Mathematical model of traffic flow at a regulated intersection

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Introduction. To begin with, consider a «intersection - a place of intersection, abutment or branching of roads at the same level, bounded by imaginary lines connecting, respectively, opposite, most distant from the center of the intersection, the beginning of the curving of the carriageway. Exits from adjacent territories are not considered as intersections»

Problem setting. Imagine an intersection with a changing mode of its passage (traffic light is on) in the form of an eight-edge geometric graph $\mathbf{G_1}$ (pic. 1). In this case, the conditions «continuity» and «flow balance » [1] will look like

$$u_1^1(l_1,t) = u_2^1(0,t) = u_3^1(l_3,t) = u_4^1(0,t) =$$
$$= u_5^1(l_5,t) = u_6^1(0,t) = u_7^1(l_7,t) = u_8^1(0,t),$$

$$d_1u_{1x}^1(l_1,t) - d_2u_{2x}^1(0,t) + d_3u_{3x}^1(l_3,t) - d_4u_{4x}^1(0,t)+$$
$$+d_5u_{5x}^1(l_5,t) - d_6u_{6x}^1(0,t) + d_7u_{7x}^1(l_7,t) - d_8u_{8x}^1(0,t) = 0,$$

$$u_{1x}^1(0,t) = u_{2x}^1(l_2,t) = u_{3x}^1(0,t) = u_{4x}^1(l_4,t) =$$
$$= u_{5x}^1(0,t) = u_{6x}^1(l_6,t) = u_{7x}^1(0,t) = u_{8x}^1(l_8,t) = 0.$$

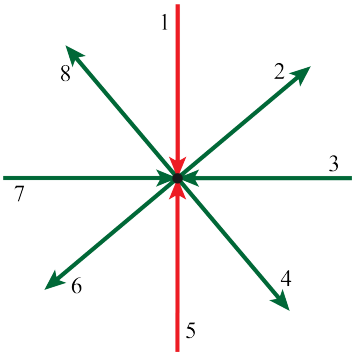

Figure 1: The intersection before the change of the traffic signal, during the time period $[\tau_{j-1}, \tau_j]$

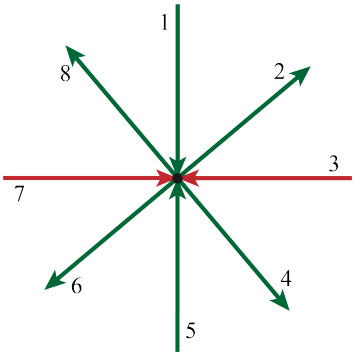

Figure 2: The intersection after changing the traffic signal, in the time period $[\tau_j, \tau_{j+1}]$

Let be k rib length l_k measured in linear metric units (kilometers or miles), however, in the mathematical model of traffic flow, the value is dimensionless. The number of lanes on the carriageway in one direction d_k will be called the capacity, similarly, in the context of the mathematical model, the value is dimensionless. Suppose that all adjacent roads at the intersection under consideration are equivalent, so we will assume that the capacity of each direction will be the same, i.e. $d_1 = d_2 = \ldots = d_8 = d$.

The traffic flow will be determined using the Oskolkov equations given on the graph $\mathbf{G_1}$

$$\lambda u_{kt}^1 - u_{ktxx}^1 = \nu u_{kxx}^1 + f_k^1, \ k = \overline{1,8}. \tag{1}$$

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Here $u_k^1 = u_k^1(x,t)$, $x \in [0, l_k]$, $t \in \overline{\mathbb{R}}_+$ ($\equiv \{0\} \cup \mathbb{R}_+$), $k = \overline{1,8}$, characterizes the average speed of the traffic flow on the set of edges E_k characterizes the average speed of the traffic flow on the set of edges of the graph $\mathbf{G_1}$. The average force that makes the wheels of vehicles spin, we will consider $f_k = f_k(x,t)$, $(x,t) \in [0, l_{ik}] \times \overline{\mathbb{R}}_+$.

Coefficient λ is equal to one divided by the retardation coefficient, which can take negative values, so we consider $\lambda \in \mathbb{R}$. The viscosity of the traffic flow, namely, its ability «extinguish» sudden changes in speed, sets the coefficient ν, by virtue of the physical meaning $\nu \in \mathbb{R}_+$.

Consider the intersection at the initial moment of turning on the traffic light, when the intersection mode changes from unregulated (yellow blinking traffic light mode) to regulated and designate this moment $t = \tau_0$. Suppose that at this moment on the first and fifth edges the traffic signal is red, i.e. for definiteness, we take the flow velocity on these edges to be equal to zero $u_1^1(x,\tau_0) = u_5^1(x,\tau_0) = 0$, on the remaining ribs $u_k^1(x,\tau_0) = u_{0k}^1(x)$, $k = 2,3,4,6,7,8$. In general, we write these conditions

$$P(u^1(x,\tau_0) - u_0^1(x)) = 0. \tag{2}$$

When the time is reached $t = \tau_1$ the traffic signal will change, the traffic flows at the intersection will be different, so we will consider a new graph $\mathbf{G_2}$ (pic. 2).

On this graph, the equations take the form

$$\lambda u_{kt}^2 - u_{ktxx}^2 = \nu u_{kxx}^2 + f_k^2,\ k = \overline{1,8}, \tag{3}$$

and the conditions «continuityPë» and «flow balance »

$$\begin{gathered} u_1^2(l_1,t) = u_2^2(0,t) = u_3^2(l_3,t) = u_4^2(0,t) = \\ = u_5^2(l_5,t) = u_6^2(0,t) = u_7^2(l_7,t) = u_8^2(0,t), \end{gathered}$$

$$\begin{gathered} du_{1x}^2(l_1,t) - du_{2x}^2(0,t) + du_{3x}^2(l_3,t) - du_{4x}^2(0,t) + \\ + du_{5x}^2(l_5,t) - du_{6x}^2(0,t) + du_{7x}^2(l_7,t) - du_{8x}^2(0,t) = 0, \end{gathered}$$

$$\begin{gathered} u_{1x}^2(0,t) = u_{2x}^2(l_2,t) = u_{3x}^2(0,t) = u_{4x}^2(l_4,t) = \\ = u_{5x}^2(0,t) = u_{6x}^2(l_6,t) = u_{7x}^2(0,t) = u_{8x}^2(l_8,t) = 0. \end{gathered}$$

When changing the traffic signal at the time $t = \tau_1$ the average speed of the third and seventh ribs will tend to zero, i.e.

$$\lim_{t \to \tau_2} u_3^2(x,t) = 0, \quad \lim_{t \to \tau_2} u_7^2(x,t) = 0.$$

In this case, on the remaining edges, the speed will be the one that is reached on the corresponding edge by the time τ_1, $u_k^2(x,\tau_1) = u_k^1(x,\tau_1) = u_{1k}^2(x)$, $k = 1,2,4,5,6,8$. In general terms, these conditions take the form

$$P(u^2(x,\tau_1) - u_1^2(x)) = 0. \tag{4}$$

Continuing the procedure for switching traffic lights at times $t = \tau_j$, $j = \overline{0,n}$, and for even n treat an intersection as a graph $\mathbf{G_1}$, and for odd n – as a count $\mathbf{G_2}$. In general, the multipoint initial-final condition takes the form

$$P(u^m(x,\tau_j) - u_j^m(x)) = 0,\ j = \overline{0,n},\ m = 1,2, \tag{5}$$

where τ_j – the moment of switching the traffic light [2].

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