The Generalized Center Optimal Location

Didier JOSSELIN , UMR ESPACE 7300, France

didier.josselin@univ-avignon.fr

Summer School on Operations Research and Applications, Nizhny Novgorod, March, 5-7, 2018

The French Group of Research in Geographical Information Science

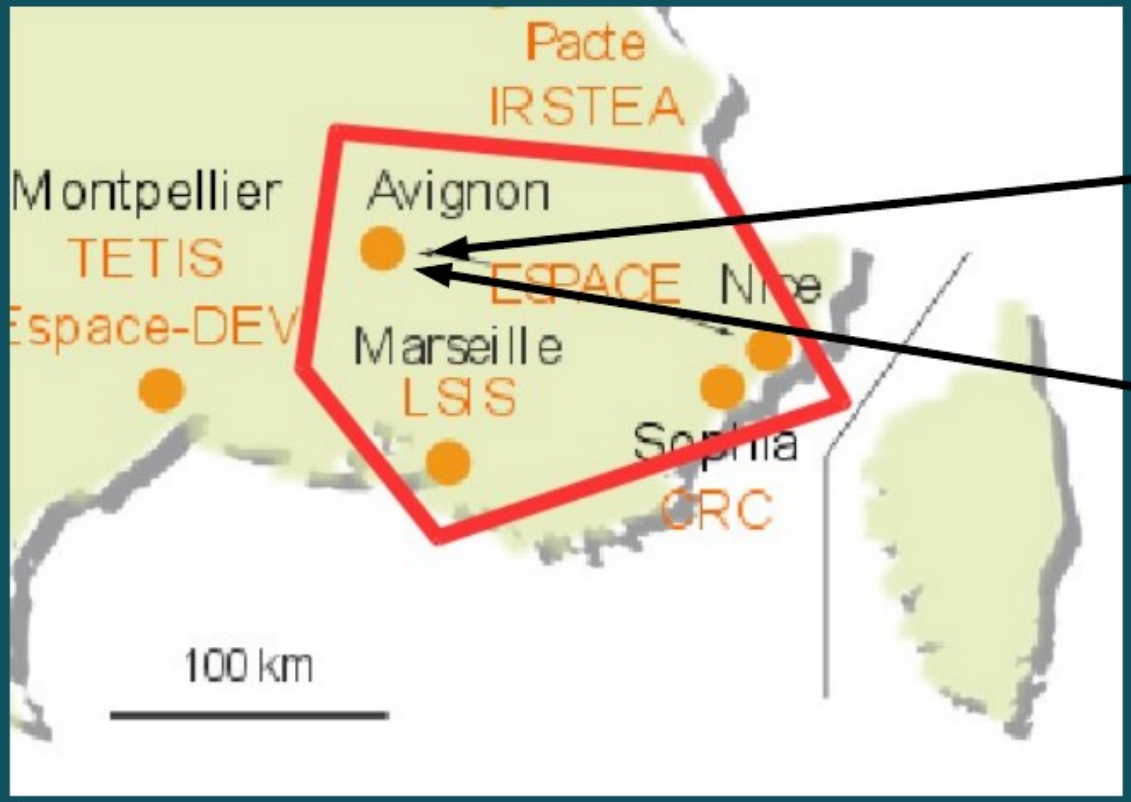
MAGIS

- A national group from CNRS
- 54 laboratories in France
- More than 300 researchers
- Interdisciplinary (geography, computer science, archeology, ecology and agronomy)
- http://gdr-magis.imag.fr/

The French Group of Research MAGIS

Europe

The University of Avignon



ESPACE (geography)

+

LMA (maths)

Project “ROLSES”

Robust and Optimal Location for Sustainable Environment and Systems

Marc Ciligot-Travain

Julio Joras-Mora

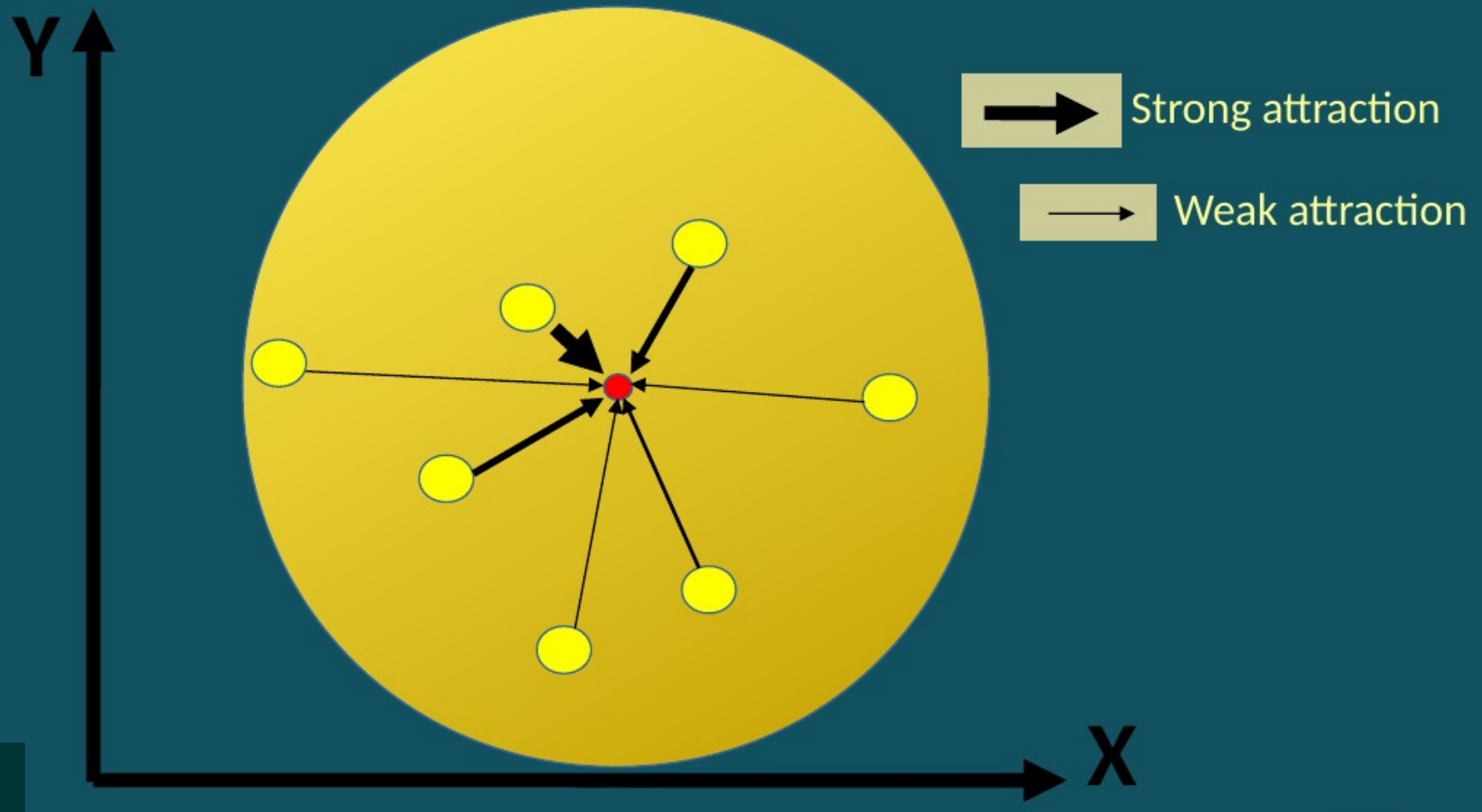
Daniel Gourion

Context and objectives

Tobler's first law of geography (1970)

« Everything is related to everything else,
but near things are more related than distant things »

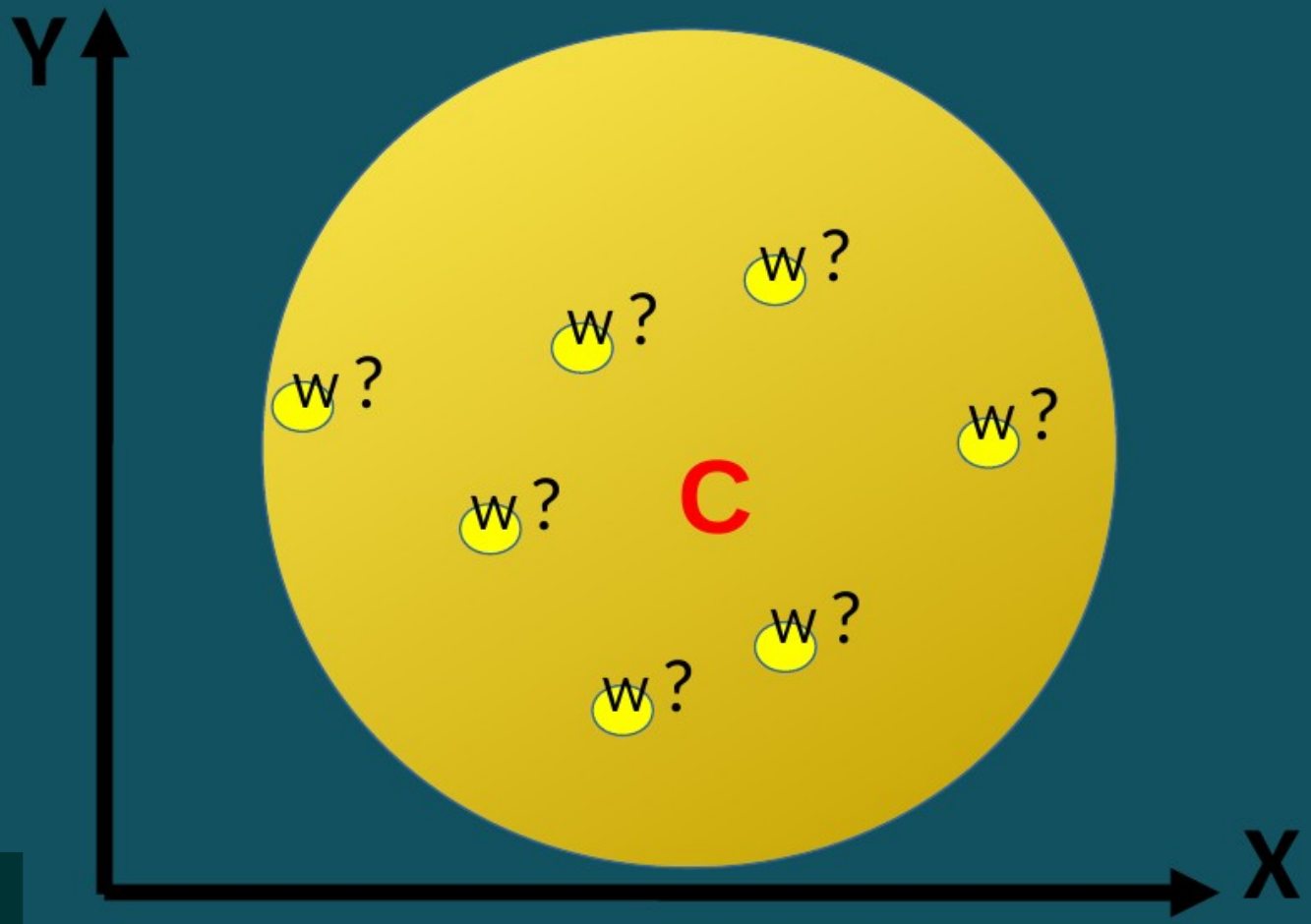
Gravity models, Tobbler's law for a center and its surrounding



D. Josselin, SSORA, March, 5-7, 2018

2D

“dual” approach: what is the influence ***w*** of each individual on the center location **C** ?



Objectives

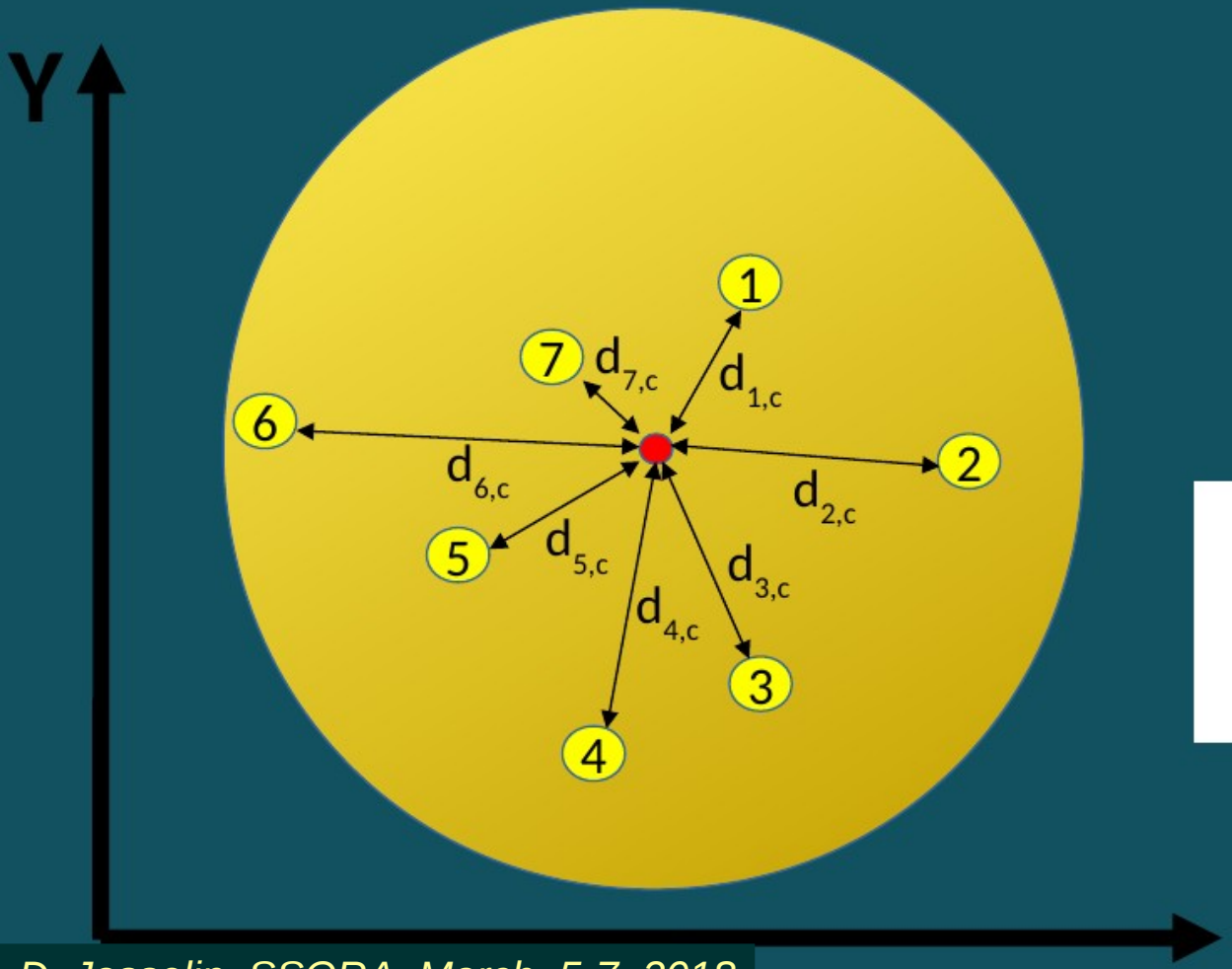
- Mesuring the influence of the metrics chosen to locate an optimal center according to a set of demands
- Generalizing the *1*-facility location problem through L_p-norms
- Assessing the meaning of a given metrics in geography for optimal location purpose (equity, equality, efficiency and compromises)

