Extinction time of stochastic SIRS epidemic models: application of Chernoff approximation for bi-continuous semigroups

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Introduction. The relationship between strongly continuous semigroups on Banach spaces and the solutions of initial value problems for partial differential equations is well established. However, very often we encounter PDEs whose corresponding semigroups are not strongly continuous. One way to treat such cases is by introducing a weaker topology to the underlying Banach space, for which to achieve that the corresponding semigroup is strongly continuous. Proposed by [4], the concept of bi-continuous semigroups describes such treatments in a general framework, with analogous generation theorems and approximation formulas.

In the area of epidemiology and population genetics, the limits of many multi-dimensional discrete stochastic models turn out to be degenerate. In this work, we focus on a particular degenerate diffusion, occurring as the limit of stochastic epidemic SIRS models, when the population size tends to infinity. In solving the hitting time problem for this diffusion, we encounter the problem of a lack of strong continuity as described above, and subsequently prove that the corresponding degenerate operator generates a bi-continuous semigroup, using a generalised Chernoff product formula [3]. Since the corresponding second-order operator is associated with the squared Bessel process and the corresponding semigroup can be computed explicitly, we obtain an explicit Chernoff equivalent to the semigroup.

Motivation. Stochastic compartmental models are widely used in modelling the spread of epidemic diseases, such as SIS, SIR and SIRS models. We consider the stochastic SIRS model, a two-dimensional continuous-time Markov chain which represents diseases with no incubation period and temporary immunity in a closed, homogeneously mixing population of size N. A stochastic SIRS model is said to be "at criticality" when its basic reproduction number $\mathcal{R}_0$ is 1. It is known that SIS and SIR models exhibit 'critical behaviours' not only at criticality, but also when the parameter converges to the criticality as $N \to \infty$. Studying such "near-critical" behaviours is regarded as one of the significant challenges for stochastic epidemic modelling, since many diseases, especially those under an eradication campaign, are near-critical [1].

In this work, we focus on the critical behaviour of the extinction time, i.e., the time taken for a population to reach zero infections. The study of the extinction time of stochastic epidemic models has drawn wide interests from both epidemiological and mathematical point of view.

We identify the critical parameter domain, where a phase transition between rapid extinction and exponential-time prevalence of the epidemic can be observed. We then apply a suitable scaling in both time and space such that the scaled stochastic SIRS model converges to a particular limit diffusion. These findings are analogous to what has been proven for stochastic SIS/ SIR models [2].

Little is known about the distribution of the extinction time of critical stochastic epidemic models in general. We contribute to the existing knowledge by expressing the tail distribution of the SIRS extinction time using bi-continuous semigroups, which allows for future exploration of its qualitative properties. The relatively simple form of the iterative approximation scheme also provides the possibility of numerical analysis.

Model setting For a closed population of size N, the stochastic SIRS model is defined as a two-dimensional continuous-time Markov chain $(I_t^N, R_t^N)_{t\geq 0}$ with parameter space

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$(\lambda_o(N), \gamma_o(N))$, where I_t^N represents the size of the infected population at time t, and R_t^N represents the size of the immune population at time t. The model is associated with the transition rates:

$$\begin{aligned}
&(i,r) \to (i+1,r), && \text{at rate } \lambda_o(N-i-r)i/N,\\
&(i,r) \to (i,r-1), && \text{at rate } \gamma_o r,\\
&(i,r) \to (i-1,r+1), && \text{at rate } i.
\end{aligned}$$

In other words, each susceptible individual is expected to contract the disease at rate $\lambda_o I_t^N/N$. Once infected, each individual is immediately infectious and will recover at rate $\mu_o = 1$ independently of other individuals. Each recovered individual loses immunity at rate γ_o and becomes susceptible independently. The basic reproduction number is $\mathcal{R}_0 = \lambda_o/\mu_o$. The extinction time is defined as $T_o^N := \inf\{t : I_t^N = 0\}$.

Method and main result. We first identify the critical scaling through a heuristic argument as

$$Y_t^N := \frac{I^N_{N^{1/3}t}}{N^{1/3}}, \quad Z_t^N := \frac{R^N_{N^{1/3}t}}{N^{2/3}},$$

and the parameters are assumed to satisfy $(1-\lambda_o(N))N^{1/3} \to \hat{\lambda} \in \mathbb{R}$, $\gamma_o(N)N^{1/3} \to \gamma \geq 0$. We denote the scaled extinction time as $T^N := \inf\{t : Y_t^N = 0\}$.

Secondly, we prove that $(Y_t^N, Z_t^N)_{t\geq 0}$ has a degenerate limit diffusion as $N \to \infty$:

$$\begin{aligned}
dY &= -(\hat{\lambda} + Z)Y\,ds + \sqrt{2Y}\,dW,\\
dZ &= (Y - \gamma Z)ds,
\end{aligned}$$

and the scaled extinction time T^N weakly converges to $T := \inf\{t : Y_t = 0\}$.

Lastly, we obtain the asymptotic distribution of T^N. Denote

$$\mathcal{L} := u\frac{\partial^2}{\partial u^2} \text{ and } \mathcal{H} := -(\hat{\lambda} + v)u\frac{\partial}{\partial u} + (u - \gamma v)\frac{\partial}{\partial v}.$$

For any $t_0 > 0$, the tail distribution of T, $\mathbb{P}[T > t_0 - t|(Y_0, Z_0) = (u,v)]$, can be expressed as the solution of a PDE

$$\frac{\partial U}{\partial t} = -\mathcal{L}U - \mathcal{H}U,$$

with the end condition $U(u,v,t_0) = \mathbf{1}_{\mathbb{R}^2_+}(u,v)$, and the boundary condition

$$\lim_{u\downarrow 0} U(u,v,t) = 0,\ t \in [0, t_0),$$

where $\mathbf{1}_{\mathbb{R}^2_+}A$ denotes the indicator function of set A.

We prove by the generalised Chernoff product formula that the bi-closure of $(\mathcal{L}+\mathcal{H}, C_c^\infty(\mathbb{R}^2_+))$ generates a bi-continuous semigroup on the Banach space of bounded continuous functions with continuous extensions to $[0,\infty)^2$, equipped with the uniform norm. The Chernoff equivalent is constructed by combining the known expression of the squared-Bessel semigroup generated by the closure of $\mathcal{L}$ and the shift operator generated by $\mathcal{H}$.

Explicitly,

$$\mathbb{P}[T > t|(Y_0, Z_0) = (u,v)] = \lim_{n\to\infty}\left(V\left(\frac{t}{n}\right)\right)^n \mathbf{1}_{\mathbb{R}^2_+}(u,v),$$

where

$$V(t)f(u,v) := \int_0^\infty g(t, ue^{-(\hat{\lambda}+v)t}; m) f(m, ve^{-\gamma t} + ut)\, dm,$$

$$g(t,u;m) := \frac{1}{t} u^{1/2} m^{-1/2} e^{-(u+m)/t} I_1\left(\frac{2m^{1/2}u^{1/2}}{t}\right),$$

and $I_1(\cdot)$ denotes the modified Bessel function of the first kind of index 1.

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