Stochastic solutions of generalized time-fractional evolution equations

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We consider a general class of integro-differential evolution equations which includes the governing equation of the generalized grey Brownian motion and the time- and space-fractional heat equation:

$$u(t,x) = u_0(x) + \int_0^t k(t,s) Lu(s,x) ds, \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{1}$$

where L is a pseudo-differential operator associated to a Lévy process and $k(t,s)$, $0 < s < t < \infty$, is a general kernel.

We present a general relation between the parameters of the equation and the distribution of any stochastic process, which provides a stochastic solution of Feynman-Kac type. More precisely, we derive a series representation in terms of the time kernel k and the symbol $-\psi$ of the pseudodifferential operator L for the characteristic function of the one-dimensional marginals of any stochastic solution. We explain how this series simplifies in the important case of homogeneous kernels which includes the kernel $k(t,s) = (t-s)^{\beta-1}/\Gamma(\beta)$ for time-fractional evolution equations and, more generally, kernels corresponding to Saigo-Maeda fractional diffintegration operators. The connection between Saigo-Maeda fractional diffintegration operators and positive random variables with Laplace transform given by Prabhakar's three parameter generalization of the Mittag-Leffler function is established. These results yield a stochastic representation for (1) with a Saigo-Maeda kernel in terms of a randomly slowed down Lévy process $(Y_{At^\beta})_{t\geq 0}$, where Y is a Lévy process with infinitesimal generator L, A is an independent random variable with Laplace transform given by the three-parameter Mittag-Leffler function, and β corresponds to the degree of homogeneity of the kernel. If Y has a stable distribution (e.g., in the case of a symmetric fractional Laplacian in space), the randomly slowed down Lévy process can be replaced by a randomly scaled linear fractional stable motion, providing a stochastic solution in terms of a self-similar process with stationary increments.

References

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