On perturbations of one-parameter semigroups
determined by covariant operator valued measures on the half-axis
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A one-parameter family of contractions \( T_t : X \to X, \ t \geq 0 \), acting on a Banach space \( X \) is said to be a semigroup if \( T_{t+s} = T_t T_s, \ t, s \geq 0, \) and \( T_0 = I \) (the identical transformation). If orbits of the semigroup \( T = (T_t) \) are continuous in some topology then there exists a linear
operator \( \mathcal{L} \) with the domain \( \mathcal{D}(\mathcal{L}) \) dense in \( X \) in the same topology such that \( T_t = \exp(t\mathcal{L}), \ t \geq 0 \). The operator \( \mathcal{L} \) is said to be a generator of the semigroup \( T \) [1]. For the case \( X = \mathcal{B}(H) \) (the algebra of all bounded operators in a Hilbert space \( H \)) a perturbation of \( \mathcal{T} \) of the semigroup \( T \) can be defined as a solution to the integral equation

\[
\hat{T}_t - \int_0^t \mathcal{M}(ds)\hat{T}_{t-s} = T_t, \ t \geq 0,
\]

where \( \mathcal{M} \) is a measure on the half-axis taking values in the set of all completely positive maps
on the algebra \( \mathcal{B}(H) \). To define a semigroup the measure \( \mathcal{M} \) should be covariant with respect
to the action of \( T \) in the sense

\[
T_r \circ \mathcal{M}([t, s]) = \mathcal{M}([t + r, s + r]), \ r \geq 0, \ s \geq t \geq 0.
\]

Let us go back to an arbitrary Banach space \( X \). If we consider a perturbation of the generator \( \mathcal{L} \) by a bounded operator \( \Delta \) in \( X \), then the operator \( \hat{\mathcal{L}} = \mathcal{L} + \Delta \) having the same domain
\( \mathcal{D}(\hat{\mathcal{L}}) = \mathcal{D}(\mathcal{L}) \) is a generator of the semigroup that is a solution to the integral equation
determined by the covariant measure

\[
\mathcal{M}([t, s]) = \int_t^s T_r \Delta dr, \ s, t \geq 0.
\]

More complicated cases that lead to a change of the domain of a generator are defined by non-
trivial cohomologies of \( T \). We consider two examples in which a crucial role is played by the
semigroup \( S = (S_t) \) consisting of right shifts in the Hilbert space \( H = L^2(\mathbb{R}_+) \). In one example,
perturbations of the semigroup \( S \) are introduced in [3-4]. The second example determines the
construction of perturbation for the semigroup \( T = (T_t) \) acting in \( X = \mathcal{B}(\mathcal{F}(H)) \) by the formula

\[
T_t(x) = \hat{S}_t x \hat{S}_t^*, \ x \in \mathcal{B}(\mathcal{F}(H)),
\]

where \( \hat{S}_t \) acts in the antisymmetric Fock space \( \mathcal{F}(H) = \{\mathbb{C}\Omega\} \oplus H \oplus H^{\otimes 2} \oplus \cdots \oplus H^{\otimes 2} \oplus \cdots \)
over one-particle Hilbert space \( H = L^2(\mathbb{R}_+) \) by the formula

\[
\hat{S}_t \Omega = \Omega, \ S_t(f_1 \Lambda f_2 \Lambda \cdots \Lambda f_n) = S_t f_1 \Lambda S_t f_2 \Lambda \cdots \Lambda S_t f_n, \ t \geq 0, \ f_j \in H.
\]

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