Facility optimal location problem

Optimal location of a center according to a set of demand points

2D

Optimal location of a center: minimizing the Minkowski distance (L_p-norm)

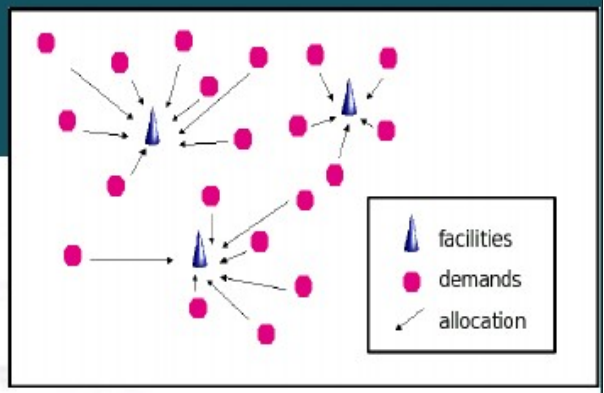


$d^p_{i,c}$ = distance between ***i*** and ***c*** given a norm p

$$c^* = min\left(\sum d_{ic}^p\right)^{\frac{1}{p}}$$

2D

Branches of the facility location problem



Networks:
Hakimi (1964)
Scott (1970)
Labbe et al. (1995)
Thomas (2002)
Nickel & Puerto (2005)

Non discrete location (fuzzy, surface...)

Discrete Facility-Location Problem

1 to k facilities

Center location

Demand location

Cost operator

- node/vertex location
  - node/vertex demand
    - min-max cost
      - center
      - barycenter
    - mini sum cost
      - median
      - minisum squared cost
  - unrestricted demand
    - min-max cost
      - general center
    - mini sum cost
      - general median
- unrestricted location
  - node/vertex demand
    - min-max cost
      - absolute center
    - mini sum cost
      - absolute median
  - unrestricted demand
    - min-max cost
      - general absolute-center
    - mini sum cost
      - general absolute-median

Generalized Lp-norm

g.a. barycenter

adapted from Evans & Minieka,1992 in Chan, 2005

Metrics to locate a center

Efficiency, Equality, Equity: illustration by the access to transport

More economic

Efficiency

Fixed lines with regural buses
In the core of downtown population

Equality

Fixed lines with regural buses
serving all the city population

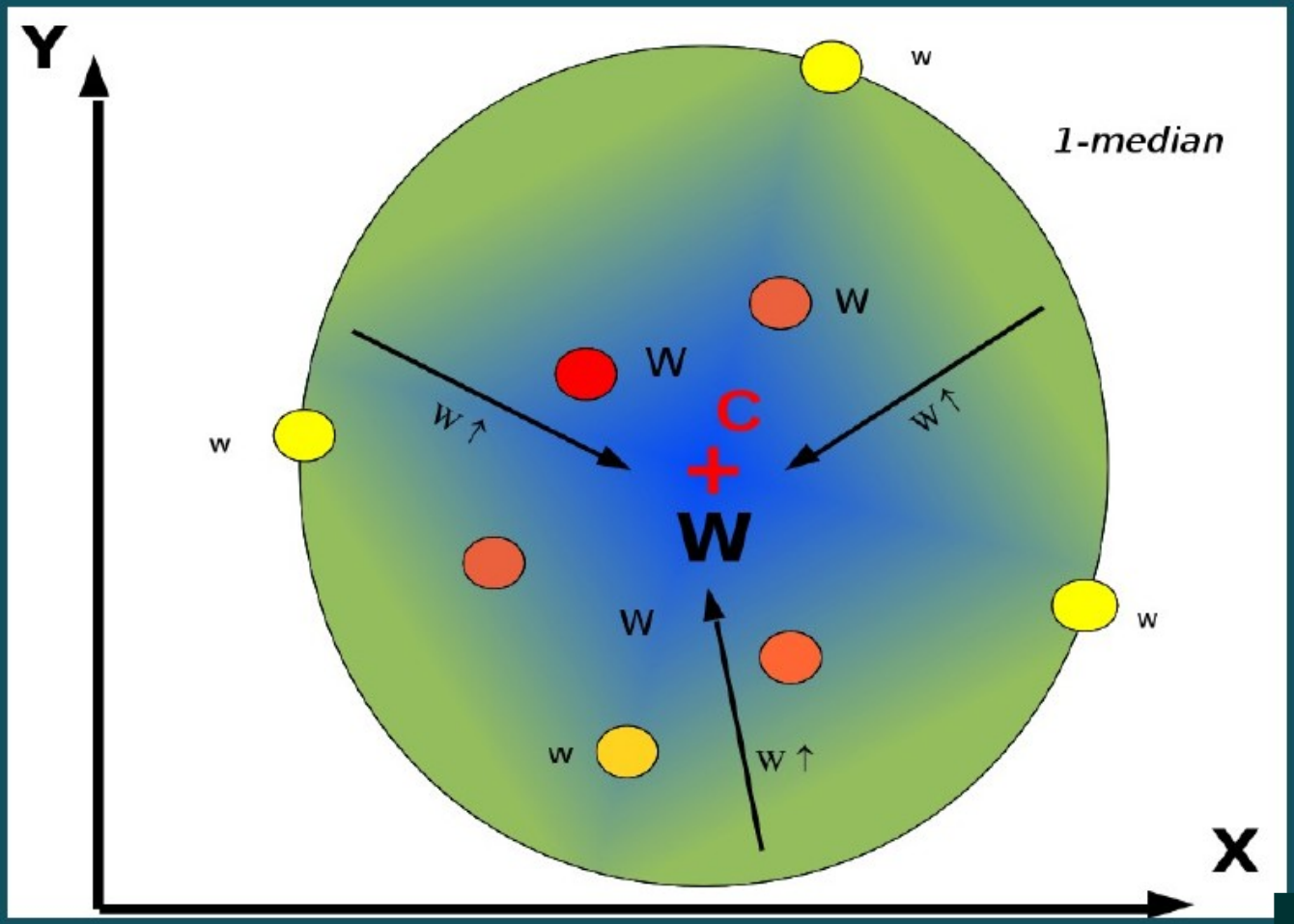
Equity

Fixed lines with regural buses
plus:
vehicles adapted to disabled people

More social

The Weber/Fermat point

- *1*-median, minisum(cost)



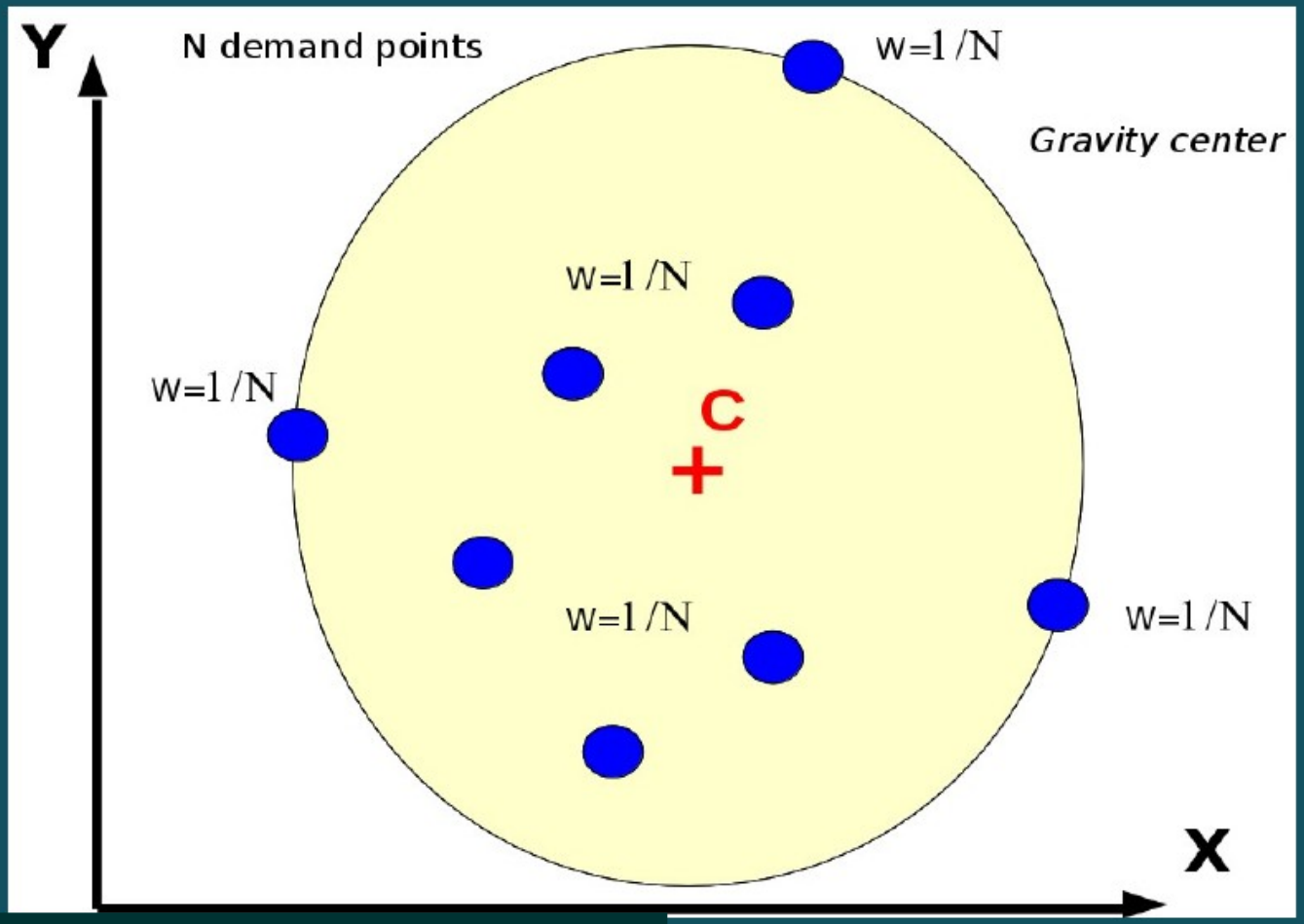
Efficiency

Ex:
a transport station

$$c^* = min \sum_{i=1}^{n} d_{ic}$$

The gravity center

- centroid, minisum^2(cost)



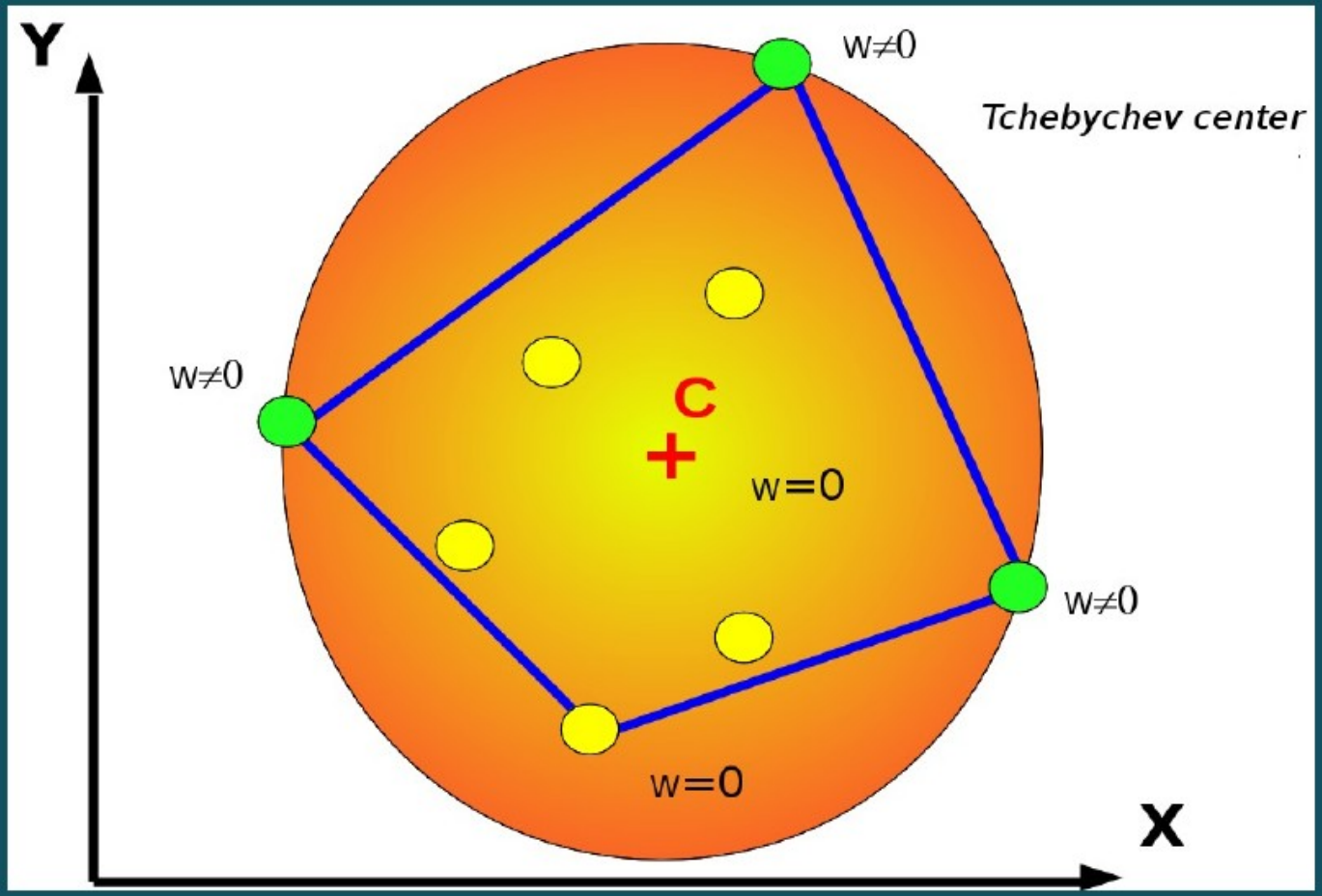
Equality

Ex:
a centroid of
a convex parcel

$$c^* = min \sqrt{\sum_{i=1}^{n} (d_{ic})^2}$$

The Tchebychev center

- *1*-center, minimax (cost)



