On the relation between Kantorovich's and Shannon's optimization problems

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Optimal transportation problems (OTPs)

Information and entropy

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Dual formulation of OTP

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Relation between $K_c[p,q]$ and $D_{KL}[p,q]$

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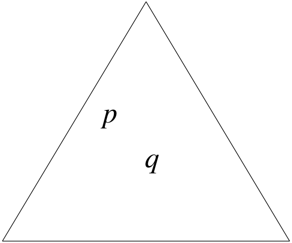

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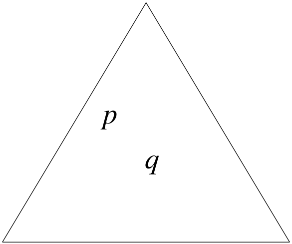

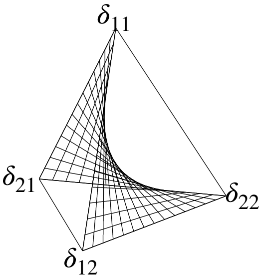

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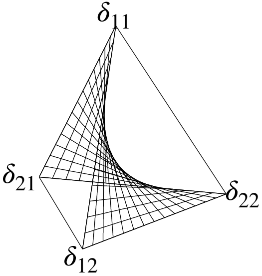

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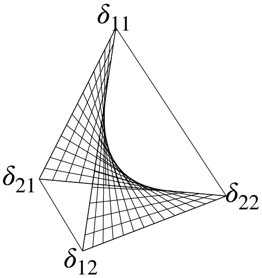

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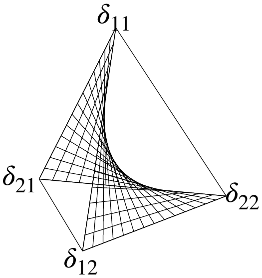

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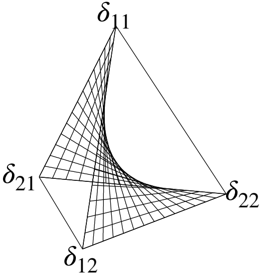
- T is determined by $w \in \mathcal{P}(X \otimes Y)$:

$$w = p(y \mid x) \otimes q$$



Monge OTP

Optimal Transportation Problem (Monge, 1781)

$$K_c[p,q] := \inf \left\{ \int_X c(x, f(x))\, dq : f : p = q \circ f^{-1} \right\}$$

where $p = q \circ f^{-1}$ is push-forward under measurable mapping $f : X \to Y$:

$$p(E) = q \circ f^{-1}(E) = q\{x : f(x) \in E\}$$

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- $p(E \mid x)$ has the form:

$$\delta_{f(x)}(E) = \begin{cases} 1 & \text{if } f(x) \in E \\ 0 & \text{otherwise} \end{cases}$$

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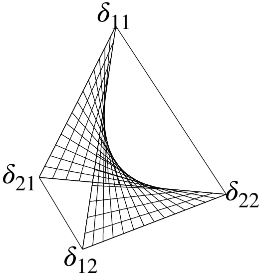
- $w_f \in \partial \mathcal{P}(X \otimes Y)$:

$$w_f(X, Y \setminus f(X)) = 0$$



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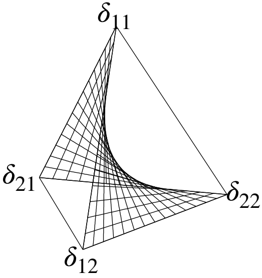

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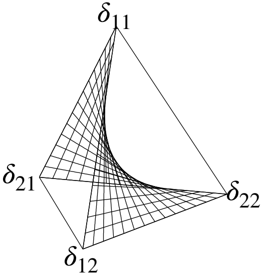
For $w \in \Gamma[q, p] \subset \mathcal{P}(X \otimes Y)$:

$$\begin{aligned} I_w\{x, y\} :=& D_{KL}[w, q \otimes p] \\ =& H[q] - H[q(x \mid y)] \\ =& H[p] - H[p(y \mid x] \end{aligned}$$



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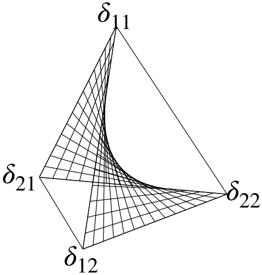
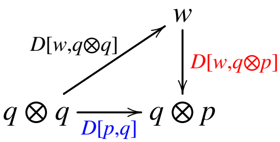
Entropy $H[p] = -\int \ln p \, dp$

$$H[p] := I_w\{y,y\} = \sup_{w : \pi_Y w = p} I_w\{x,y\}$$




w
$D[w,q\otimes q]$
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$q \otimes q$
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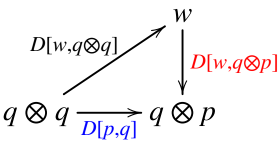
Theorem (Shannon-Pythagorean)

