Linear dynamical quantum systems: New directions and opportunities

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Introduction. The state-space representation of linear dynamical systems, deterministic and stochastic, lie at the heart of modern mathematical systems and control theory [1]. The stochastic case is described by linear stochastic differential equations of the form,

$$dx(t) = Ax(t)dt + Bdw(t) + Eu(t)dt,\ dy(t) = Cx(t)dt + Ddv(t) + Fu(t)dt,$$

where x is the state, u is the input signal, y is the output signal, w and v are the system and measurement Wiener noise vectors, respectively, and A, B, C, D are the real *system matrices* (of the appropriate dimensions). This class of models is the basis for the Kalman-Bucy stochastic filtering theory and linear quadratic Gaussian control [2].

In the quantum setting, one encounters a similar class of models, known as linear quantum systems [3]. In the Heisenberg picture of quantum mechanics, the model takes the form of a linear quantum stochastic differential equation (QSDE) [3,4],

$$dx(t) = Ax(t)dt + Bdw(t) + Eu(t)dt,\ dy(t) = Cx(t)dt + Ddw(t) + Fu(t)dt. \tag{1}$$

In this case, $x(t)$, $y(t)$ and $w(t)$ are vectors consisting of operators on the Hilbert space $\mathfrak{h}_n \otimes \Gamma_s(L^2([0,\infty);\mathbb{C}^m))$, where $\mathfrak{h}_n$ is the Hilbert space for n quantum harmonic oscillators and $\Gamma_s(L^2([0,\infty);\mathbb{C}^m))$ is the boson Fock space over the space of square-integrable complex functions on $[0,\infty)$ taking values in $\mathbb{C}^m$. In particular, $x(t) = (q_1(t), p_1(t), \ldots, q_n(t), p_n(t))^T$, $w(t) = (Q_1(t), P_1(t), \ldots, Q_m(t), P_m(t))^T$ and $y(t) = (Q_1'(t), P_1'(t), \ldots, Q_r'(t), P_r'(t))^T$ $(r \leq m)$, where $q_j(t)$ and $p_j(t)$ are the position and momentum operators of the j-th oscillator, $Q_j(t)$ and $P_j(t)$, and $Q_j'(t)$ and $P_j'(t)$, are the amplitude and phase quadratures of the j-th input and output boson fields (taken to be in a Gaussian state), respectively. In Eq. (1), $u(t)$ is a classical (non-quantum) deterministic or stochastic input signal. Quantum mechanical constraints require that canonical commutation relations be preserved. This leads to algebraic constraints on the A, B, C, D matrices, not encountered in the classical setting, known as the *physical realizability* constraints, of the form [3]:

$$A\mathbb{J}_n + \mathbb{J}_n A^T + B\mathbb{J}_m B^T = 0,\ \mathbb{J}_n C^T + B\mathbb{J}_m D^T = 0,\ D\mathbb{J}_m D^T = \mathbb{J}_r, \tag{2}$$

where $\mathbb{J}_k = I_k \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Only linear QSDEs (1) with system matrices satisfying (2) correspond to the dynamics of physically valid systems. Note that when the joint initial state of the oscillators is a quantum Gaussian state, then $x(t)$ and $y(t)$ remain in Gaussian states at all times $t \geq 0$.

Physical relevance. Linear quantum systems accurately describe a wide-class of linear quantum devices of interest in many applications, including, for instance, quantum optical cavities, quantum optical parametric oscillators, microwave superconducting resonators, gravitational wave interferometers and optical quantum memories [3]. Since linear quantum systems preserve Gaussian states, they are of interest in Gaussian quantum information processing. The linear structure of the QSDEs of linear quantum systems has motivated the adaptation

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and extension of powerful notions and methods from classical linear systems and control theory [1,2] to address problems for linear quantum systems [3].

Future directions. This talk is intended as an overview talk (rather than a technical talk), on the modelling of linear quantum systems and their applications. In particular, it will discuss some new research directions for this class of systems. This includes the problem of system identification of linear quantum systems [5,6], and developing a theory for infinite-dimensional linear quantum systems, as a quantum analogue of stochastic distributed parameter systems represented by stochastic linear PDEs. The latter is motivated by, for instance, fully quantum modeling of a class of quantum memories called photon-echo memories, such as gradient echo memories [7].

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