Convergence the solution of initial-boundary value problem for one partial differential equation in the *-weak sense

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Introduction. Let $\Omega \subset \mathbb{R}^n$ be a domain with the boundary $\partial\Omega$ of class C^∞ and $T \in \mathbb{R}_+$. In the cylinder $C = \Omega \times (0, T)$, consider the modified Boussinesq equation

$$(\lambda - \Delta)u_{tt} - \alpha^2 \Delta u + u^3 = 0, \quad (x,t) \in \Omega \times (0,T) \tag{1}$$

with Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \tag{2}$$

and Cauchy conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3}$$

where $\lambda, \alpha \in \mathbb{R}$. The equation has many applications in various fields of natural science. For example, it simulates wave propagation in shallow water, taking into account capillary effects. In this case, the function $u = u(x,t)$ determines the wave height. In monograph [1] a linear mathematical model of wave propagation in shallow water is constructed. A (modified) mathematical model of wave propagation in shallow water in a one-dimensional region was investigated in [2] and a soliton solution of equation (1) was obtained. The existence of a unique global solution to the Cauchy problem for equation (1) was proved [3] for $\lambda = 1, \alpha = 1$. In [4], a similar solution was obtained for describing the interaction of shock waves. In all the works listed above, an essential condition is the continuous invertibility of the operator at the highest derivative with respect to t. However, the operator $\lambda - \Delta$ can be degenerate. Equations that are not solvable with respect to the highest time derivative, according to [5] are called Sobolev type equations.

Using the theory of p-bounded operators developed by G.A. Sviridyuk and his disciples [6, 7], it was shown in [8] that in appropriately chosen spaces the problem (1)–(3) can be reduced to the initial value problem

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \tag{4}$$

for an abstract semilinear second-order Sobolev type equation

$$L\ddot{u} - Mu + N(u) = 0, \tag{5}$$

where $\dot{u}, \ddot{u}$ are the first and the second derivatives with respect to t, $L = \lambda - \Delta, M = \alpha^2\Delta, N(u) = u^3$. Then, using the phase space method, a theorem on the existence of a unique local solution was proved.

Definition 1. The set $\mathfrak{P}$ is called the phase space of equation (5) if
1) for any $(u_0, u_1) \in T\mathfrak{P}$ ($T\mathfrak{P}$ is the tangent bundle of $\mathfrak{P}$) there is a unique solution to problem (4), (5);
2) any solution $u = u(t)$ of equation (5) lies in $\mathfrak{P}$ as a trajectory.
Moreover, the notation $(u_0, u_1) \in T_{u_0}\mathfrak{P}$ should be understood as $u_0 \in \mathfrak{P}$ and $u_1 \in T_{u_0}\mathfrak{P}$.

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Let $\ker L \neq \{0\}$ and the operator M be $(L, 0)$ bounded, then, by the splitting theorem [7], equation (7) can be reduced to an equivalent system of equations

$$\begin{cases} 0 = (\mathbb{I} - Q)(M + N)(u), \\ \ddot{u}^1 = L_1^{-1} Q(M + N)(u), \end{cases}$$

where $u^1 = Pu$, P is some projector along $\ker L$. Then the phase space $\mathfrak{P}$ of equation (5) is the set [8]

$$\mathfrak{P} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(M + N)(u) = 0\}.$$

It was also noted that in the case of monotonicity of the operator N, the phase space would be a simple manifold.

Main result. Let us formulate and prove a theorem that answers the question on how to find a solution to (1) – (3).

Let the operator $L : H^1(\Omega) \to H^{-1}(\Omega)$ be given by formula

$$\langle Lu, v\rangle = \int_\Omega (\nabla u \nabla v + \lambda uv)dx.$$

Denote $B = L^4(\Omega) \cap H_0^1(\Omega)$ and $D = H^1(\Omega) \cap \operatorname{coim} L$ (where $\operatorname{coim} L = H^1(\Omega) \ominus \ker L$).

In addition, define spaces of distributions (functions with values in a Banach space) $L^\infty(0, T; B)$ and $L^\infty(0, T; L^2(\Omega))$. Construct the conjugate spaces using the Dunford – Pettis theorem: $(L^\infty(0, T; B))^* \simeq L^1(0, T; L^{\frac{4}{3}}(\Omega) \cup H^{-1}(\Omega))$ and $(L^\infty(0, T; D))^* \simeq L^1(0, T; D^*)$.

Let λ_k be the eigenvalues of the homogeneous Dirichlet problem (2) for the Laplace operator, numbered nonincreasingly taking into account their multiplicity, and φ_k be the corresponding eigenfunctions. Note that the linear span of $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ for $m \to \infty$ is dense in B and orthonormal (in the sense of the inner product in $L^2(\Omega)$).

Theorem 1. Let $\lambda \in [\lambda_1, +\infty)$, $u_0 \in B$ and $u_1 \in D$ and $(u_0, u_1) \in T_{u_0}\mathfrak{P}$. Then there exists a solution $u = u(x, t)$ to problem (1)–(3) such that $u \in L^\infty(0, T; B)$ and $\dot{u} \in L^\infty(0, T; D)$.

Theorem 2. Under the conditions of Theorem 1 and Ω being such that embedding $H^1(\Omega) \subset L^4(\Omega)$ is compact, the solution to problem (1)–(3) is unique.

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