Topological conjugacy Morse-Smale flows with finite number of moduli on surfaces

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Introduction. Two flows $f^t, f'^t \colon M \to M$ on a manifold M are called *topologically equivalent* if there exists a homeomorphism $h\colon M \to M$ sending trajectories of f^t into trajectories of f'^t preserving orientations of the trajectories. Two flows are called *topologically conjugate* if h sends trajectories into trajectories preserving not only directions but in addition the time of moving. To find an invariant showing the class of topological equivalence or topological conjugacy of flows in some class means to get a *topological classification* for one. Note, that for some classes their classifications in sense of equivalence and conjugacy coincide; for other classes these classifications completely differ.

The *Morse-Smale flows* were introduced on the plane for the first time in the classical paper of A.A. Andronov and L.S. Pontryagin in [1]. The non-wandering set of such flows consists of a finite number of hyperbolic fixed points and finite number of hyperbolic limit cycles, besides, saddle separatrices cross-sect only transversally, which means on surfaces that there is no a trajectory connecting saddle points. The most important for us combinatorial invariants for Morse-Smale flows are the *Leontovich-Maier's scheme* [2], [3] for flows on the plane, the *Peixoto's directed graph* [4] for Morse-Smale flows on any closed surface and the *Oshemkov-Sharko's molecule* [5] for Morse-Smale flows on any closed surface.

J. Palis in [6] proved that the class of topological equivalence of a flow can contain any volume of topological conjugacy classes, describing by parameters called *moduli*. For example, a modulus appears when a flow has a separatrix common for two saddle points.

Obviously, any limit cycle generates a modulus equals to the period of one. Additionally, in [7] it was proved that the presence of a cell bounded by limit cycles gives infinite number of moduli connected with the uniqueness of invariant foliation in the basin of the limit cycle.

The results. The first result solves the problem of a flow class with a finite number of modulus.

Theorem 1. A Morse-Smale surface flow has finite number of moduli iff it has no a trajectory going from one limit cycle to another.

Second, we use the complete topological classification with respect to equivalence for Morse-Smale surface flows [5], [8] by means of an *equipped graph* $\Upsilon^*_{\phi^t}$ describing dynamics of ϕ^t.

To distinguish topological conjugacy classes we add to the equipped graph an information on the periods of the limit cycles. It gives a new equipped graph $\Upsilon^{**}_{\phi^t}$. In this way we get the following result.

Theorem 2. Morse-Smale surface flows ϕ^t, ϕ'^t without trajectories going from one limit cycle to another are topologically conjugate iff the equipped graphs $\Upsilon^{**}_{\phi^t}$ and $\Upsilon^{**}_{\phi'^t}$ are isomorphic.

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