- $w \in \mathcal{P}(X \otimes Y)$, $\pi_X w = q$, $\pi_Y w = p$

$$D_{KL}[w, q \otimes q] = D_{KL}[w, q \otimes p] + D_{KL}[p, q]$$

(Belavkin, 2013a)



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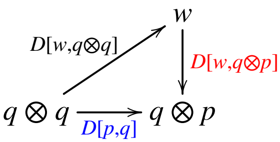

Proof.

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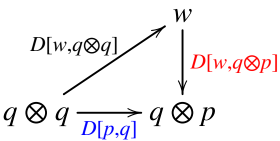

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Cross-Information (Belavkin, 2013a)

$$D_{KL}[w, q \otimes q] = \underbrace{-\langle \ln q, p \rangle}_{\text{Cross-entropy}} - \underbrace{\Big(H[p] - D_{KL}[w, q \otimes p]\Big)}_{H[p(y|x)]}$$

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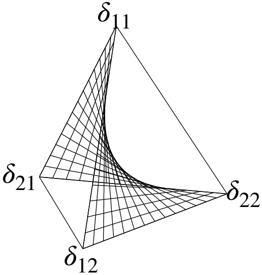

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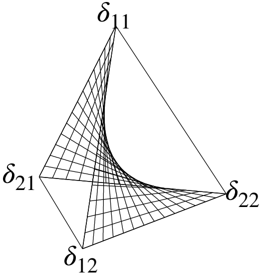
Exponential family solutions

- Optimal $T : \mathcal{P}(X) \to \mathcal{P}(Y)$ is defined by

$$w = e^{-\beta c - \ln Z}\, q \otimes p\,, \quad \beta^{-1} = -dR_c[q](\lambda)/d\lambda$$



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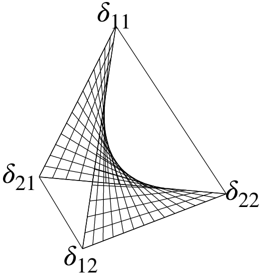

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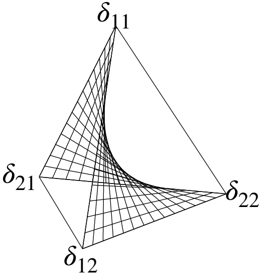

Value of Information (Stratonovich, 1965)

$$V(\lambda) := R_c[q](0) - R_c[q](\lambda) = \sup\{\mathbb{E}_w\{u\} \,:\, I_w\{x,y\} \leq \lambda\}$$

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- q and p have entropies $H[q]$ and $H[p]$ and

$$0 \leq I_w\{x,y\} \leq \min\{H[q], H[p]\}$$

- $K_c[q,p]$ has implicit constraint $I_w\{x,y\} \leq \lambda = \min\{H[q], H[p]\}$ and

$$R_c[q](\lambda) \leq K_c[q,p](\lambda)$$

Inverse Optimal Values

Inverse of the OCP Value

$$R_c^{-1}[q](\upsilon) := \inf\left\{ I_w\{x,y\} : \pi_X w = q,\ \int c\, dw \le \upsilon \right\}$$

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- These inverse values represent the smallest amount of Shannon's information required to achieve expected cost $\int c\,dw = \upsilon$.
- If $\upsilon = K_c[q,p](\lambda)$, then

$$R_c^{-1}[q](\upsilon) \leq K_c^{-1}[q,p](\upsilon)$$

Common Solution

Theorem

Let w_{OCP} *and* $w_{OTP} \in \mathcal{P}(X \times Y)$ *be optimal solutions to OCP and OTP problems with the same constraint* $I(x,y) \leq \lambda$. *Then* $R_c[q](\lambda) = K_c[p,q](\lambda)$ *if and only if* $w_{OCP} = w_{OTP} \in \Gamma[p,q]$.

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Proof.

