N-soliton solutions in the problem of Rogue waves

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The rogue wave problem was originally related to abnormally high oceanic waves which occurred suddenly on the sea surface, seemingly without clear precursors, and had resulted in a number of fatalities including ship sinking, damages of off-shore platforms, coastal constructions and large ships; people deaths [1]. Shortly later rogue waves became a hot topic of research in optics. These waves represent electromagnetic fields of extreme intensity which can breakdown the fiber in data transmission lines, when occur accidentally [2]. In the recent time, kindred researches commenced in application to other physical realms [3].

The rogue wave studies in diverse physical fields have much in common. A significant part of the research is developing *nonlinear* rogue wave models assuming that the corresponding dynamics is due to fast unstable nonlinear effects, when the wave intensity / energy gets concentrated within some location and time interval. For example, the nonlinear cubic Schrödinger (NLS) equation is the first-order approximation to the wave processes on the surface of a deep sea, and also to the waves of light in optical fibers. The quadratic in nonlinearity Korteweg – de Vries (KdV) equation describes water waves under the shallow water condition, while its modified version with the cubic nonlinearity is applicable to the nonlinear optical fibers.

Remarkably, the mentioned above equations, the nonlinear Schrödinger equation, and the two kinds of the Korteweg – de Vries equations (and also the extended, quadratic-cubic KdV, called the Gardner equation), are completely integrable by the Inverse Scattering Transform (IST) nonlinear partial differential equations [4,5]. They all possess soliton solutions which correspond to the nonlinear limit of stationary waves (or wave groups) of the permanent shape. Due to the nonlinear nature, the solitons may exhibit drastically different dynamical and probabilistic properties compared to linear waves, hence they have received much attention in the context of the rogue wave studies, and have inspired a new branch of the research in the field of mathematics. Conventionally, *rogue wave solutions* in mathematics are the ones which demonstrate fast localized growth of the solution; they often appear as a result of small perturbation of a uniform solution, but behave qualitatively similar within some range of the perturbation parameters (the issue of robustness [6]).

Within the IST solitons correspond to the discrete spectrum of the associated scattering problem, while the continuous spectrum specifies the other (small-amplitude) waves. Since the associated scattering problem is isospectral, the spectrum may be used to characterize the wave system at any instant. The IST may be understood as a nonlinear generalization of the Fourier transform. The limit of purely discrete spectrum is often considered as a reasonable problem statement. Fortunately, in this case the machinery of the IST is significantly simpler and allows some analytic results.

The concept of *soliton gas*, i.e., ensembles of solitons with irregular parameters (such as amplitudes and locations) represents the counterpart of the traditional paradigm of independent

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linear waves, which leads to the Gaussian statistics. In particular, the stochastic sea wave dynamics in many cases may be represented as a linear superposition of linear Fourier modes (the Gaussian sea), or of nonlinear (Stokes) modes. However, the recent advances in the rogue wave problem suggest that nonlinear interactions between the modes and coherent soliton-like structures may be the main reason of occurrence of abnormally high waves in the ocean.

Since we are interested in the statistical properties of the waves, the application of kinetic equations for solitons [7], which describe the transport of the soliton density, is limited. The direct numerical simulation of soliton ensembles is a popular efficient tool to consider the problem when solitons interact occasionally. However, the simulation of simultaneous interaction between many soliton is a strongly nonlinear process, even within formally weakly nonlinear frameworks. It represents a heavy problem to the direct numerical simulation [8]. These very rare events may correspond to the appearance of extremely large waves. Exact analytic solutions to integrable equations help to investigate the problem systematically.

The general form of a multi-soliton / multi-breather solution of the integrable KdV-type equations, which is even with respect to any variable when one of the variables is put equal to zero, was derived and analyzed in [9,10]. Physically, this solution corresponds to the most synchronized wave sequence (focusing of solitons). It was shown that the focused wave increases in amplitude only when sign-alternating solitons interact. Hence, unipolar KdV-type solitons do not exhibit extreme dynamics. This conclusion was also confirmed in direct numerical simulations, e.g. [11].

Let us focus on the example of the classic Korteweg – de Vries (KdV) equation, which may be presented in the standard dimensionless form as follows,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where the solution $u(x,t) \in \Re$ describes the perturbation, $x \in (-\infty;\infty)$ has the meaning of the spatial coordinate, and $t \in (-\infty;\infty)$ is the time. The dynamics of irregular waves within (1) (waves of the discrete and continuous spectra) was simulated numerically in [12], where the regimes of strongly non-Gaussian statistics were found. The effect of two-soliton interactions on the statistical properties of the solution was analyzed analytically in [13]. The evolution of rarefied soliton gas was modeled numerically in a number of publications, e.g. [11].

The multisoliton solution of (1) can be obtained with the help of the Darboux transformation

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \ln W_N(\psi_1, \psi_2, ..., \psi_N), \tag{2}$$

where W_N is the Wronskian of N so-called seed functions $\psi_j(x,t)$, $j = 1, ..., N$ (see e.g. in [10]),

$$\psi_j = \cosh k_j(x - 4k_j^2 t - x_j), \quad \text{if } j \text{ is odd}, \tag{3}$$

$$\psi_j = \sinh k_j(x - 4k_j^2 t - x_j), \quad \text{if } j \text{ is even}.$$

The parameters $k_j > 0$ specify N solitons with amplitudes $A_j = 2k_j^2 > 0$; x_j are constants which determine locations of the solitons. Though the solution (2) is explicit, its computation is technically difficult when N is large.

The use of extra-high precision of the numerical subroutine (similar to [14]) helps to construct the multisoliton solutions with larger N. We have implemented this approach to calculate the solution (2) to the KdV equation (1), using typically the mantissa of the length of 100 digits. The solution (2) may be used for constructing dense soliton gas within a finite interval. The generated gas states may possess the property of uniformity within some shorter interval if the density is not too large [15]. We also suggest the physically consistent way to calculate the soliton density function explicitly using the Darboux transformation approach. Then the spectral moments $\mu_n = < u^n >$, $n = 1, 2, ...$, where the angle brackets mean ensemble averaging, may be calculated.

We particularly consider the situation of synchronous collisions of KdV solitons, which corresponds to the choice $x_j = 0$, $j = 1, ..., N$ in (3). Then the solution (2) is characterized by the symmetries $u(x, 0) = u(-x, 0)$ and $u(0, t) = u(0, -t)$. We choose for certainty the particular distribution of the soliton amplitudes $A_j = 1/d^{j-1}$, $j = 1, ..., N$, and vary the constant $d > 1$. The appearance of the evolving solution, and the evolution of the third and the fourth statistical moments, μ_3, μ_4, have been examined. The value $d = 3$ corresponds to the marginal situation in the case $N = 2$, when the exchange and overtaking scenarios of the soliton interaction change; however it does not correspond to a peculiar solution when $N > 2$.

Though we consider solutions when the number N is always limited, the obtained solution is able to approximate the limit $N \to \infty$ in some range of parameters, what allows us direct examination of this ultimate limit of soliton interactions.

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