Equity

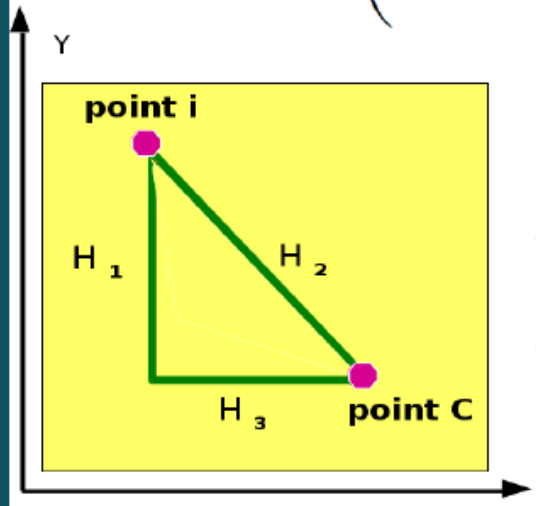
Ex:
a hospital

$$c^* = minmax \sum_{i=1}^{n} (d_{ic})$$

Main L_p-norms from a geometric point of view

Maths
(3 main norms)

$$D_{ic} = \left(|x_i - x_c|^{p'} + |y_i - y_c|^{p'} \right)^{\frac{1}{p'}}$$



L_1 : Manhattan distance **(p'=1)**

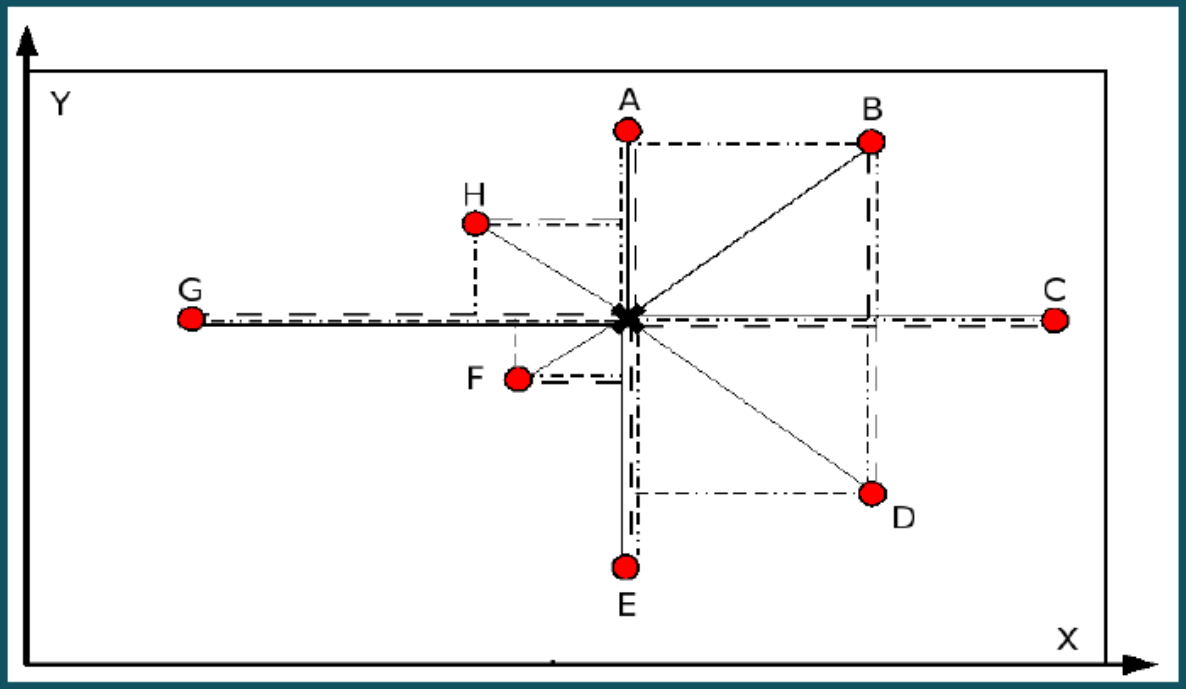
$D_{ic} = H_1 + H_3$

L_2 : Euclidian distance **(p'=2)**

$D_{ic} = (H_1^2 + H_3^2)^{1/2} = H_2$

L_∞ : Tchebychev distance **(p'=∞)**

$D_{ic} = \max\{H_1, H_3\} = H_1$



Geography
(2 main distances : L_1, L_2)

Two types of L_p-norms involved in optimal center location

- the **distance** considered (**p'**) between the demand points ***i*** and the center ***c***
- the **function** which aggregates these distances (**p**)

$$c^* = \arg\min_c \left(\sum_{i=1}^{n} \left(\left(|x_i - x_c|^{p'} + |y_i - y_c|^{p'} \right)^{\frac{1}{p'}} \right)^{p} \right)^{\frac{1}{p}}$$

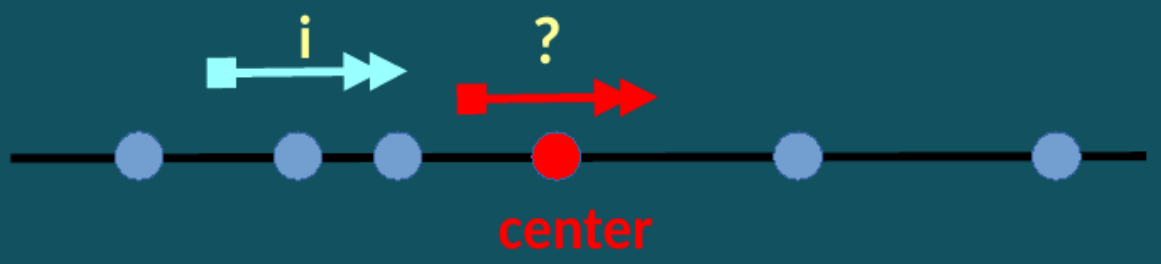
(Josselin & Ciligot-Travain, 2014)

Crossing L_p-norms with p and p'

		norm of the distance		
		Manhattan	*Euclidian*	*«Tchebychev»*
Norm of the aggregation	$p'=$ / $p=$	**1**	**2**	**∞**
Efficiency	**1**	Median of coordinates	***k-median***	?
Equality	**2**	?	***k- barycenter*** Mean of coordinates	?
Equity	**∞**	?	***k-center***	Minimax of coordinates

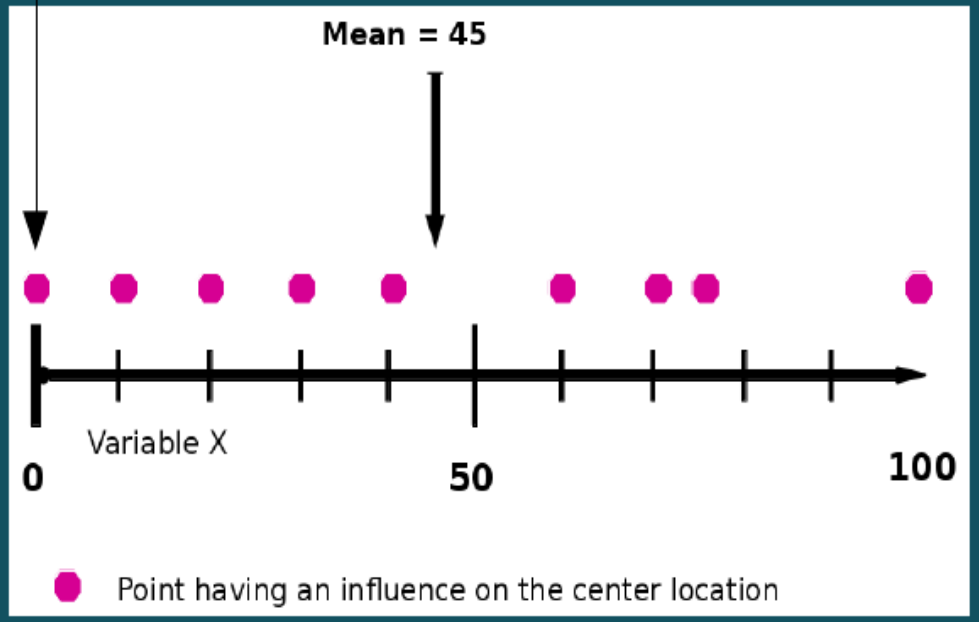
Center sensitivity analysis and demand point influence assessment

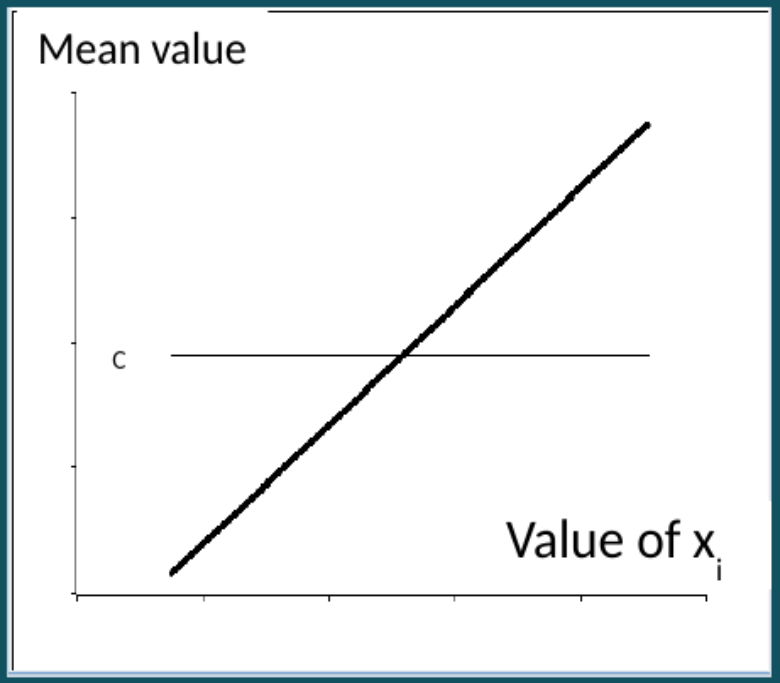
Moving *i* to get the influence on the center *c*: the case of the mean



