands, which find various applications; see [Dafermos (1982)], [Nagurney (1999), Chapter IV] and references therein.

Let us be given a graph with finite sets of nodes $\mathcal{M}$ and oriented arcs $\mathcal{A}$ which join the nodes so that any arc $a = (i, j)$ has origin i and destination j. Next, among all the pairs of nodes of the graph we extract a subset of origin-destination (O/D) pairs $\mathcal{N}$ of the form $s = (i \to j)$. Each pair $s \in \mathcal{N}$ is associated with the set of paths $\mathcal{P}_s$ which connect the origin and destination for this pair. Also, denote by x_p the path flow for the path p. Given a flow vector $x = (x_p)_{p \in \mathcal{P}_s,\, s \in \mathcal{N}}$, one can determine the value of the arc flow

$$f_a = \sum_{s \in \mathcal{N}} \sum_{p \in \mathcal{P}_s} \alpha_{pa} x_p \tag{8}$$

for each arc $a \in \mathcal{A}$, where

$$\alpha_{pa} = \begin{cases} 1 & \text{if arc } a \text{ belongs to path } p, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

If the vector $f = (f_a)_{a\in\mathcal{A}}$ of arc flows is known, one can determine the dis-utility value $c_a(f)$ for each arc. Then one can compute the dis-utility value for each path p:

$$g_p(x) = \sum_{a\in\mathcal{A}} \alpha_{pa} c_a(f). \tag{10}$$

In the known elastic demand models, each (O/D) pair $s \in \mathcal{N}$ is associated with one variable value of flow demand and hence one inverse demand (dis-utility) function; see e.g. [Nagurney (1999), Chapter IV] and references therein. However, several active agents (users) with different dis-utility functions may have the same physical location for many networks arising in applications.

For this reason, we now consider the generalization, where each (O/D) pair $s \in \mathcal{N}$ may have several pairs of active users hence it is associated with the set of such pairs $\mathcal{B}_s$ so that each pair of users $j \in \mathcal{B}_s$ has its particular flow demand y_j and dis-utility function h_j, which can be in principle dependent of the flow demand y, i.e. $y = (y_j)_{j \in \mathcal{B}_s, s \in \mathcal{N}}$. Then one can define the feasible set of flows:

$$W = \left\{ w = (x, y) \left| \begin{array}{c} \sum_{p \in \mathcal{P}_s} x_p = \sum_{j \in \mathcal{B}_s} y_j, \\ x_p \geq 0, \; p \in \mathcal{P}_s, \\ 0 \leq y_j \leq \gamma_j, \; j \in \mathcal{B}_s; \; s \in \mathcal{N} \end{array} \right. \right\}. \quad (11)$$

We say that a feasible flow / demand pair $(x^*, y^*) \in W$ is an *equilibrium point* if it satisfies the following conditions:

$$\forall s \in \mathcal{N},\ \exists \lambda_s \text{ such that } g_p(x^*) \begin{cases} \geq \lambda_s & \text{if} \quad x_p^* = 0, \\ = \lambda_s & \text{if} \quad x_p^* > 0, \end{cases} \qquad (12)$$
$$\forall p \in \mathcal{P}_s;$$

and

$$h_j(y^*) \begin{cases} \leq \lambda_s & \text{if} \quad y_j^* = 0, \\ = \lambda_s & \text{if} \quad y_j^* \in (0, \gamma_j), \\ \geq \lambda_s & \text{if} \quad y_j^* = \gamma_j, \end{cases} \quad \forall j \in \mathcal{B}_s. \qquad (13)$$

Clearly, the equilibrium conditions in (12)–(13) represent some implementation of the multi-commodity two-sided market equilibrium model (1)–(2), where each commodity is associated with an (O/D) pair $s \in \mathcal{N}$, its set of traders (carriers) with price functions $g_p(x)$ is represented by the paths $p \in \mathcal{P}_s$, whereas its set of buyers with price functions $h_j(y)$ is represented by the pairs of users $j \in \mathcal{B}_s$. We observe that the prices here are not fixed, the dependence of volumes for offer price functions g_p is given in (8)–(10) and caused by the complexity of the system topology and by the fact that carriers of different (O/D) pairs can utilize the same links (arcs).