- $w_{OCP} \in \partial D^*[-\beta c, q \otimes p]$ of subdifferential of D^* at $u = -\beta\, c$:

$$D^*[u, q \otimes p] = \ln \int_{X \times Y} e^{u(x,y)}\, dq(x)\, dp(y)$$

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Optimal transportation problems (OTPs)

Information and entropy

Optimal channel problem (OCP)

Dual formulation of OTP

Geometry of information divergence and optimization

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- Consider $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ such that

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- Can $J_c[p, q]$ be related to $D_{KL}[p, q]$?

KL-divergence

- $$\begin{aligned} D[p,q] &= D[p,r] + D[r,q] - \int_X \ln \frac{dq(x)}{dr(x)} \, [dp(x) - dr(x)] \\ &= D[p,r] - D[q,r] - \int_X \ln \frac{dq(x)}{dr(x)} \, [dp(x) - dq(x)] \end{aligned}$$

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- Let $f(x) - g(y) \leq c(x,y)$ satisfy also:
  $$\beta f(x) = \nabla D[p,r] = \ln \frac{dp(x)}{dr(x)}\,, \qquad \beta \geq 0$$
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- Thus $dp(x) = e^{\beta f(x) - \kappa[\beta f]} \, dr(x)$, $dq(x) = e^{\alpha \, g(x) - \kappa[\alpha g]} \, dr(x)$ and
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- Let us denote

$$J_{c,\varepsilon}[p,q] := \frac{1}{\varepsilon}\left[\beta\mathbb{E}_p\{f\} - \alpha\mathbb{E}_q\{g\}\right]$$

where $\varepsilon = \inf\{\epsilon \geq 0 : \beta f(x) - \alpha g(y) \leq \epsilon\, c(x,y)\}$.

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Common solution for Dual OTP

Theorem

Let (f, g) be the solution to the dual OTP. If there exists a reference measure $r \in \mathcal{P}(X)$ such that $f = \nabla D[p, r]$ and $g = \nabla D[q, r]$, then

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Proof.

- $f = \nabla D[p, r]$ and $g = \nabla D[q, r]$ imply that $\alpha = \beta = 1$, and

$$p = \exp(f - \kappa[f])\,r\,, \quad q = \exp(g - \kappa[g])\,r$$

- The result follows, because

$$\mathbb{E}_p\{f\} - \mathbb{E}_q\{g\} = J_c[p, q] = K_c[p, q]$$

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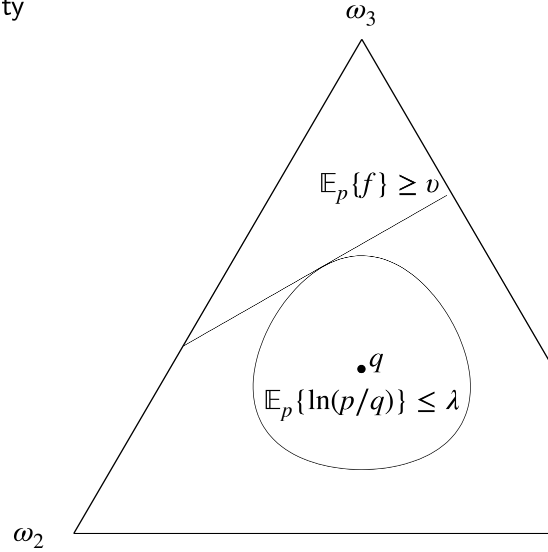
Problems on Conditional Extremum

- $\mathbb{E}_p\{u\} = \langle u, p \rangle$ expected utility



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- $\mathbb{E}_p\{u\} = \langle u, p \rangle$ expected utility
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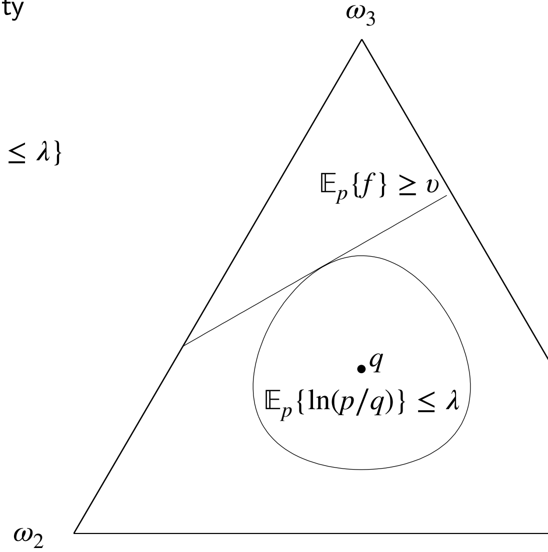