1D

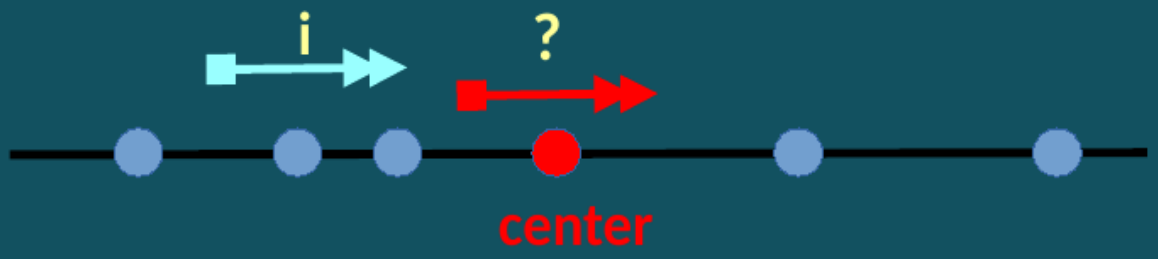
Point moved





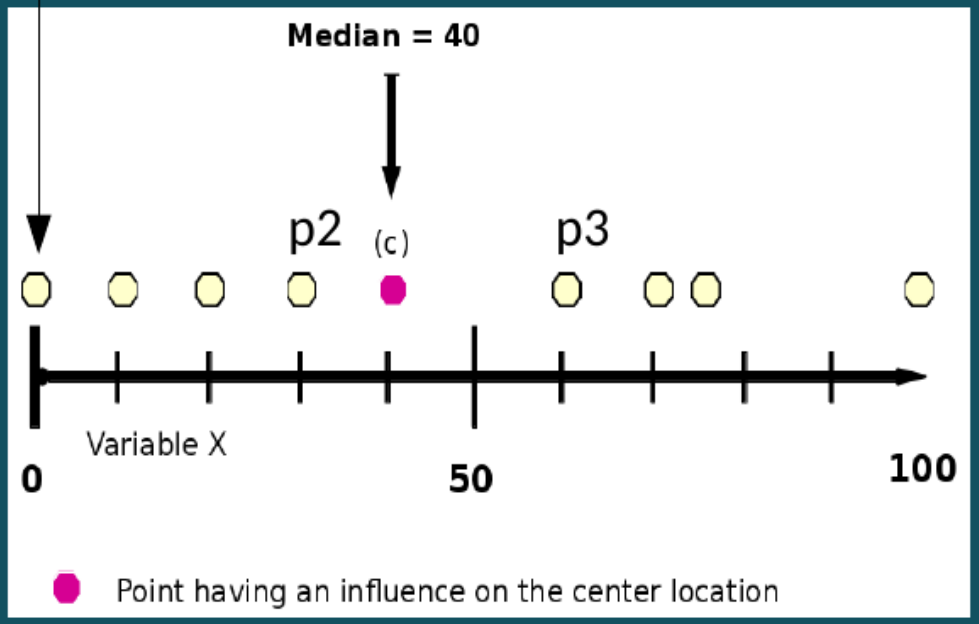
Influence curve

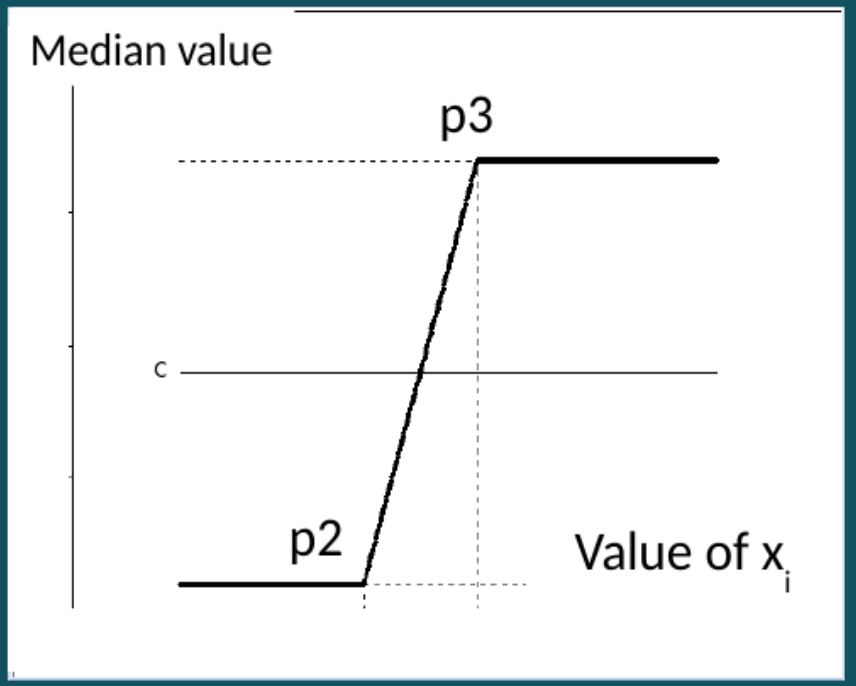
Moving *i* to get the influence on the center *c*: the case of the median



1D

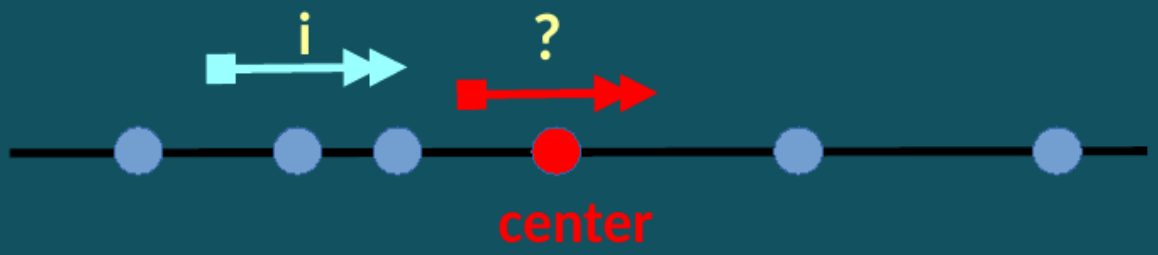
Point moved





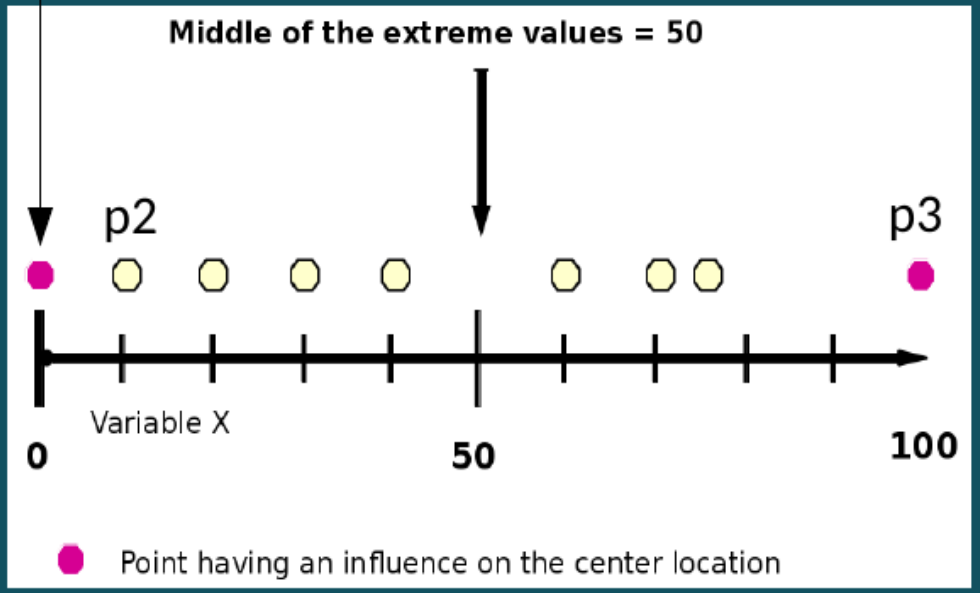
Influence curve

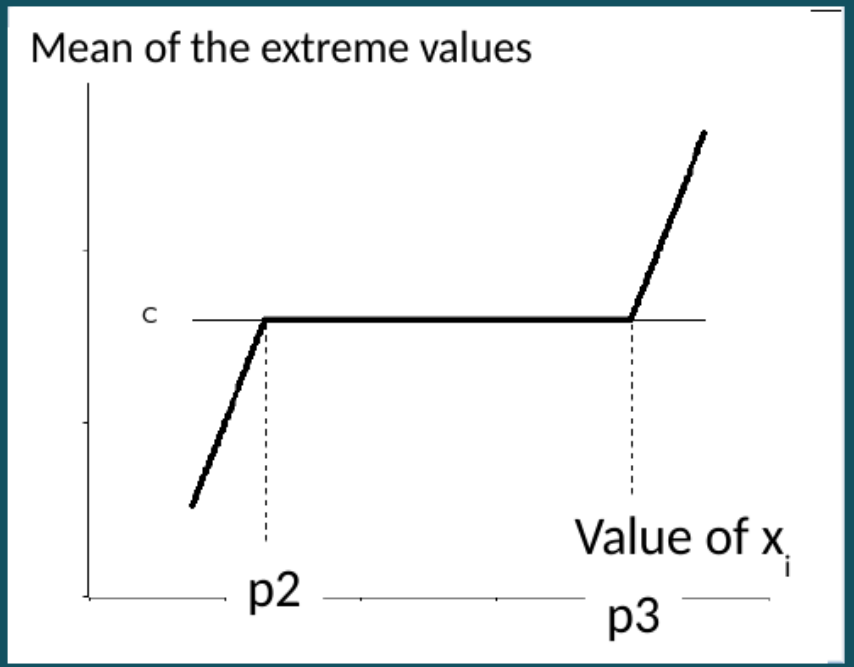
Moving *i* to get the influence on the center *c*: the Tchebytchev center



1D

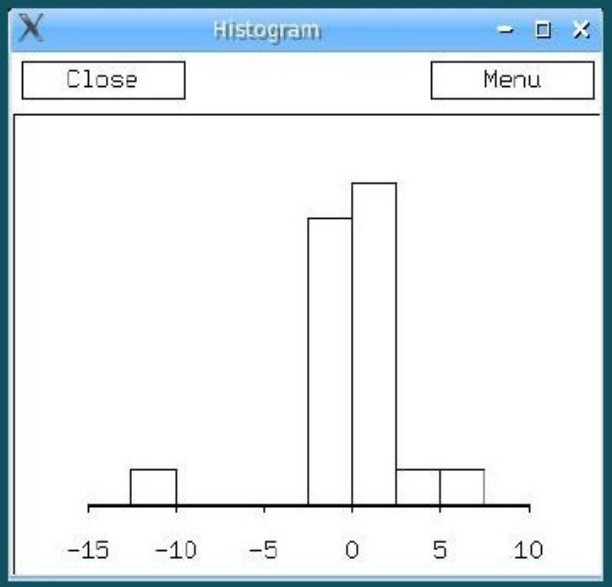
Point moved

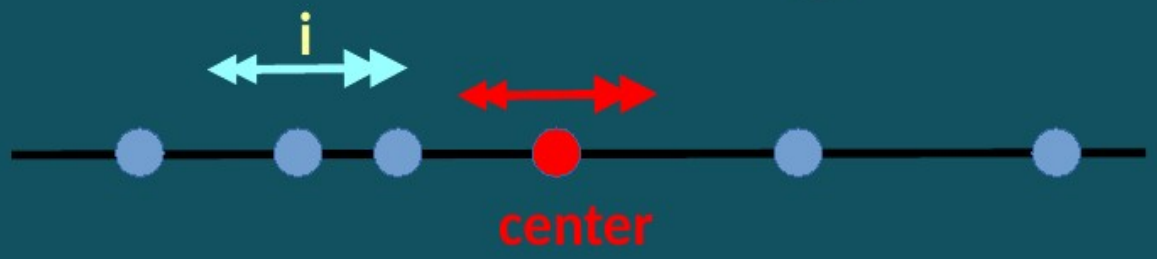




Influence curve

Learning the vector of weights in 1D



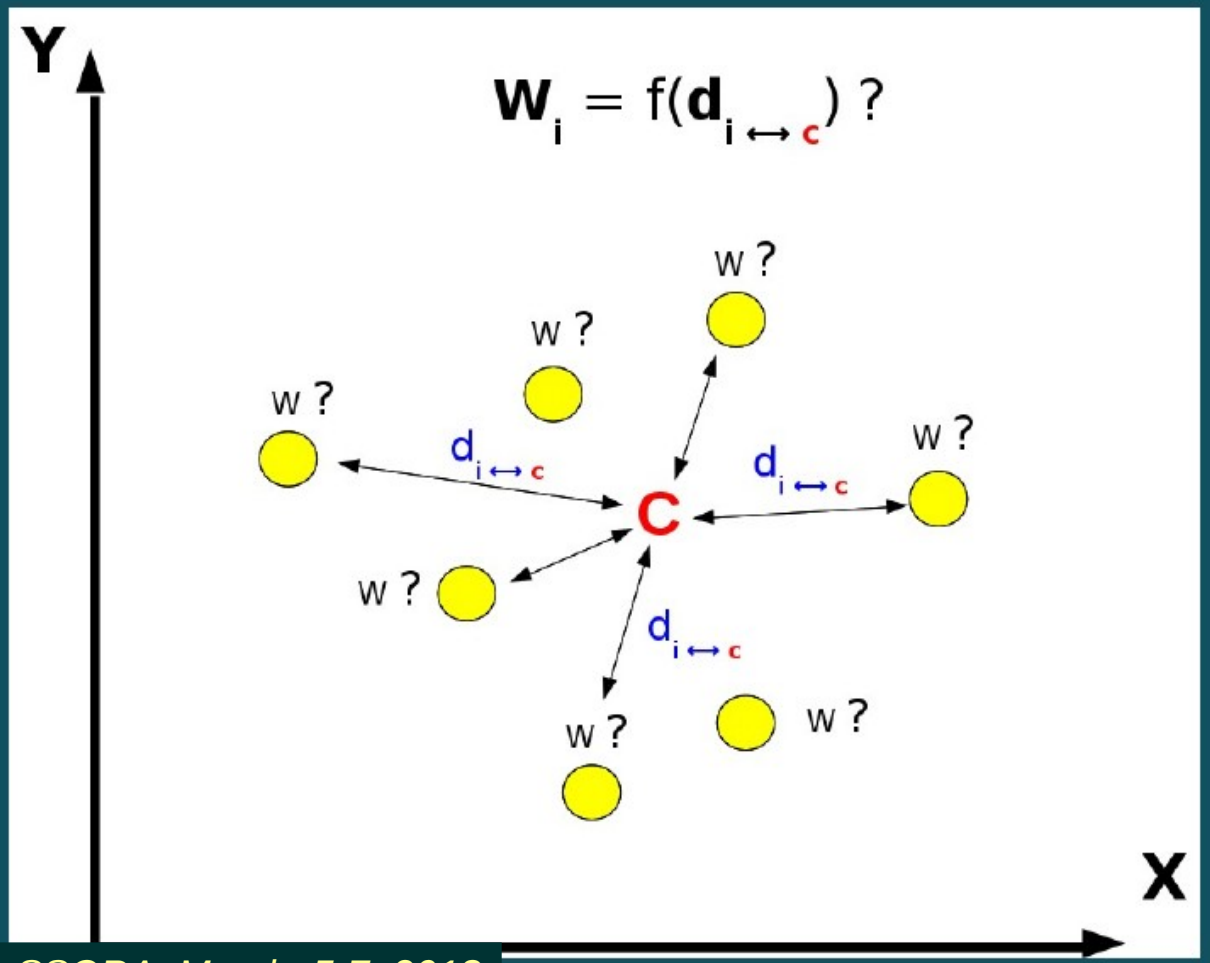


Center *Value*	*Exact value (truncated)*	*Estimate (truncated)*
Mean μ	-0.2	-0.3
Median M	0.097	0.132
Tchebytchev center T	-1.92	-1.84

Center *Point value* Status	*1* min	*4*	*5*	*7* med.	*8*	*9*	*52* max	*6*	*5*	*8*	*7* med.
Absolute and normalized weights for the Mean μ	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**	9.09 **1/11**
Absolute and normalized weights for the Median M	0 0	0 0	0 0	0.25 **1/2**	0 0	0 0	0 0	0 0	0 0	0 0	0.25 **1/2**
Absolute and normalized weights for the Tchebytchev center T	0.25 **1/2**	0 0	0 0	0 0	0 0	0 0	0.25 **1/2**	0 0	0 0	0 0	0 0

Influence W_i of the points *i* on the center *c*

- What is the relation between W_i and d_{ic}?