We can show that conditions (11)–(13) can be equivalently rewritten in the form of a VI: Find a pair $(x^*, y^*) \in W$ such that

$$\sum_{s \in \mathcal{N}} \left[\sum_{p \in \mathcal{P}_s} g_p(x^*)(x_p - x_p^*) - \sum_{j \in \mathcal{B}_s} h_j(y^*)(y_j - y_j^*) \right] \geq 0 \quad (14)$$
$$\forall (x, y) \in W.$$

Proposition 4.1 *A pair $(x^*, y^*) \in W$ solves VI (14) if and only if it satisfies conditions (12)–(13).*

If each (O/D) pair is attributed to only one pair of users, we obtain the custom network equilibrium problems with elastic (inverse) demands; see e.g. [Nagurney (1999), Chapter IV]. If all the (O/D) traffic demands in this model are not restricted with upper bounds, we obtain the model considered in [Dafermos (1982)]. Let us insert the same condition in our model:

$$\gamma_j = +\infty \quad \forall j \in \mathcal{B}_s, \ s \in \mathcal{N}. \tag{15}$$

Then (13) reduces to the following condition:

$$h_j(y^*) \begin{cases} \leq \lambda_s & \text{if} \quad y_j^* = 0, \\ = \lambda_s & \text{if} \quad y_j^* > 0; \end{cases} \quad \forall j \in \mathcal{B}_s. \tag{16}$$

We can also write some other equivalent network equilibrium conditions, for instance,

$$g_p(x^*) - h_j(y^*) \begin{cases} = 0 & \text{if} \ \ x_p^* > 0 \quad \text{and} \quad y_j^* > 0, \\ \geq 0 & \text{if} \ \ x_p^* = 0 \quad \text{or} \quad y_j^* = 0; \end{cases} \tag{17}$$
$$\forall p \in \mathcal{P}_s, \ j \in \mathcal{B}_s, \ s \in \mathcal{N}.$$

Proposition 4.2 *Let (15) hold. Then, for any pair* $(x^*, y^*) \in W$*, condition (17) is equivalent to (12) and (16).*

It is easy to see that conditions (17) can be replaced with the following:

$$g_p(x^*) - h_j(y^*) \begin{cases} > 0 & \Longrightarrow \quad x_p^* = 0 \quad \text{or} \quad y_j^* = 0, \\ \geq 0 & \Longleftrightarrow \quad x_p^* \geq 0 \quad \text{and} \quad y_j^* \geq 0; \end{cases} \tag{18}$$
$$\forall p \in \mathcal{P}_s, \ j \in \mathcal{B}_s, \ s \in \mathcal{N}.$$

Proposition 4.3 *Let (15) hold. Then, for any pair* $(x^*, y^*) \in W$*, condition (18) is equivalent to (12) and (16).*

The equivalent VI formulation of network equilibrium problems enables us to obtain the existence of solutions rather easily. The feasible set W of the network equilibrium problem defined in (11) is bounded if $\gamma_j < +\infty$ for all $j \in \mathcal{B}_s$, $s \in \mathcal{N}$. Then VI (14) and hence the equivalent network equilibrium problem are solvable if all the mappings c_a, $a \in \mathcal{A}$ and h_j, $j \in \mathcal{B}_s$, $s \in \mathcal{N}$ are continuous. Let us turn to the above pure unbounded case (15). Then the feasible set W in (11) is unbounded. We now deduce a new existence result for VI (14) and hence for the equivalent network equilibrium problem by a direct application of Proposition 2.2. We need the proper following coercivity condition; cf. $(\mathbf{C})$.

(C1) *There exists a number* $r > 0$ *such that for any point* $w = (x, y) \in W$ *and for each* $s \in \mathcal{N}$ *it holds that*

$$\exists j \in \mathcal{B}_s, \ y_j > r \Longrightarrow \exists p \in \mathcal{P}_s$$

such that

$$x_p > 0 \text{ and } g_p(x) \geq h_j(y).$$

We observe that condition **(C1)** implies condition **(C)** for VI (14) and we obtain the desired existence result.

Theorem 4.1 *Suppose that (15) holds, the set* W *defined in (11) is nonempty, all the functions* c_a *and* h_j *are continuous for all* $a \in \mathcal{A}$*,* $j \in \mathcal{B}_s$*, and* $s \in \mathcal{N}$*. If condition* **(C1)** *is fulfilled, then VI (14) has a solution.*

5 Implementation of partial linearization methods for integrable network equilibrium problems

In Section 3, several versions of partial linearization (PL) methods for special decomposable optimization problems over Cartesian product sets were described for the general multi-commodity market equilibrium model of Section 2 in the integrable case. Hence, PL methods can be also applied to integrable network equilibrium problems with elastic demands of Section 4.

