On the Liouville and strong Liouville properties for a class of non-local operators

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Introduction. A \(C^2\)-function \(f : \mathbb{R}^n \to \mathbb{R}\) is called harmonic, if \(\Delta f = 0\) for the Laplace operator \(\Delta\). It is known that any bounded harmonic function is constant. Often it is helpful to understand \(\Delta f\) as a Schwartz distribution in \(\mathcal{D}'(\mathbb{R}^n)\) and to re-formulate the Liouville problem in the following way: The operator \(\Delta\) enjoys the Liouville property if

\[
\forall \phi \in C_0^\infty(\mathbb{R}^n) : \langle \Delta f, \phi \rangle := \langle f, \Delta \phi \rangle = 0 \implies f \equiv \text{const}
\]

holds; \(\langle \cdot , \cdot \rangle\) denotes the (real) dual pairing used in the theory of distributions. An excellent account on the history and the importance of the Liouville property can be found in the paper [1] by Alibaud et al. If the condition ‘\(f \in L^\infty(\mathbb{R}^n)\)’ in (1) can be replaced by ‘\(f \geq 0\)’, we speak of the *strong Liouville property*. In this talk we prove a necessary and sufficient condition for the Liouville and strong Liouville properties of the infinitesimal generator of a Lévy process and subordinate Lévy processes. The talk is based on the preprint [2].

Main result. A Lévy process is a stochastic process with independent, stationary increments and right-continuous paths with finite left-hand limits. It is well-known that the infinitesimal generator \(L_\psi\) of a Lévy process is a pseudo-differential operator

\[
\mathcal{L}_\psi u(\xi) = -\psi(\xi)\hat{u}(\xi), \quad u \in \mathcal{S}(\mathbb{R}^n)
\]

where \(\hat{u}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix\cdot x} u(x) \, dx\) is the Fourier transform and \(\mathcal{S}(\mathbb{R}^n)\) is the Schwartz space of rapidly decreasing smooth functions. The symbol \(\psi : \mathbb{R}^n \to \mathbb{C}\) is a continuous and negative definite function which is uniquely characterized by its Lévy–Khintchine representation

\[
\psi(\xi) = -ib \cdot \xi + \frac{1}{2} Q \xi \cdot \xi + \int_{\mathbb{R}^n \setminus \{0\}} \left( 1 - e^{i\xi \cdot x} + i\xi \cdot x 1_{(0,1)}(|x|) \right) \nu(dx);
\]

where \(b \in \mathbb{R}^n\), \(Q \in \mathbb{R}^{n \times n}\) (a positive semidefinite matrix) and \(\nu\) (a Radon measure on \(\mathbb{R}^n \setminus \{0\}\) such that \(\int_{\mathbb{R}^n \setminus \{0\}} \min\{|x|^2, 1\} \nu(dx) < \infty\)) uniquely describe \(\psi\). Our main result states the following:

*Theorem 1.* Let \(L_\psi\) be the generator of a Lévy process with symbol \(\psi\). The operator \(L_\psi\) has the Liouville property if, and only if, the zero-set of the symbol satisfies \(\{\psi = 0\} = \{0\}\).

References


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