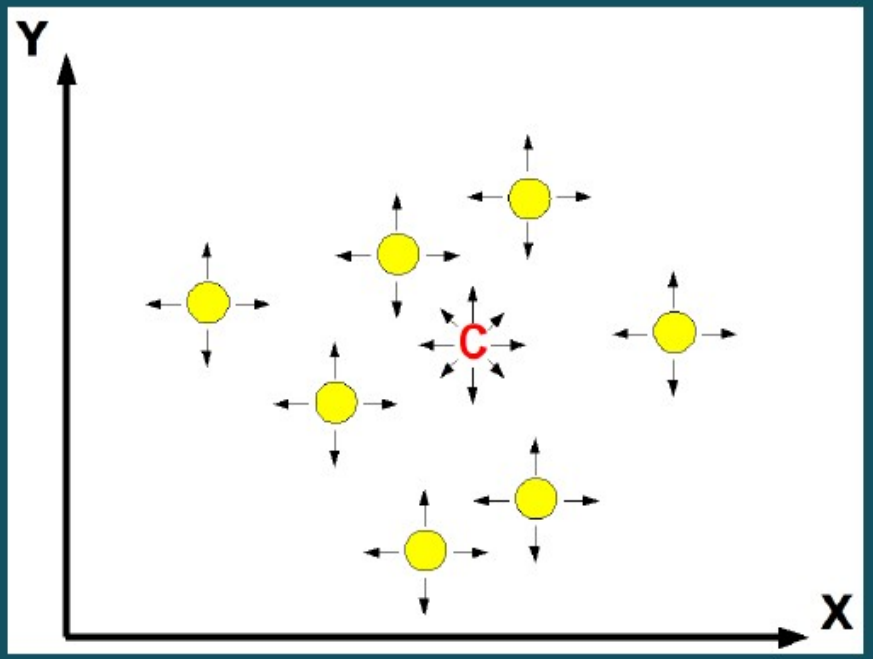
$$c^* = min\left(\sum d_{ic}^p\right)^{\frac{1}{p}}$$

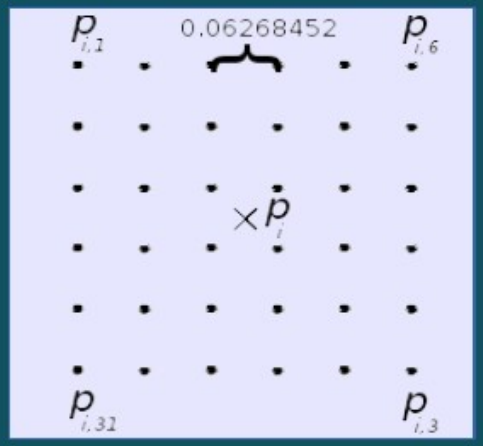
d_{ii} = distance between *i* and *c*

2D

Measurement of influence

- Moving each point to assess the center location sensitivity





2D

$$w_i = \frac{\frac{1}{n}\sum_k \frac{\delta_{c\to c'}}{\delta_{i\to i'}}}{\sum_i \left(\frac{1}{n}\sum_k \frac{\delta_{c\to c'}}{\delta_{i\to i'}}\right)}$$

Relation between point influence and distance to the center

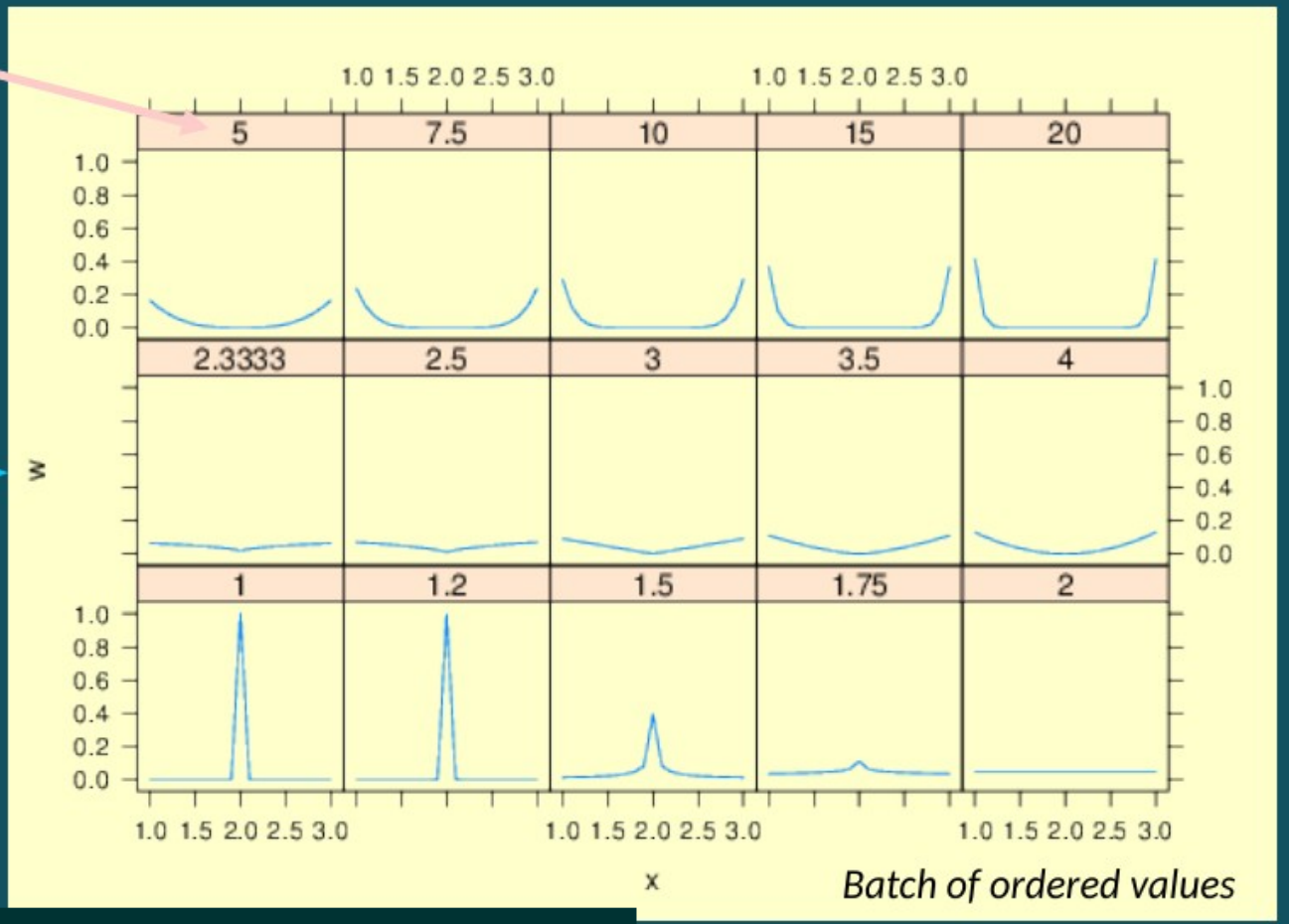
(Josselin & Ciligot-Travain, Environment & Planning B, 2015)

Results

- Plot of W_i and the ordered values, given the L_p-norm

1D

Weight

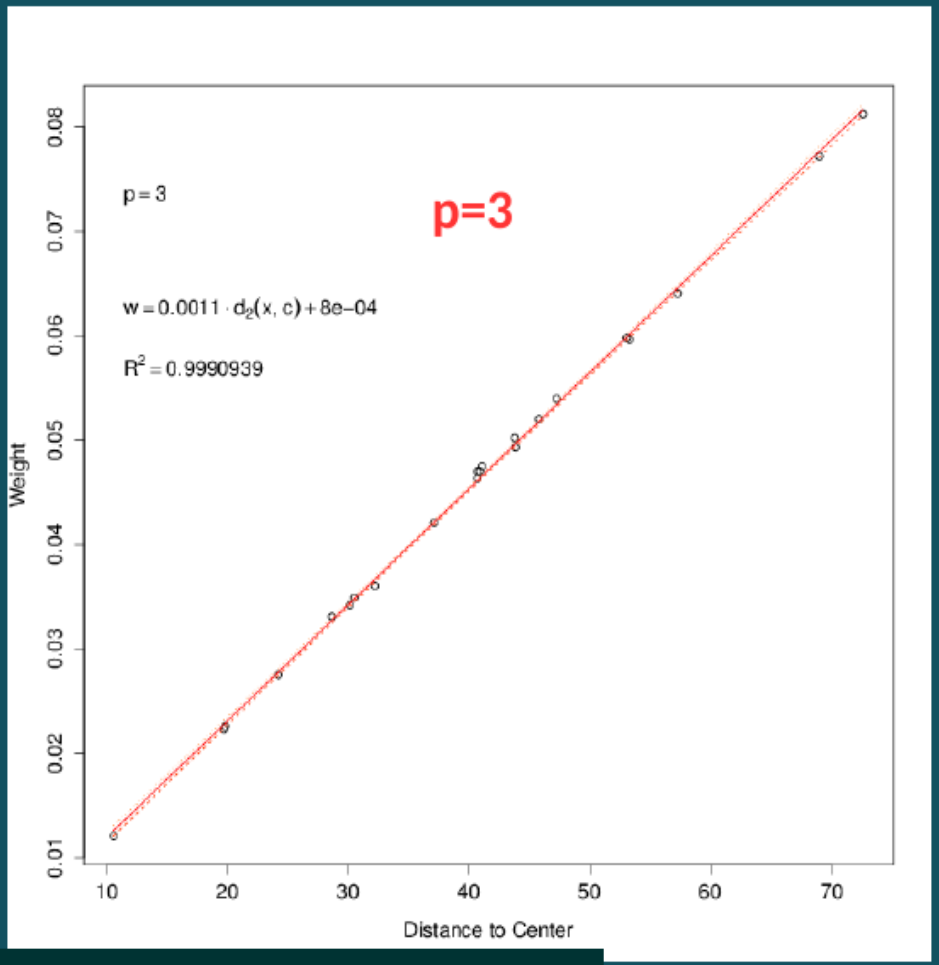


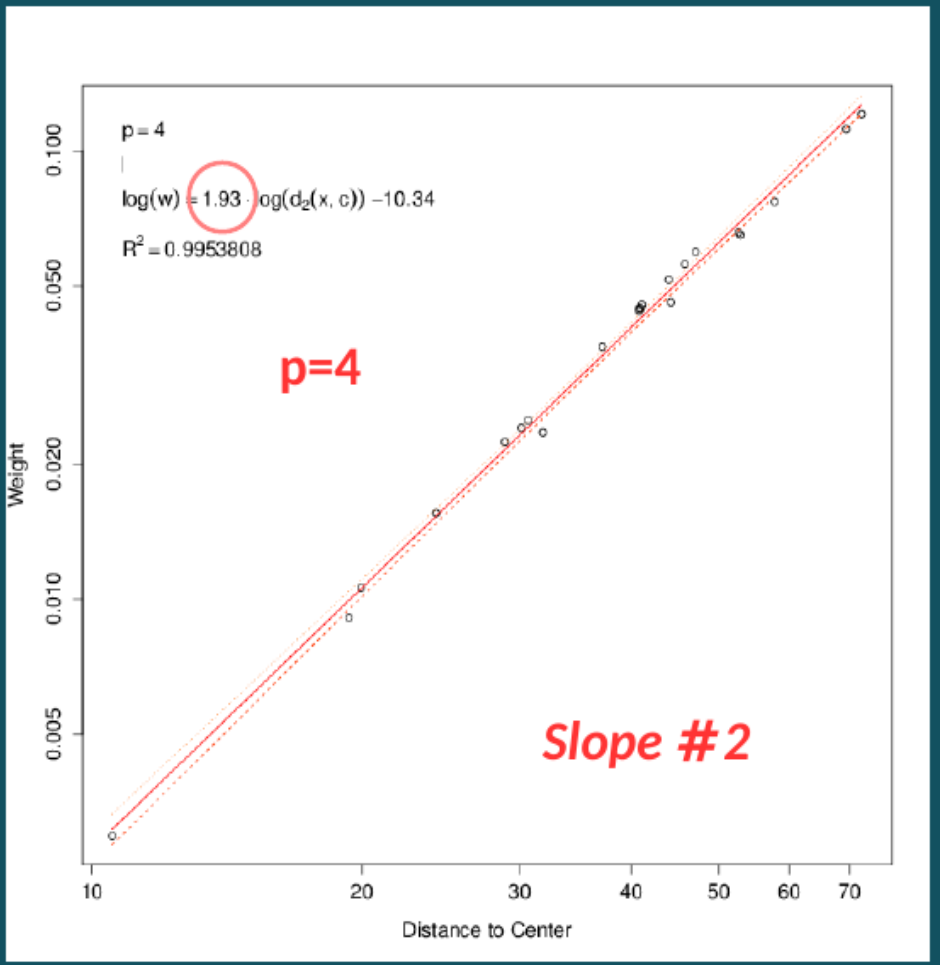
$$c^* = min\left(\sum d_{ic}^p\right)^{\frac{1}{p}}$$

Results

p'=2 (Euclidian distance between points i and center c)