Therefore, we now will suppose that all the functions c_a and h_j are continuous and separable, i.e., $c_a(f) = c_a(f_a)$ and $h_j(y) = h_j(y_j)$. Besides, we assume that $c_a(f_a)$ and $-h_j(y_j)$ are monotone increasing functions. Next, we assume that

$$\gamma_j < +\infty \quad \forall j \in \mathcal{B}_s, \ s \in \mathcal{N};$$

then the feasible set W is non-empty, convex, and compact and

$$\begin{aligned} W &= \prod_{s \in \mathcal{N}} W_s, \text{ where} \\ W_s &= \left\{ w_{(s)} = (x_{(s)}, y_{(s)}) \;\middle|\; \begin{array}{l} \sum_{p \in \mathcal{P}_s} x_p = \sum_{j \in \mathcal{B}_s} y_j, \\ x_p \geq 0, p \in \mathcal{P}_s, \\ 0 \leq y_j \leq \gamma_j, \ j \in \mathcal{B}_s \end{array} \right\}; \\ &\text{for } s \in \mathcal{N}. \end{aligned}$$

Here $x_{(s)} = (x_p)_{p \in \mathcal{P}_s}$, $y_{(s)} = (y_j)_{j \in \mathcal{B}_s}$.

Due to the separability of the functions c_a and h_j, their continuity implies integrability, i.e., then there exist functions

$$\begin{aligned} \mu_a(f_a) &= \int\limits_0^{f_a} c_a(t)dt \ \forall a \in \mathcal{A}, \\ \eta_j(y_j) &= \int\limits_0^{v_j} h_j(t)dt \ \forall j \in \mathcal{B}_s, \ s \in \mathcal{N}. \end{aligned}$$

Taking into account (8), we see that VI (14) gives a necessary and sufficient optimality condition for the following optimization problem:

$$\min_{(x,y)\in W} \to \left\{ \sum_{a\in\mathcal{A}} \mu_a(f_a) - \sum_{s\in\mathcal{N}} \sum_{j\in\mathcal{B}_s} \eta_j(y_j) \right\}. \qquad (19)$$

However, this problem falls into the basic format (6) and the suggested PL methods can be applied to (19).

We describe the solution of the basic direction finding problem (7). It now consists in finding an element $\bar{w}_{(s)} = (\bar{x}_{(s)}, \bar{y}_{(s)}) \in W_s$, which solves the optimization problem

$$\min_{(x_{(s)}, y_{(s)}) \in W_s} \to \left\{ \sum_{p \in \mathcal{P}_s} g_p(x^k) x_p - \sum_{j \in \mathcal{B}_s} \eta_j(y_j) \right\} \tag{20}$$

for some selected pair $s \in \mathcal{N}$. The solution of (20) can be found with the simple procedure below, which is based on optimality conditions (12)–(13).

First we calculate the shortest path $q \in \mathcal{P}_s$ with the minimal cost. Set $\tilde{\lambda}_s = g_q(x^k)$, $\bar{x}_p = 0$ for all $p \in \mathcal{P}_s$.

For each $j \in \mathcal{B}_s$ we verify three possible cases.

Case 1. If $h_j(0) \leq \tilde{\lambda}$, then set $\bar{y}_j = 0$. Otherwise go to Case 2.

Case 2. If $h_j(\gamma_j) \geq \tilde{\lambda}$, set $\bar{y}_j = \gamma_j$, $\bar{x}_q = \bar{x}_q + \gamma_j$. Otherwise go to Case 3.

Case 3. We have $h_j(\gamma_j) < \tilde{\lambda} < h_j(0)$. By continuity of h_j, we find the value $\bar{y}_j \in [0, \gamma_j]$ such that $h_j(\bar{y}_j) = \tilde{\lambda}$, set $\bar{x}_q = \bar{x}_q + \bar{y}_j$.

Therefore, the suggested PL methods can be implemented rather easily.

