Asymptotic behaviour of biharmonic heat equations on unbounded domains

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Abstract. It is well-known that the heat equation $u_t - \Delta u = 0$ on $\mathbb{R}^n$ enjoys a *positivity preserving property*: if $u_0 \geq 0$ is a non-trivial function, then the solution $u = u(t,x)$ to the heat equation with initial datum u_0 satisfies $u(t,x) > 0$ for all $t > 0$. This phenomenon is closely connected with maximum principles for the Laplacian $-\Delta$ and second-order elliptic operators in general. On the other hand, one cannot expect the positivity preserving property to hold for higher-order elliptic operators. Nevertheless, it seems that positivity in some sense is "almost" preserved. As a particular case, the biharmonic heat equation $u_t + (-\Delta)^2 u = 0$ on $\mathbb{R}^n$ displays *local eventual positivity*, as shown in [5] and elaborated in [4]. Roughly speaking, this means that given non-trivial initial datum $u_0 \geq 0$, for every compact set $K \subset \mathbb{R}^n$, there exists a time $T > 0$ such that the corresponding solution $u = u(t,x)$ is positive on K for all $t \geq T$. Intuitively, this phenomenon occurs as a result of the oscillatory behaviour of the fundamental solution, and so far, local eventual positivity for these equations has been studied via explicit analysis of such biharmonic heat kernels.

From an abstract perspective, solutions to evolution equations may be studied via an appropriate semigroup of linear operators. The study of positive operator semigroups is by now a classic topic. However, as remarked above, this framework does not apply directly to higher-order evolution equations. One may instead use the theory of *eventually positive semigroups*, which was first developed systematically in [2,3]. Very recently, a localised version of the theory was initiated in [1]. The results of these papers are effective in particular for studying positivity in elliptic or parabolic problems on bounded domains where the associated operator has a simple principal eigenvalue. For this reason, however, the methods cannot be adapted to evolution equations on unbounded domains.

The current work therefore lies at the crossroads of the explicit methods of [4,5] and the semigroup methods in [1,2,3]. We study the asymptotic behaviour of solutions to the biharmonic heat equation on $\mathbb{R}^n$ as well as on 'infinite cylinders' of the form $\mathbb{R} \times \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^∞ boundary. The main tools are the Fourier transform and a classical blow-up argument, which reveals the asymptotic profile of the solutions. As a consequence of our results, we demonstrate how the local eventual positivity of solutions may be obtained qualitatively (i.e. without use of explicit heat kernels), and for a larger class of initial data than was previously considered. The analysis on the infinite cylinder is modelled loosely on the $\mathbb{R}^n$ problem, but in addition uses various properties of a family of fourth-order eigenvalue problems, which may be of independent interest.

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