- Linear relationship between W_i and d_{ic}

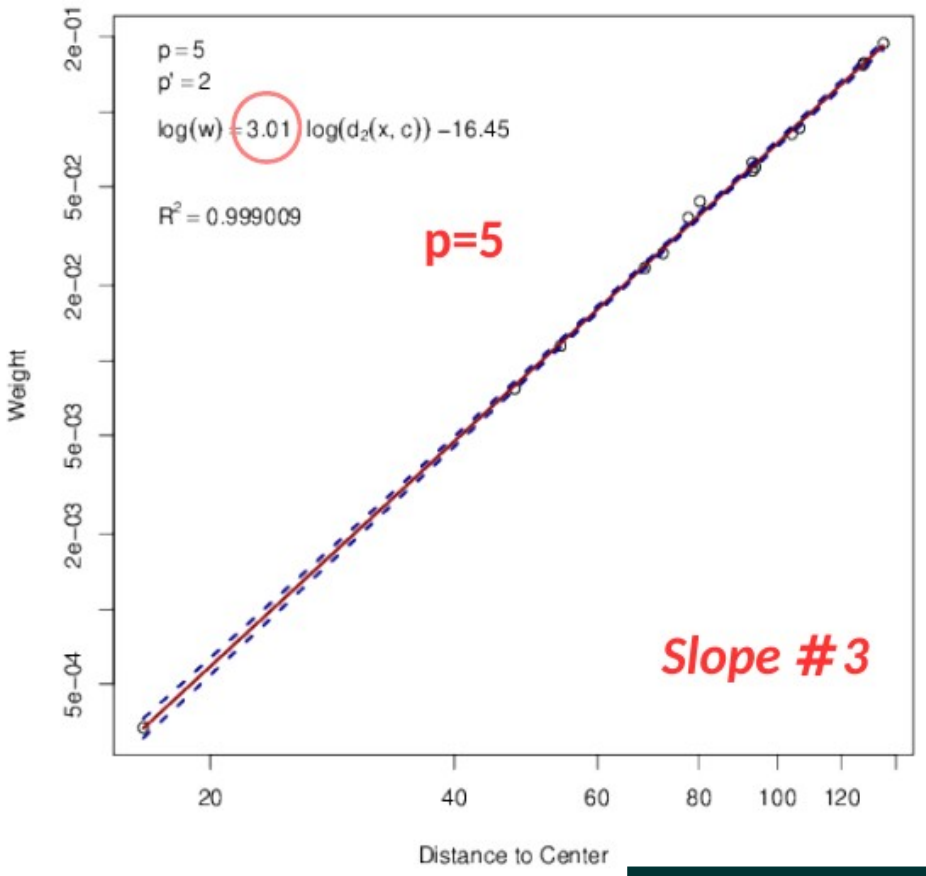


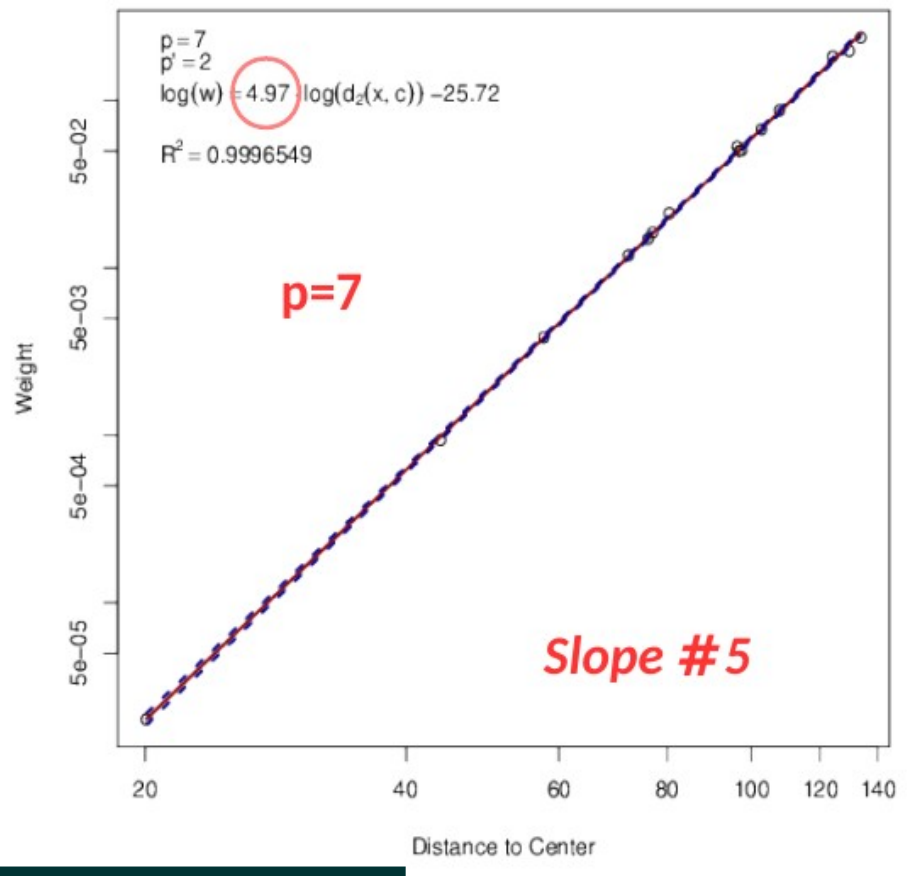


Results

p'=2 (Euclidian distance between points i and center c)

- Linear relationship between W_i and d_{ic}

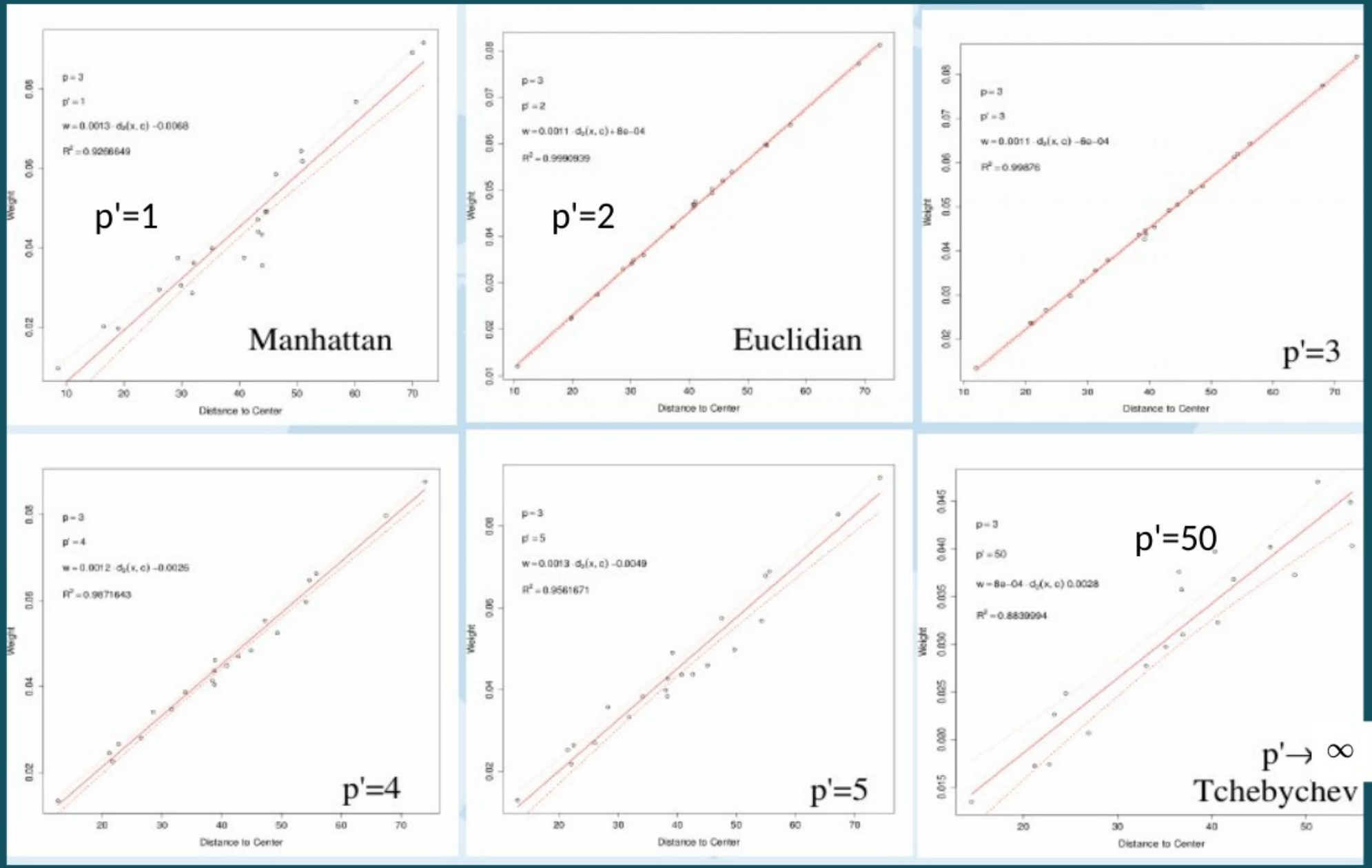




Results

p=3 (norm of the distances aggregation)

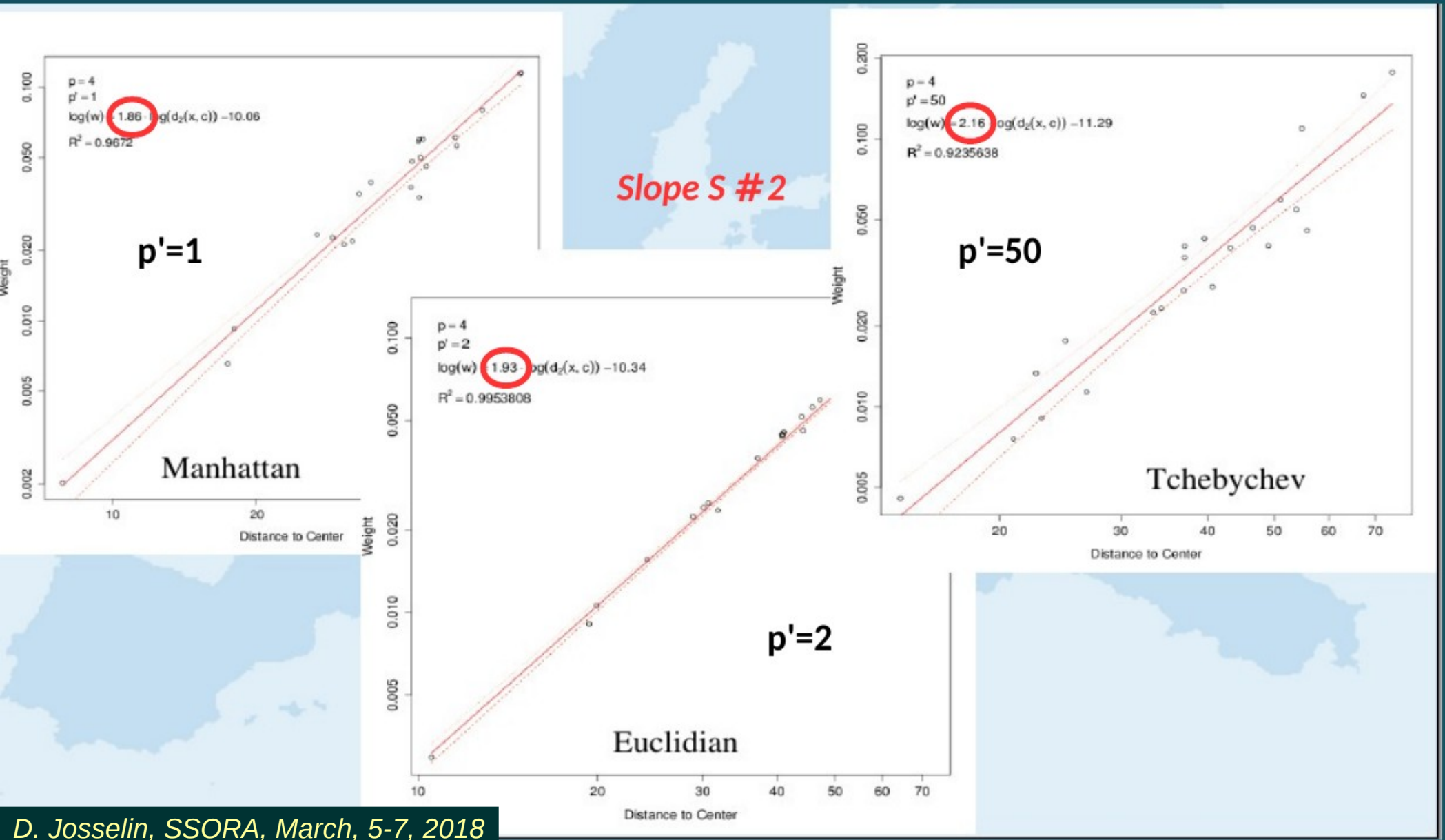
- Linear relations between W_i and d_{ic}, whatever p'



Results

p=4 (norm of the distances aggregation)

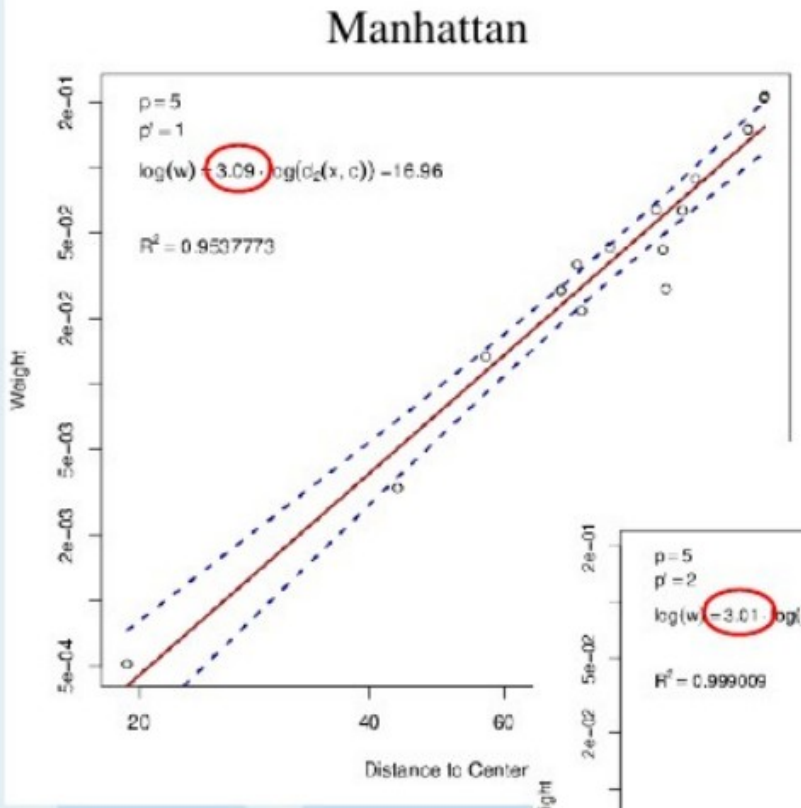
- Linear relations between W_i and d_{ic}, whatever p'