6 Application of market models to resource allocation in wireless networks

In contemporary wireless networks, increasing demand of services leads to serious congestion effects, whereas significant network resources (say, bandwidth and batteries capacity) are utilized inefficiently for systems with fixed allocation rules. This situation forces one to apply more flexible market type allocation mechanisms. Due to the presence of conflict of interests, most papers on allocation mechanisms are devoted to pure game-theoretic models reflecting imperfect competition; see, e.g., [Leshem and Zehavi (2009)], [Raoof and Al-Raweshidy (2010)]. However, certain lack of information about the participants is typical for wireless telecommunication networks (see, e.g., [Iosifidis and Koutsopoulos (2010)], [Raoof and Al-Raweshidy (2010)]), and some other market models may be suitable here because they can be utilized under minimal information requirements on involved users.

We now consider the problem of allocation of services of several competitive wireless network providers for a large number of users, which is very essential for contemporary communication systems. This problem was investigated in [Hayrapetyan and Tardos (2007)], [Korcak et al. (2012)], [Maillé et al. (2012)], [Zhang and Zhang (2013)] for wired and wireless network settings, where game-theoretic models for competitive providers were presented. An alternative model, which is based on some VI formulation and uses proper equilibrium conditions, was suggested for this problem in [Konnov (2015b), Section 6]. We now propose its extension that admits different kinds of users' behavior.

Namely, we suppose that there are m wireless network providers and that all the users are divided into n classes, that is, the users belonging to the same class j are considered as one service consumer with a price function $h_j(y_j)$ and a scalar bid volume $y_j \in [0, \beta_j]$ for $j \in N = \{1, \dots, n\}$. Next, each provider i announces his/her price function $b_i(x_i)$ depending on the offer volume $x_i \in [0, \alpha_i]$ for $i \in M = \{1, \dots, m\}$. However, such joint consumption of wireless network resources yields the additional dis-utility $l_i(x)$ for users consuming resources of provider i, where $x = (x_1, \dots, x_s)^\top$; see [Korcak et al. (2012)], [Maillé et al. (2012)], [Zhang and Zhang (2013)] for more detail. Hence, the actual price function of provider i for users becomes $g_i(x) = b_i(x_i) + l_i(x)$. We can thus define the feasible set of offer/bid values

$$D = \left\{ (x, y) \;\middle|\; \begin{array}{l} x_i \in [0, \alpha_i], i \in M, \\ y_j \in [0, \beta_j], j \in N; \end{array} \sum_{i \in M} x_i = \sum_{j \in N} y_j; \right\};$$

where $y = (y_1, \dots, y_n)^\top$.

Then we can write the two-sided equilibrium problem that consists in finding a feasible pair $(\bar{x}, \bar{y}) \in D$ and a price λ such that

$$
\begin{array}{l}
g_i(\bar{x}) \begin{cases} \geq \lambda, & \text{if } \bar{x}_i = 0, \\ = \lambda, & \text{if } \bar{x}_i \in (0, \alpha_i), \ \ i \in M; \\ \leq \lambda, & \text{if } \bar{x}_i = \alpha_i, \end{cases} \\
h_j(\bar{y}_j) \begin{cases} \leq \lambda, & \text{if } \bar{y}_j = 0, \\ = \lambda, & \text{if } \bar{y}_j \in (0, \beta_j), \quad j \in N. \\ \geq \lambda, & \text{if } \bar{y}_j = \beta_j, \end{cases}
\end{array}
\tag{21}
$$

Clearly, it is a particular case of those in (1)–(2). Due to Proposition 2.1, (21) can be replaced with the equivalent VI: Find $(\bar{x}, \bar{y}) \in D$ such that

$$\sum_{i \in M} g_i(\bar{x})(x_i - \bar{x}_i) - \sum_{j \in N} h_j(\bar{y}_j)(y_j - \bar{y}_j) \geq 0 \quad \forall (x, y) \in D. \quad (22)$$

This property enables us to establish existence of solutions for the above problem and develop efficient iterative solution methods. In fact, if all the price functions are continuous and the set D is nonempty and bounded, then VI (22) has a solution. In the unbounded case, some coercivity condition is necessary. For instance, let us consider the case where $\alpha_i = +\infty$ for $i \in M$ and $\beta_j = +\infty$ for $j \in N$ and take the following condition; cf. **(C)**.

