Pointwise convergence of integral kernels for Feynman-Trotter path integrals

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This contribution is based on joint work with F. Nicola (Polytechnic University of Turin).

Introduction. The Feynman path integral formulation of quantum mechanics is universally recognized as a milestone of modern theoretical physics. Roughly speaking, the core principle of this picture provides that the integral kernel of the time-evolution operator shall be expressed as a "sum over all possible histories of the system". This phrase entails a sort of integral on the infinite-dimensional space of suitable paths, to be interpreted in some sense as the limit of a finite-dimensional short-time approximation procedure. In spite of the suggestive heuristic insight, the quest for a rigorous theory of Feynman path integrals is far from over, as evidenced by the wide variety of mathematical approaches developed over the last seventy years - cf. [1] and the references therein for a broad introductory account.

Lagrangian formulation via the Trotter formula. Among the several proposed frameworks, the closest one to Feynman's original intuition is probably the time-slicing approximation due to E. Nelson [4]. In short, if $U(t)$ is the Schrödinger time evolution operator with Hamiltonian $H = H_0 + V$ (free particle plus a suitable potential perturbation), then the Trotter product formula holds for all $f \in L^2(\mathbb{R}^d)$:

$$U(t)f = e^{-\frac{i}{\hbar}t(H_0+V)}f = \lim_{n\to\infty} E_n(t)f, \quad E_n(t) = \left(e^{-\frac{i}{\hbar}\frac{t}{n}H_0}e^{-\frac{i}{\hbar}\frac{t}{n}V}\right)^n.$$

Integral representations for the approximate propagators $E_n(t)$ can be derived, so that the Trotter formula allows one to give a precise meaning to path integrals by means of a sequence of integral operators.

The problem of pointwise convergence. Notwithstanding the convergence results in suitable operator topologies, a closer inspection of Feynman's writings suggests that his original intuition underlay the much more difficult and widely open problem of the pointwise convergence of the integral kernels of the approximation operators $E_n(t)$ to that of $U(t)$. In the recent paper [5] we addressed this problem by means of function spaces and techniques arising in the context of time-frequency analysis. The toolkit of Gabor analysis has been fruitfully applied to the study of path integrals only in recent times, leading to promising outcomes [6,7,8].

Main results. With reference to the notation above, we consider a setting where H_0 is the Weyl quantization of a real quadratic form, hence covering fundamental examples such as the free particle or the harmonic oscillator. In addition, we introduce a bounded potential perturbation V whose regularity is characterized in terms of the decay in phase space of its windowed Fourier transform (such levels of regularity are encoded by the so-called modulation spaces). This setting covers, and in fact extends, a case that is often met in the literature on mathematical path integrals - namely, the harmonic oscillator plus a bounded perturbation which is the Fourier transform of a complex (finite) measure (see for instance the pioneering works by K. Itô and the line of research developed by S. Albeverio, R. Høegh-Krohn and S. Mazzucchi).

We exploit techniques of Gabor analysis of pseudodifferential operators to prove that the problem of pointwise convergence has a positive answer under the previous assumptions. Precisely, we prove stronger convergence results which imply uniform convergence on compact subsets for the integral kernels in the Trotter formula.

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Our results hold for any fixed value of $t \in \mathbb{R} \setminus \mathfrak{E}$, where $\mathfrak{E}$ is a discrete set of exceptional times - in that case the integral kernels are genuine distributions. In the recent contribution [2] we characterized the properties of such distribution kernels (precisely, they are "mild distributions" in the sense of Feichtinger's Banach-Gelfand fundamental triple of harmonic analysis, cf. e.g. [3]) and we derived weaker convergence results in the sense of modulation spaces even for $t \in \mathfrak{E}$.

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