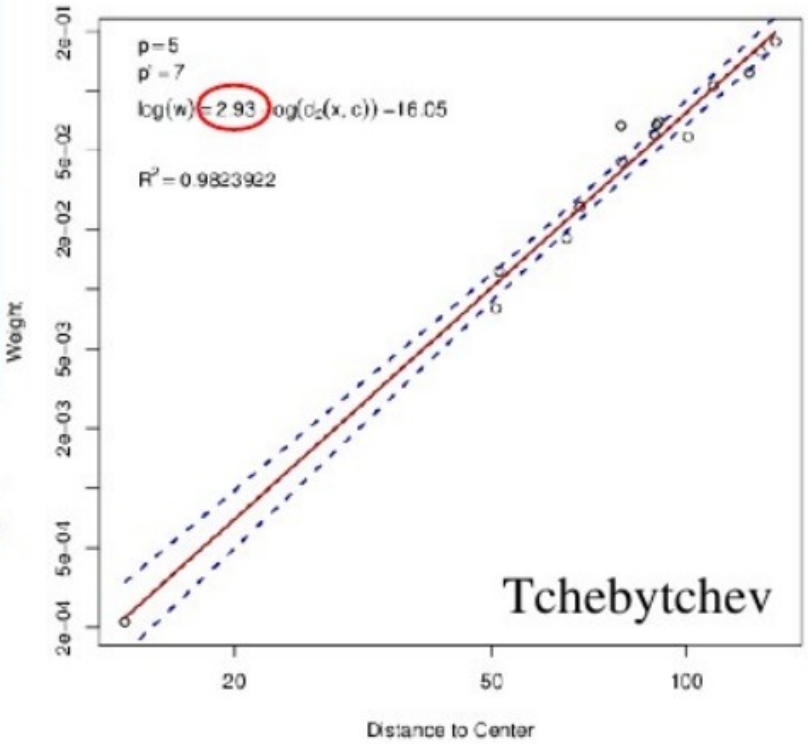


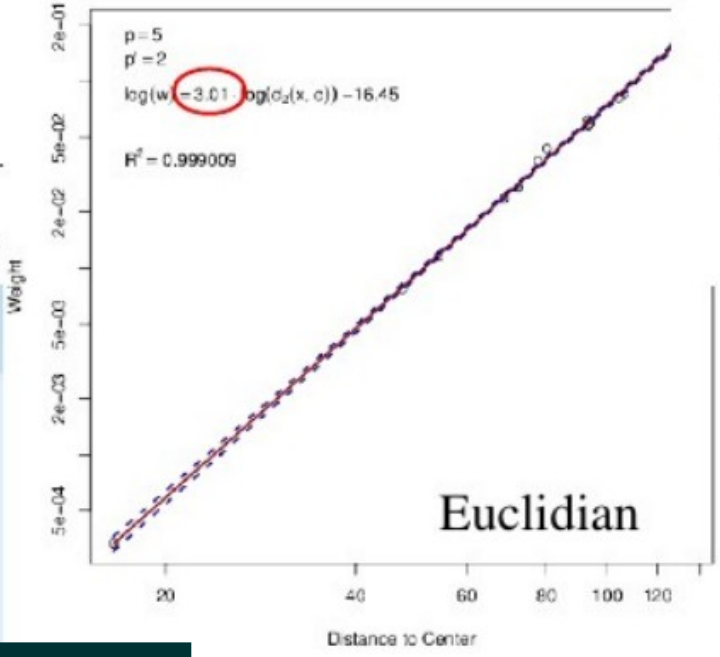
Results

p=5 (norm of the distances aggregation)

- Linear relations between W_i and d_{ic}, whatever p'





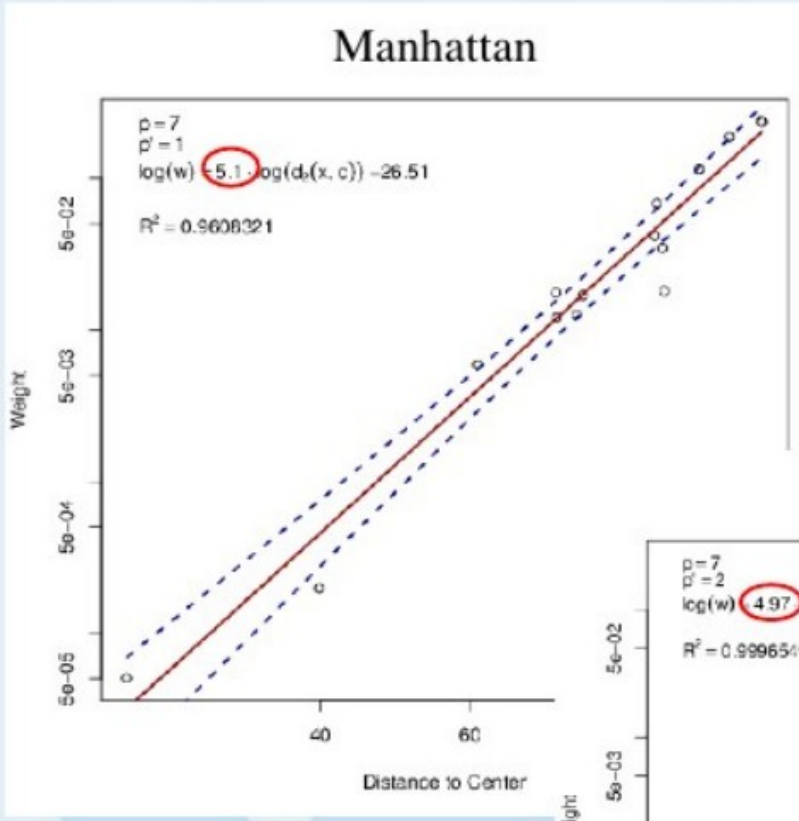


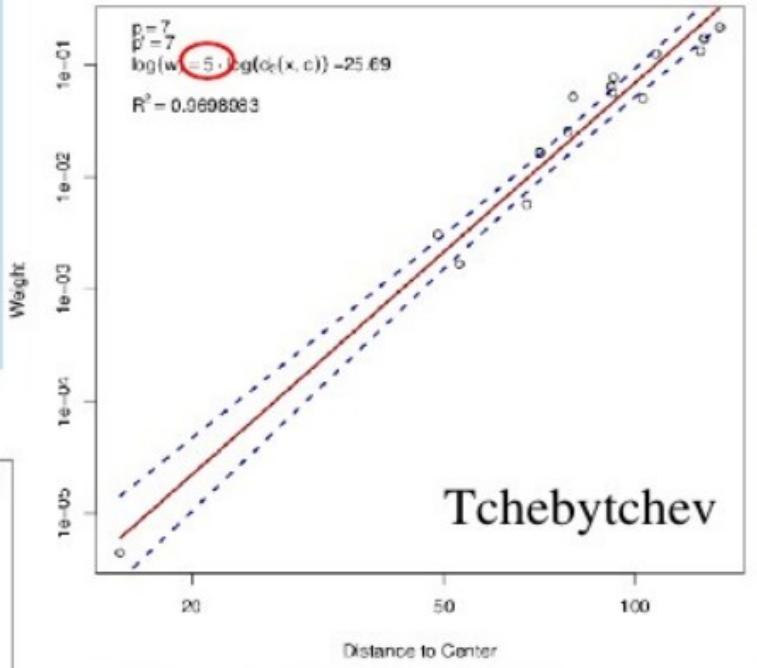
Slope S # 3

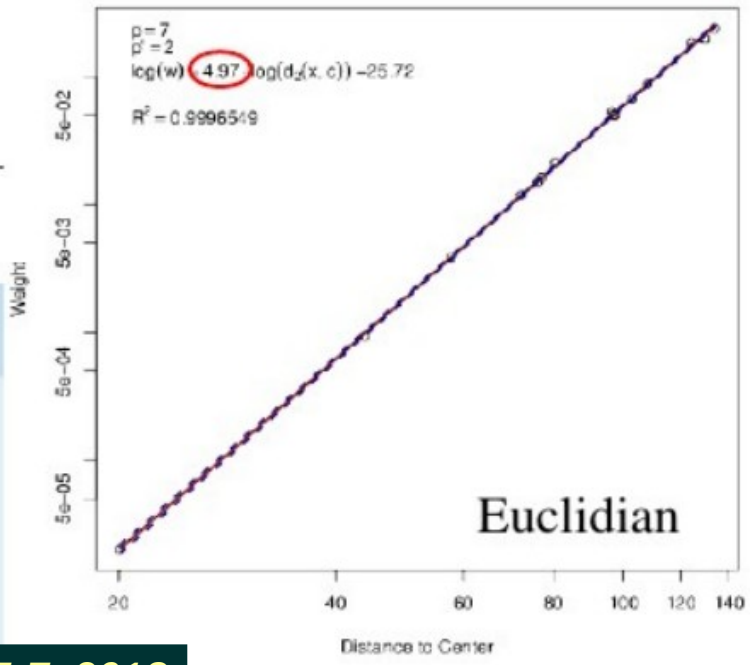
Results

p=7 (norm of the distances aggregation)

- Linear relations between W_i and d_{ic}, whatever p'







Slope S # 5

Generalization to L_p-norms

(Josselin et al., 2017)

Generalization

- Relationship between W_i and d_{ic}

$$ln(w_i) = (p-2)ln(d_{ic}) + r$$

with $d_{ic} > 0$

and $p \geq 1$

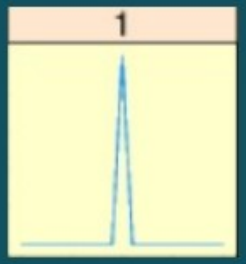
which is equivalent to:

$$w_i = \frac{d_{ic}^{(p-2)}}{e^{-r}}$$

$$r = \frac{ln(w_i)}{(p-2)ln(d_{ic})} = \frac{1}{p-2} log_{d_{ic}}(w_i)$$

Examples with standard L_p-norms

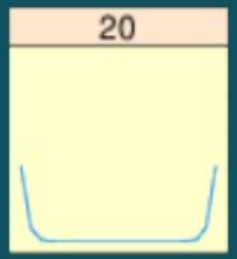
- *p=1*; *k*-median; efficiency



$$w_i = \frac{1}{d_{ic}e^{-r}}$$

$$r = -\frac{1}{2} log_{d_{ic}}(w_i)$$

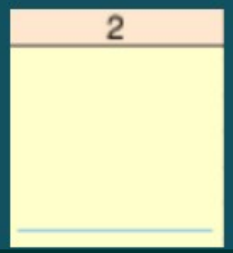
- $p \rightarrow \infty$; *k*-center; equity



$$w_i = \frac{d_{ic}^{(p-2)\rightarrow\alpha}}{e^{-r}}$$

$$r = \left[\frac{1}{(p-2)\rightarrow\alpha} log_{d_{ic}}(w_i)\right] \rightarrow 0$$

- *p=2*; *k*-barycenter; equality

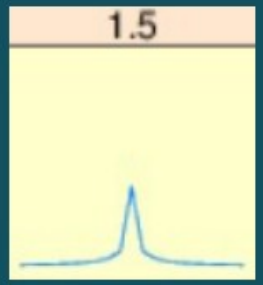


$$w_i = \frac{1}{e^{-r}}$$

$$r = ln(w_i)$$

Examples with other L_p-norms

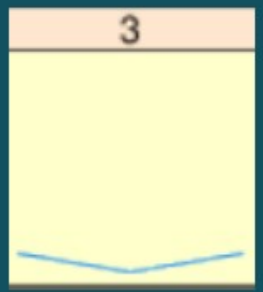
- *p=1.5* (compromise *efficiency-equality*)



$$w_i = \frac{d_{ic}^{-0.5}}{e^{-r}} = \frac{1}{e^{-r}\sqrt{d_{ic}}}$$

$$r = 2.log_{d_{ic}}(w_i)$$

- *p=3* (compromise *equality-equity*)



$$w_i = \frac{d_{ic}}{e^{-r}}$$

$$r = log_{d_{ic}}(w_i)$$

Mathematical modelling

(Gourion et al., 2017)

Derivation of the Minkowski function

$$c^* = \arg\min_c \left(\sum_{i=1}^{n} |d_{ic}|^p \right)^{\frac{1}{p}} = \arg\min_c \sum_{i=1}^{n} |d_{ic}|^p$$

$$f(x) = |x|^p \quad \Rightarrow \quad f'(x) = p|x|^{p-2}x$$

with $p \geq 1$ and $x \neq 0$

Influence of the demand point

$$\frac{\delta f}{\delta c} = p.d_{ic} \sum_{i=1}^{n} |d_{ic}|^{p-2}$$

1D

With the theorem of the implicit functions:

$$w_j = \frac{\delta c_p}{\delta x_j} = \frac{|d_{jc}|^{p-2}}{\sum_{i=1}^{m} |d_{ic}|^{p-2}}$$

with $p \geq 2$

Case of a geographical space (first results)

With *F* the function that links the center C_p and the point location X_i

$$\mathbb{R}^2 \Rightarrow \mathbb{R}^2$$ 2D

$$F(x_1, \ldots, x_m, c_p) = \sum_{i=1}^{m} \|c_p - x_i\|^{p-2} (c_p - x_i) = 0.$$