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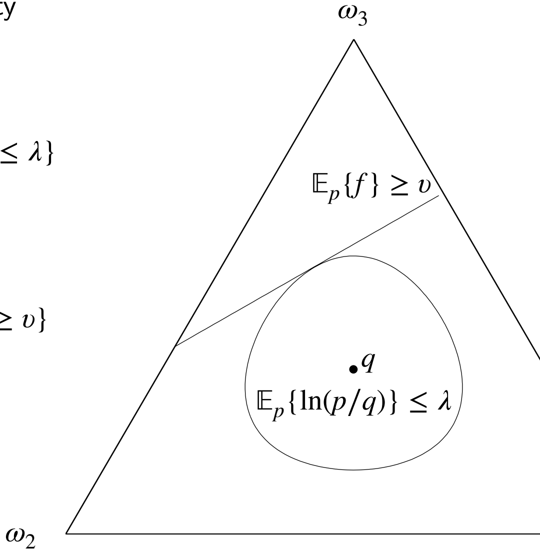

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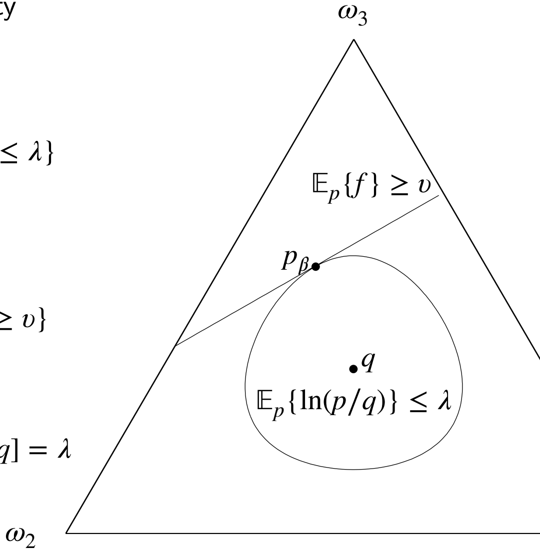

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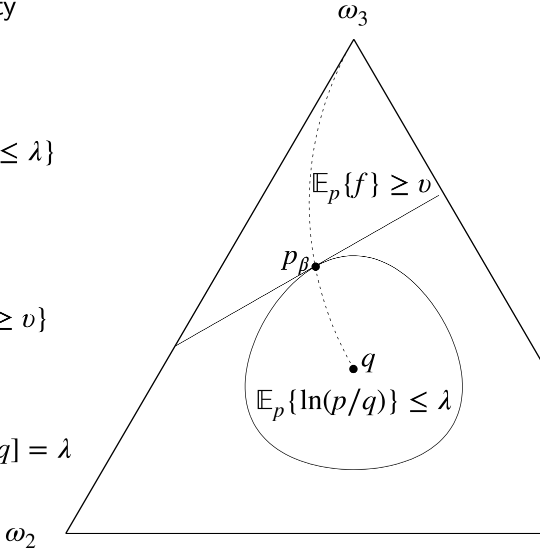

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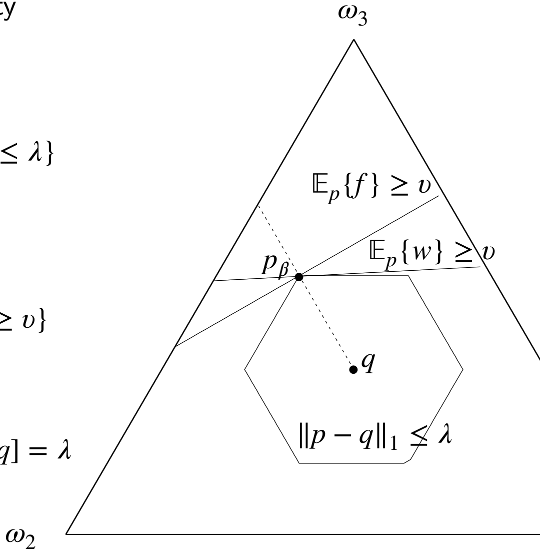

General Solution

- Lagrange function for $\upsilon_u(\lambda) := \sup\{\langle u, p\rangle : F[p,q] \leq \lambda\}$ $\big(\lambda_u(\upsilon) := \inf\{F[p,q] : \langle u, p\rangle \geq \upsilon\}\big)$:

$$L(p, \beta^{-1}) = \langle u, p\rangle + \beta^{-1}(\lambda - F[p,q])$$

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- Optimal solutions are subgradients of $F^*[u,q] = \sup\{\langle u, p\rangle - F[p,q]\}$:

