Resonant localized patterns in the Swift-Hohenberg equation

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The generalized Swift-Hohenberg equation (briefly, SHE)

$$u_t = \alpha u - (1+\Delta)^2 u + \beta u^2 - u^3$$

is the well known pattern-forming model equation and studying its solutions with different spatial structure is a very interesting problem both theoretically and in view of its various applications. We are interested here in its solutions with the localization property

$$\lim_{r\to\infty} u(t,\mathbf{r}) = 0, \quad \mathbf{r} = (x,y), \quad r^2 = x^2 + y^2.$$

Such solutions are important in many applications and were found in a various experiments, both natural and numerical ones.

In the Sobolev space of $H^2(\mathbf{R^2})$ this equation defines a gradient-like differential equation with the functional

$$\mathcal{F} = \int\limits_{\mathbf{R^2}} \{\frac{1}{2}[(1+\Delta)u]^2 - \frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \frac{1}{4}u^4\} dxdy,$$

therefore its stationary solutions are of the primary importance.

The existence of localized stationary solutions being rotationally invariant or radial, i.e. when u depends only on r, $u(r)$, is rather well studied ([1], [2], [3], and others). But numerical and natural experiments demonstrate also an existence of non-radial localized stationary patterns to this equation. Thus, it is an interesting problem to understand a genesis of such solutions and their possible shape.

Our strategy of searching non-radial stationary localized solution is to select some branch of radial solutions, for instance, fix β and vary α, and find those points on the branch, where the linearization of the equation on that radial solution experiences a bifurcation of the appearance of non-radial solution with the lesser symmetry group. The latter means a possible appearance of a solution being invariant w.r.t. a discrete symmetry group $\mathbf{Z}_n$ instead of symmetry group $\mathbf{S^1}$. Appearance of such solutions reminds the resonance phenomena when an elliptic equilibrium passes through the resonance of frequencies, from here is the title.

When studying the phenomenon we found the localized resonant solutions with invariant w.r.t the group $\mathbf{Z}_n$, $n = 2,3,4,5,6$. After a related bifurcation we continued the newborn solution numerically. To substantiate the results, we address to the polar coordinates and use the Galerkin approximations expanding solutions in the Fourier series in angular variable φ. This gives a system of differential equations in radial variable that is investigated.

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