Jacobian matrix of *F*
relating to the coordinates of *C*

Coordinates in 2D: *k=1,2*

$$J_{F_c} = \begin{pmatrix} \frac{\partial F_1}{\partial c_1} & \frac{\partial F_1}{\partial c_2} \\ \frac{\partial F_2}{\partial c_1} & \frac{\partial F_2}{\partial c_2} \end{pmatrix}$$

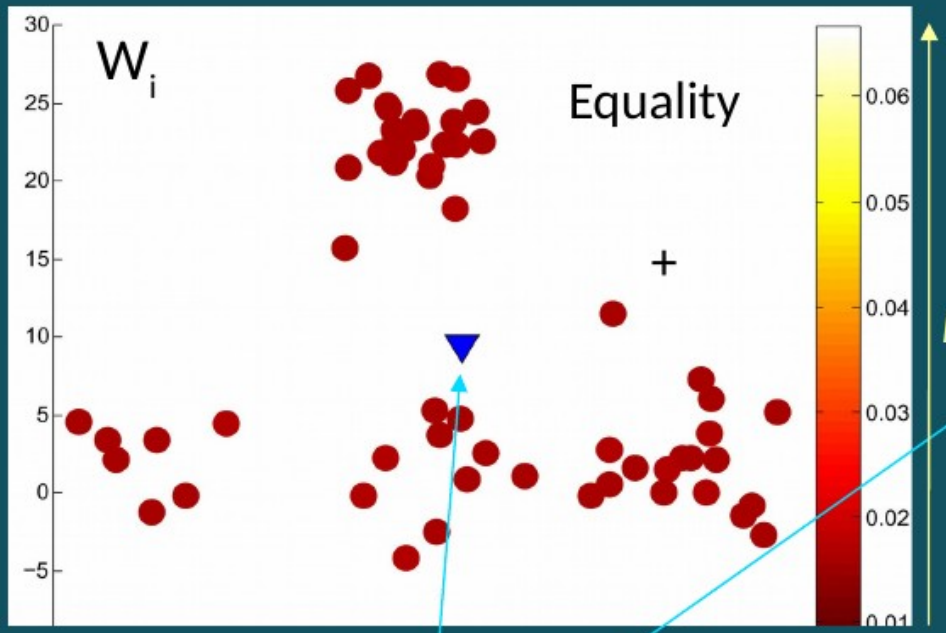
Applications in geographical space

Application in forest fire surveillance

Continuous space

Demand points

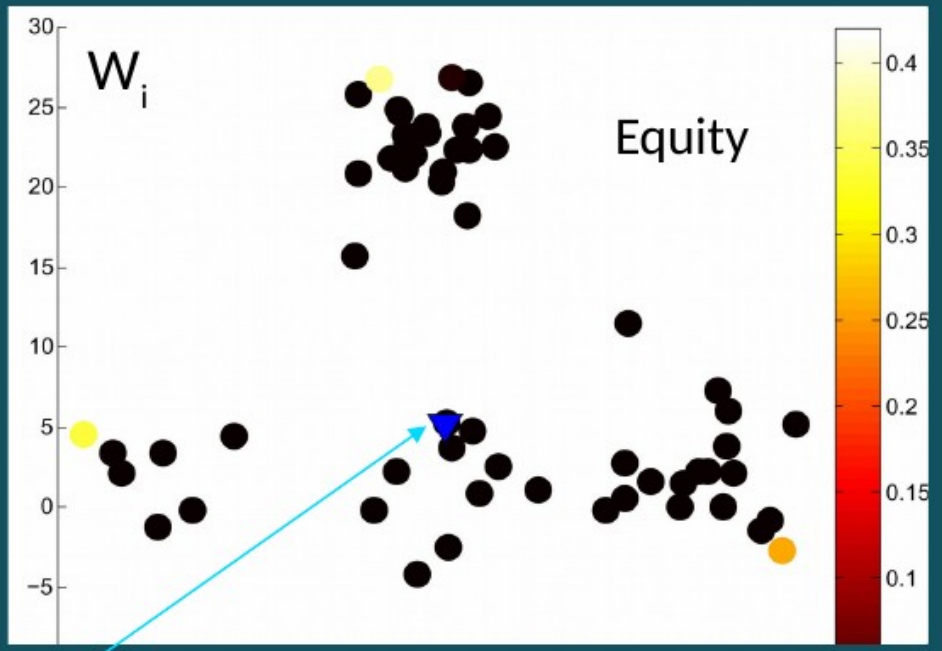
p=2



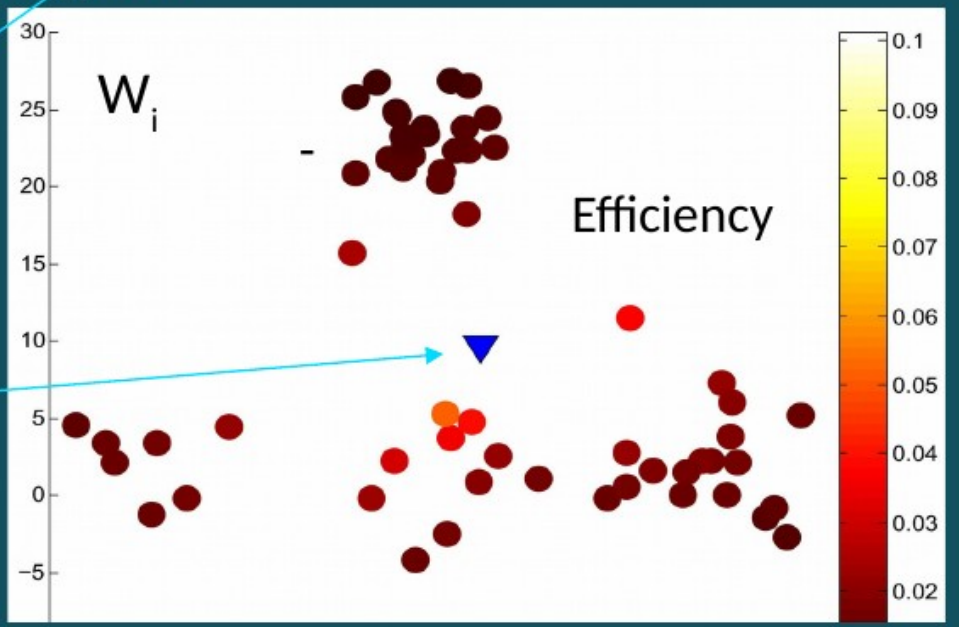
Center C

p=1

$p \to \infty$



More influence



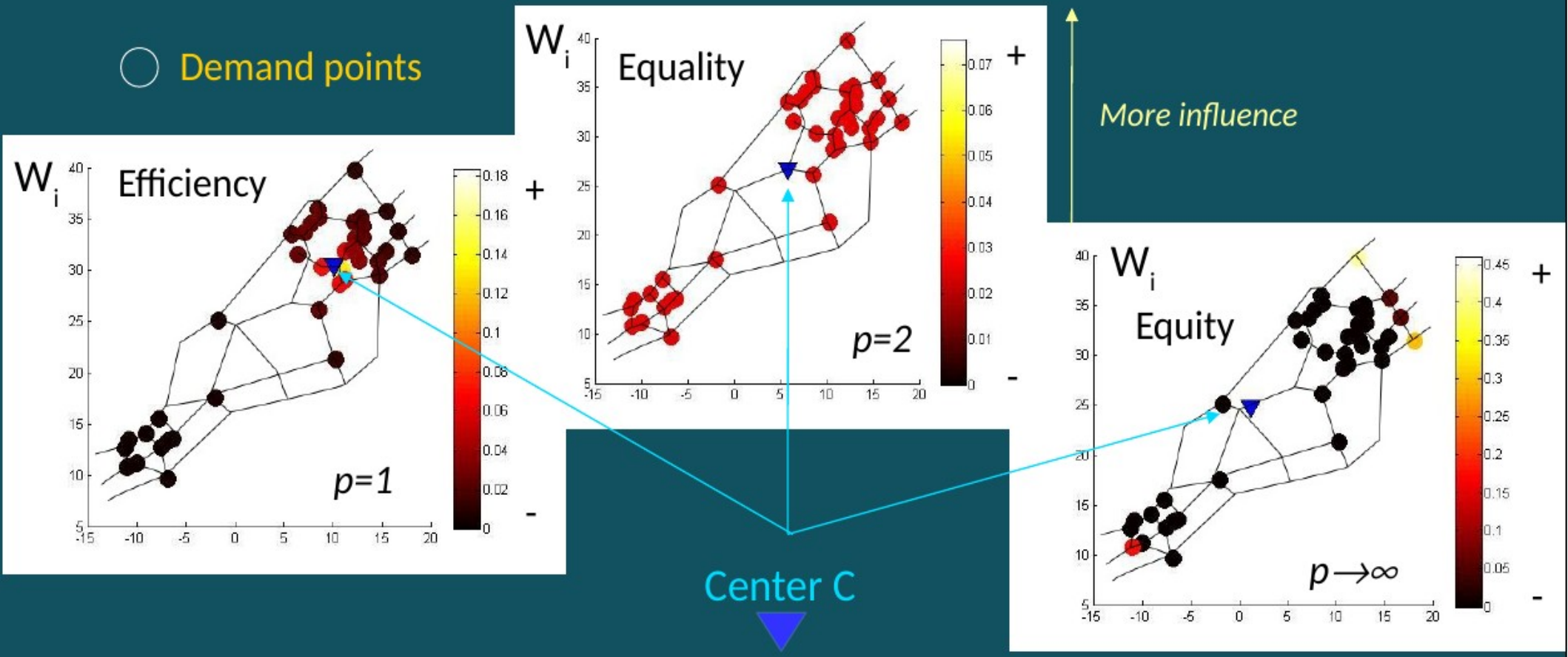
D. Josselin, SSORA, March, 5-7, 2018

Application on networks

Discrete space

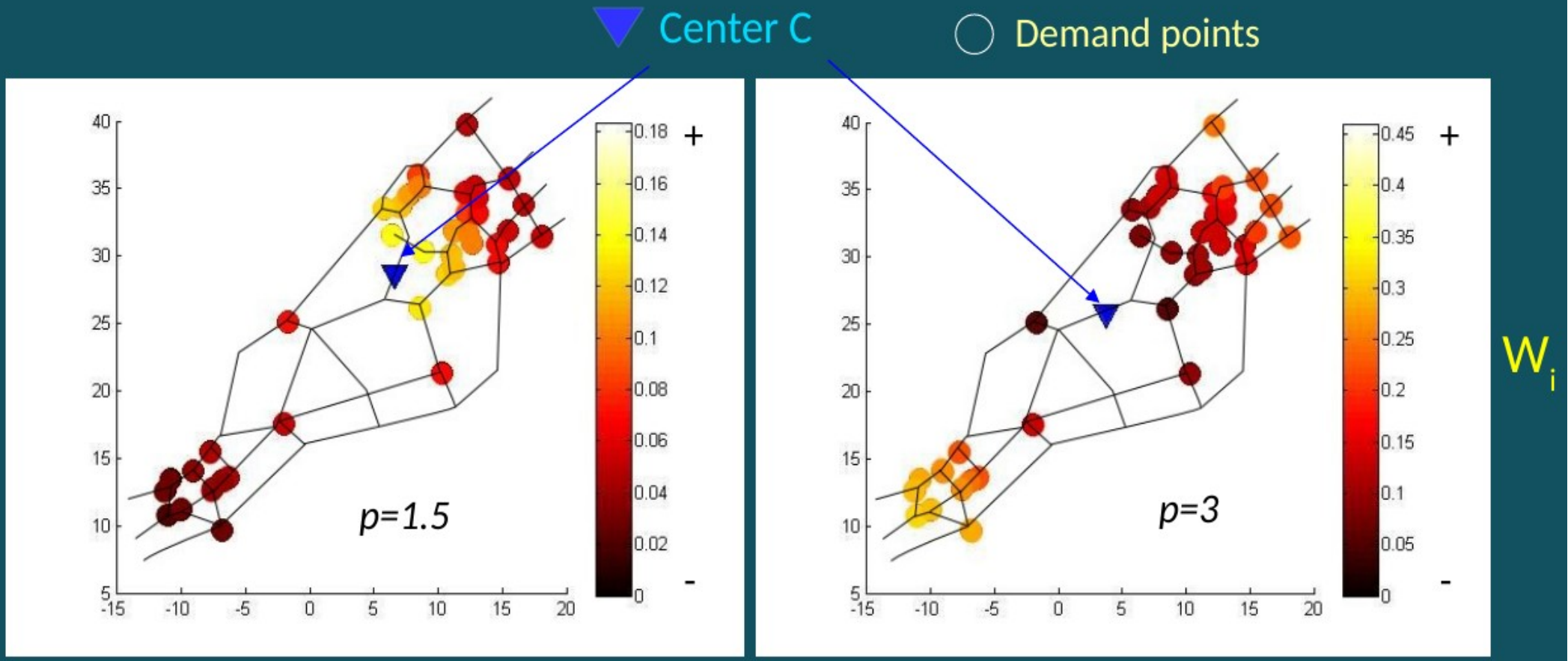
- Standard L_p-norms ($p=1$; $p=2$; $p \to \infty$)

Demand points



Application on networks

- Non-standard L_p-norms ($p=1.5$; $p=3$)



Conclusion

Conclusion (1)

- There exists a clear linear relation between the demand point influence and the distance to the center, whatever the metric
- This influence, related to center location sensitivity, can be assessed and mapped
- This method can be used to tune the metric used to fit a specific objective in transportation planning

Conclusion (2)

- The Tobbler's law can be revisited in a facility optimal location framework: because the norm chosen wears a part of the relation between the center and the demand points
- The type of distance *p'* considered between demand points and center locations does not really matter
- A new compromise (“equility”) norm with *p=3* appears to be very interesting (linear relationship)

The Generalized Center Optimal Location

Didier JOSSELIN , UMR ESPACE 7300, France

didier.josselin@univ-avignon.fr

Summer School on Operations Research and Applications, Nizhny Novgorod, March, 5-7, 2018