# On the Liouville and strong Liouville properties for a class of non-local operators

D. Berger[1], R. L. Schilling[2]

**Keywords:** Characteristic exponent; Lévy generator; Liouville property; strong Liouville property.

**MSC2010 codes:** *Primary:* 60G51, 35B53. *Secondary:* 31C05, 35B10, 35R09, 60J35.

**Introduction.** A $C^2$-function $f:\mathbb{R}^n\to\mathbb{R}$ is called *harmonic*, if $\Delta f=0$ for the Laplace Operator $\Delta$. It is kown that any bounded harmonic function is constant. Often it is helpful to understand $\Delta f$ as a Schwartz distribution in $\mathcal{D}'(\mathbb{R}^n)$ and to re-formulate the Liouville problem in the following way: The operator $\Delta$ enjoys the **Liouville property** if

$$f\in L^\infty(\mathbb{R}^n)\ \text{ and }\ \forall\phi\in C_c^\infty(\mathbb{R}^n)\ :\ \langle\Delta f,\phi\rangle:=\langle f,\Delta\phi\rangle=0\implies f\equiv\text{const}\tag{1}$$

holds; $\langle\cdot,\cdot\rangle$ denotes the (real) dual pairing used in the theory of distributions. An excellent account on the history and the importance of the Liouville property can be found in the paper [1] by Alibaud *et al.* If the condition '$f\in L^\infty(\mathbb{R}^n)$' in (1) can be replaced by '$f\geq 0$', we speak of the **strong Liouville property**. In this talk we prove a necessary and sufficient condition for the Liouville and strong Liouville properties of the infinitesimal generator of a Lévy process and subordinate Lévy processes. The talk is based on the preprint [2].

**Main result.** A Lévy process is a stochastic process with independent, stationary increments and right-continuous paths with finite left-hand limits. It is well-known that the infinitesimal generator $\mathcal{L}_\psi$ of a Lévy process is a **pseudo-differential operator**

$$\widehat{\mathcal{L}_\psi u}(\xi)=-\psi(\xi)\widehat{u}(\xi),\quad u\in\mathcal{S}(\mathbb{R}^n)\tag{2}$$

where $\widehat{u}(\xi)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{-i\xi\cdot x}u(x)\,dx$ is the Fourier transform and $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing smooth functions. The **symbol** $\psi:\mathbb{R}^n\to\mathbb{C}$ is a **continuous and negative definite** function which is uniquely characterized by its **Lévy–Khintchine representation**

$$\psi(\xi)=-ib\cdot\xi+\frac{1}{2}Q\xi\cdot\xi+\int_{\mathbb{R}^n\setminus\{0\}}\left(1-e^{i\xi\cdot x}+i\xi\cdot x1_{(0,1)}(|x|)\right)\nu(dx);\tag{3}$$

where $b\in\mathbb{R}^n$, $Q\in\mathbb{R}^{n\times n}$ (a positive semidefinite matrix) and $\nu$ (a Radon measure on $\mathbb{R}^n\setminus\{0\}$ such that $\int_{\mathbb{R}^n\setminus\{0\}}\min\{|x|^2,1\}\,\nu(dx)<\infty$) uniquely describe $\psi$. Our main result states the following:

*Theorem 1.* Let $\mathcal{L}_\psi$ be the generator of a Lévy process with symbol $\psi$. The operator $\mathcal{L}_\psi$ has the Liouville property if, and only if, the zero-set of the symbol satisfies $\{\psi=0\}=\{0\}$.

## References

[1] N. Alibaud, F. del Teso, J. Endal, E.R. Jakobsen. The Liouville theorem and linear operators satisfying the maximum principle. // Journal des Mathematiques Pures et Appliquees. 2020. V. 142. P. 229–242

[2] D. Berger, R.L. Schilling. On the Liouville and strong Liouville properties for a class of non-local operators. // arXiv:2101.01592.

[1]TU Dresden, Institut für mathematische Stochastik, Germany, Dresden. Email: david.berger2@tu-dresden.de

[2]TU Dresden, Institut für mathematische Stochastik, Germany, Dresden. Email: rene.schilling@tu-dresden.de