(C2) *There exists a number $r > 0$ such that for any pair $(x, y) \in D$ it holds that*

$$y_l > r \Longrightarrow \exists k \in M \text{ such that } x_k > 0 \text{ and } g_k(x) \geq h_l(y_l).$$

Clearly, **(C2)** implies **(C)** for VI (22) and Proposition 2.2 provides the existence result.

Theorem 6.1 *Suppose that the set D is nonempty, the functions g_i and h_j are continuous for all $i \in M$, $j \in N$. If condition **(C2)** is fulfilled, then VI (22) has a solution.*

7 The partial linearization method for resource allocation problems in wireless networks

Iterative solution methods for solving VI of form (22) in general require additional monotonicity assumptions for convergence; see e.g. [Patriksson (1999)], [Konnov (2007a)], [Konnov (2013b)]. Additional solution methods appear in the integrable case where

$$g_i(x) = \frac{\partial \mu(x)}{\partial x_i}, \ i \in M; \ h_j(y_j) = -\eta_j'(y_j), \ j \in N.$$

Then, VI (22) gives the optimality condition for the optimization problem:

$$\begin{array}{l} \min\limits_{w \in D} \to f(w), \\ f(w) = f(x,y) = \{\mu(x) + \eta(y)\}, \\ \eta(y) = \sum\limits_{j \in N} \eta_j(y_j); \end{array} \tag{23}$$

cf. (4) and (6). In particular, conditional gradient, gradient projection, and Uzawa type methods then can be utilized; see e.g. [Konnov (2013a)], [Konnov (2015a)]. We now only describe a way to implement the custom PL method since the problem is not separable. We suppose in addition that the function μ is smooth and convex, $\alpha_i = +\infty$ for all $i \in M$, and $0 \le \beta_j < +\infty$ for all $j \in N$. Then the feasible set D is non-empty, convex, and compact.

For more clarity, we rewrite the PL method for problem (23). We define the gap function

$$\varphi(w) = \varphi(x,y) = \max_{(x',y')\in D} \{\langle x - x', \mu'(x)\rangle + \eta(y) - \eta(y')\}.$$

Method (PL).
Choose a point $w^0 \in D$, numbers $\beta \in (0,1)$ and $\theta \in (0,1)$, set $k = 0$. At the k-th iteration, $k = 0, 1, \ldots$, we have a point $w^k \in D$. Find a solution $v^k = (\bar{x}^k, \bar{y}^k)$ of the problem

$$\min_{v\in D} \to \left\{\langle \mu'(x^k), v\rangle + \eta(v)\right\}. \tag{24}$$

If $v^k = w^k$, stop. Otherwise set $d^k = v^k - w^k$, find p as the smallest number in $\mathbb{Z}_+$ such that

$$f(w^k + \theta^p d^k) \le f(w^k) - \beta\theta^p \varphi(w^k),$$

set $\sigma_k = \theta^p$, $w^{k+1} = w^k + \sigma_k d^k$, and $k = k + 1$.

The solution of the basic direction finding problem (24) can also be found with the simple procedure, which is similar to that from Section 5 and based on the optimality conditions.

First we calculate an index $q \in M$ that corresponds to the minimal value

$$g_q(x^k) = \min_{i \in M} g_i(x^k)$$

and set $\tilde{\lambda} = g_q(x^k)$, $\bar{x}_i^k = 0$ for all $i \in M$.

For each $j \in N$ we verify three possible cases.

Case 1. If $h_j(0) \leq \tilde{\lambda}$, then set $\bar{y}_j^k = 0$. Otherwise go to Case 2.

Case 2. If $h_j(\beta_j) \geq \tilde{\lambda}$, set $\bar{y}_j^k = \beta_j$, $\bar{x}_q^k = \bar{x}_q^k + \beta_j$. Otherwise go to Case 3.

Case 3. We have $h_j(\beta_j) < \tilde{\lambda} < h_j(0)$. By continuity of h_j, we find the value $\bar{y}_j^k \in [0, \beta_j]$ such that $h_j(\bar{y}_j^k) = \tilde{\lambda}$, set $\bar{x}_q^k = \bar{x}_q^k + \bar{y}_j^k$.

Let us now consider the case where $0 \leq \alpha_i < +\infty$ for all $i \in M$ and $0 \leq \beta_j < +\infty$ for all $j \in N$. Then the feasible set D is also non-empty, convex, and compact. Hence, the above PL method can be applied to (23), however, we should then take more complex procedures for solution of problem (24). However, we can eliminate the upper bounds for the variables x_i via a suitable penalty approach.