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Example: Exponential Solution

- For $F[p,q] = D_{KL}[p,q]$:

$$L(p,\beta^{-1}) = \langle u,p\rangle + \beta^{-1}(\lambda - \langle \ln(p/q),p\rangle + \langle 1,p-q\rangle)$$

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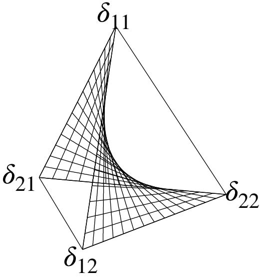
- Optimal solutions are gradients of $D^*_{KL}[u,q] = \ln\langle e^u, q\rangle$:

$$p(\beta) = e^{\beta\, u - \ln Z[\beta u]}\, q\,, \qquad D_{KL}[p(\beta),q] = \lambda$$

Solution to Shannon's OCP

- The solution for $I_w\{x,y\} = D_{KL}[w, q \otimes p] \leq \lambda$:

$$w(\beta) = e^{\beta u - \ln Z[\beta u]}\, q \otimes p$$



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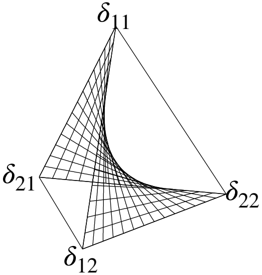

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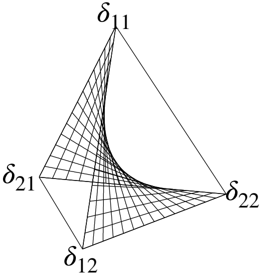

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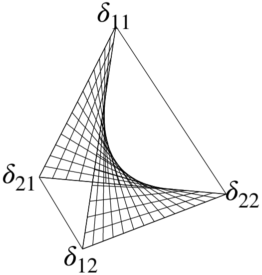
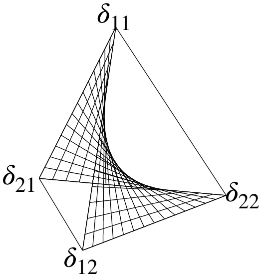


Solution to Kantorovich's OTP

- $\Gamma[q, p]$ is convex:

$$\pi_X[(1-t)w_1 + tw_2] = (1-t)q + tq = q$$



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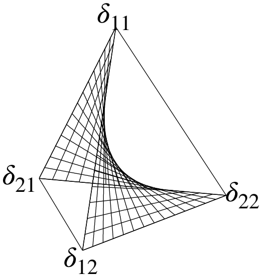

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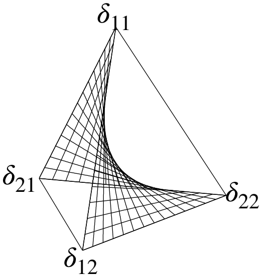

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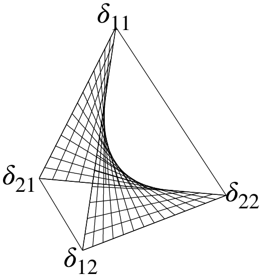

Monge-Amper equation

$$q = p \circ \nabla\varphi |\nabla^2 \varphi|$$

where $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is convex, and $\nabla\varphi : X \to Y$ is such that $p = q \circ (\nabla\varphi)^{-1}$ (McCann, 1995; Villani, 2009).

Strict Inequalities

Theorem (Belavkin, 2013b)

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$$\textit{maximizing } \mathbb{E}_w\{u\} \textit{ on sets } \{w : F[w] \leq \lambda\}\,,\ \forall\, \lambda = F[w]$$

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Strict Bounds for Monge OTP

Corollary

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- *If* w_f *and* $w(\beta)$ *have equal* $I_{w_f}\{x,y\} = I_{w(\beta)}\{x,y\} = \lambda < \sup I_w\{x,y\}$*, then*

$$K_c[p,q](\lambda) > R_c[q](\lambda) > 0$$

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- *If* w_f *and* $w(\beta)$ *achieve equal values* $K_c[p,q] = R_c[q](\lambda) = \upsilon > 0$, *then*

$$K_c^{-1}[p,q](\upsilon) > R_c^{-1}[q](\upsilon)$$

Optimal transportation problems (OTPs)

Information and entropy

Optimal channel problem (OCP)

Dual formulation of OTP

Geometry of information divergence and optimization

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