For instance, replace problem (23) with the sequence of auxiliary problems of the form

$$\begin{array}{l} \min\limits_{w \in D} \to \Phi(w,\tau), \\ \Phi(w,\tau) = \mu(x) + \tau\varphi(x) + \eta(y), \\ \varphi(x) = 0.5 \sum\limits_{i \in M} \max\{x_i - \alpha_i, 0\}^2; \end{array} \tag{25}$$

where $\tau > 0$ is a penalty parameter, the functions μ and η are defined as above. Under the standard assumptions the sequence of solutions of (25) will approximate a solution of (23) if $\tau \to +\infty$; see e.g. [Konnov (2013b)]. Next, each problem (25) has the previous format without the upper bounds for the variables x_i. Hence, we can apply directly the above version of the PL method to (25) with replacing $f(w)$ by $\Phi(w,\tau)$. Clearly, (24) is replaced by

$$\min_{v \in D} \to \left\{ \langle \mu'(x^k) + \tau\varphi'(x^k), v \rangle + \eta(v) \right\}.$$

We also have to substitute each function $g_i(x)$ with $\tilde{g}_i(x) = g_i(x) + \tau \max\{x_i - \alpha_i, 0\}$ in the procedure of finding its solution. This gives us an alternative way to solve such resource allocation problems in wireless networks.

8 Computational experiments with network equilibrium test problems

In order to compare the performance of the PL methods we carried out preliminary series of computational experiments on network equilibrium test problems of form (11)–(13) or (14). We took their adjustment described in Section 5.

For comparison we took proper extensions of the known test examples of network equilibrium problems with elastic demands, namely, each (O/D) pair was associated with two pairs of active users. We used the arc cost functions $c_a(f_a) = 1 + f_a$ for all $a \in \mathcal{A}$ and the minimal path cost (dis-utility) functions $h_{j1(s)}(y_{j1}) = 30 - 0.5y_{j1(s)}$ and $h_{j2(s)}(y_{j2(s)}) = 28 - 0.3y_{j2(s)}$, where $\mathcal{B}_s = \{j1(s), j2(s)\}$ for all $s \in \mathcal{N}$. We took

$$\Delta_k = \varphi(w^k) = \sum_{s \in \mathcal{N}} \varphi_s(w^k)$$

as accuracy measure for the methods. Both the PL and CPL methods were implemented with the Armijo line-search rule where $\beta = \theta = 0.5$. Due to the above description we see that we can take the total number of blocks where the linesearch procedure was utilized as unified complexity measure for both the methods, which will be called block iterations. Hence we reported this value in the tables for attaining different accuracies. The methods were implemented in C++ with double precision arithmetic.

The topology of Example 1 was taken from [Bertsekas and Gafni (1982)]. The graph contains 25 nodes, 40 arcs, and 5 O/D pairs. We used two rules for changing the parameter δ_l with $\delta_0 = 10$ in CPL. The performance results are given in Table 1.

Table 1: Example 1. The numbers of block iterations

accuracy	PL	CPL	CPL
		$\delta_{l+1} = \delta_l/2$	$\delta_l = \delta_0/l$
0.2	4970	4427	3519
0.1	10785	8747	6411
0.05	21260	17284	13425

The topology of Example 2 was taken from [Nagurney (1984), Network 26]. The graph contains 22 nodes, 36 arcs, and 12 O/D pairs. We used the rule $\delta_l = \delta_0/l$ with $\delta_0 = 10$ in CPL. The performance results are given in Table 2.

Table 2: Example 2. The numbers of block iterations

accuracy	PL	CPL
0.2	420	233
0.1	468	246
0.05	504	256

In Example 3, the data were generated randomly. The graph contained 20 nodes, 114 arcs, and 10 O/D pairs. We used the rule $\delta_l = \delta_0/l$ with $\delta_0 = 10$ in CPL. The results are given in Table 3. In all the cases, CPL showed certain preference over PL in the number of block iterations.

Table 3: Example 3. The numbers of block iterations

accuracy	PL	CPL
1	135730	106308
0.5	271830	217932
0.2	662220	531032
0.1	1329910	1082449

The author is grateful to Olga Pinyagina for her assistance in carrying out computational experiments.

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