Investigation of positive solutions to the Sobolev-type equations in sequence spaces

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Introduction. We investigate positive solutions to the Cauchy and Showalter–Sidorov problem based on the properties of Banach lattices in sequence spaces, the theory of degenerate holomorphic groups of operators [1] and the theory of the positive semigroups of operators [2]. When studying positive solutions, we rely on the theory of positive degenerate holomorphic groups of operators generated by linear and continuous operators L and M, besides the operator M is (L, p)-bounded. Using these groups, we obtain positive solutions to linear homogeneous and inhomogeneous Sobolev type equations. As an example, we consider positive degenerate holomorphic groups in Sobolev spaces of sequences. Our aim is to find the necessary and sufficient conditions for the positivity of such groups. In the future, we intend to adapt the theory of positive degenerate groups for non-classical models of equations of mathematical physics [3], including the Barenblatt–Zheltov–Kochina model with a multipoint initial-final condition.

Main result. Let us consider Banach lattice $\mathcal{B} = (\mathcal{B}, \mathcal{C})$. Here $\mathcal{C}$ is a proper generating cone, and note that $\mathcal{C}$ may not coincide with *the canonical cone* $\mathcal{B}_+$. The operator $A \in \mathcal{L}(\mathcal{B})$ is called *positive* if $Ax \geq 0$ for any $x \in \mathcal{C}$. The group of operators $V^\bullet = \{V^t : t \in \mathbb{R}\}$, acting on the space $\mathcal{B}$, is called *positive*, if $V^t x \geq 0$ for any $x \in \mathcal{C}$ and $t \in \mathbb{R}$. If $V^\bullet$ is a degenerate group, then its unit V^0 is a projector that splits the space $\mathcal{B}$ into a direct sum $\mathcal{B} = \mathcal{B}^0 \oplus \mathcal{B}^1$, where $\mathcal{B}^0 = \ker V^0$ and $\mathcal{B}^1 = \mathrm{im} V^0$. Since $V^t = V^0 V^t V^0$, then $\mathcal{B}^0 = \ker V^t$, and $\mathcal{B}^1 = \mathrm{im} V^t$. Hence, we can define $\ker V^\bullet = \mathcal{B}^0$ and $\mathrm{im} V^\bullet = \mathcal{B}^1$. If the degenerate group $V^\bullet$ is, in addition, positive, then $\mathcal{B}^1$ is Banach lattice with proper generating cone $\mathcal{C}^1 = \{x \in \mathcal{C} : V^0 x = x\} = \mathcal{B}^1 \cap \mathcal{C}$. If it turns out that the space $\mathcal{B}^0$ is also Banach lattice with a generating cone $\mathcal{C}^0$ and an order relation $\succeq$, then the cone $\mathcal{C}^* = \mathcal{C}^0 \oplus \mathcal{C}^1$ can generate a new Banach lattice of the space $\mathcal{B}$ with an order relation $\textcircled{\geq}$, that is

$$(x \textcircled{\geq} y) \leftrightarrow (x_0 \succeq y_0) \wedge (x_1 \geq y_1).$$

The main goal of the discussion is to indicate the conditions for the positivity of degenerate holomorphic groups $U^\bullet = \{U^t : t \in \mathbb{R}\}$ and $F^\bullet = \{F^t : t \in \mathbb{R}\}$. Further we assume that the Banach space $\mathcal{U}(\mathcal{F})$ is a Banach lattice $\mathcal{U} = (\mathcal{U}, \mathcal{C}_\mathcal{U})$ $(\mathcal{F} = (\mathcal{F}, \mathcal{C}_\mathcal{F}))$, where $\mathcal{C}_\mathcal{U}$ $(\mathcal{C}_\mathcal{F})$ is the proper generating cone.

Moreover, as Banach spaces $\mathcal{U}$ and $\mathcal{F}$ we take the Sobolev spaces of sequences

$$l_q^m = \{u = \{u_k\} : \sum_{k=0}^{\infty} \lambda_k^{\frac{mq}{2}} |u_k|^q < \infty\}, \ m \in \mathbb{R}, \ q \in [1, +\infty).$$

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Theorem 1. Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the following statements are equivalent.

(i) $(\mu R^L_\mu(M))^{p+1}$ $((\mu L^L_\mu(M))^{p+1})$ is positive for all sufficiently large $\mu \in \mathbb{R}_+$.

(ii) Degenerate holomorphic group $U^\bullet$ $(F^\bullet)$ is positive.

Let $\mathcal{U} = (\mathcal{U}, \mathcal{C}_\mathcal{U})$ and $\mathcal{F} = (\mathcal{F}, \mathcal{C}_\mathcal{F})$ be Banach lattices, where $\mathcal{C}_\mathcal{U}$ and $\mathcal{C}_\mathcal{F}$ are own generating cones. Let $L \in \mathcal{L}(\mathcal{U}, \mathcal{F})$ and $M \in \mathcal{C}l(\mathcal{U}, \mathcal{F})$ be operators. Consider the linear homogeneous Sobolev type equation

$$L\dot{u} = Mu. \tag{1}$$

A vector function $u \in C^1(\mathbb{R}, \mathcal{U})$, satisfying this equation is called *a solution to equation* (1). A solution $u = u(t)$ is called *a solution to the Cauchy problem*, if it satisfies the condition

$$u(0) = u_0. \tag{2}$$

for some $u_0 \in \mathcal{U}$. A solution $u = u(t)$ is called *a solution to the Showalter B– Sidorov problem* [4], if it satisfies the condition

$$P(u(0) - u_0) = 0. \tag{3}$$

The vector function $u(t) = U^t u_0$ is solution to equation (1) for any $u_0 \in \mathcal{U}$, and it is also a solution to problem (3) also for any $u_0 \in \mathcal{U}$. Now we arrive at the question about the existence and uniqueness of the solution to problem (1), (2) and the question about the uniqueness of the solution to problem (1), (3).

Here $\{U^t : t \in \mathbb{R}\}$ is a degenerate holomorphic group of operators of the form

$$U^t = \frac{1}{2\pi i} \int\limits_\gamma R^L_\mu(M) e^{\mu t} d\mu, \quad t \in \mathbb{R},$$

$$F^t = \frac{1}{2\pi i} \int\limits_\gamma L^L_\mu(M) e^{\mu t} d\mu, \quad t \in \mathbb{R},$$

defined on the spaces $\mathcal{U}$ and $\mathcal{F}$ respectively. Here the contour $\gamma \in \mathbb{C}$, $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$ $R^L_\mu(M) = (\mu L - M)^{-1} L$ is *the right L-resolvent of the operator* M and $L^L_\mu(M) = L(\mu L - M)^{-1}$ is *the left one.*

To solve the question about the existence and the uniqueness of solutions, recall that a set $\mathcal{B} \subset \mathcal{U}$ is called the phase space of equation (1) if any of its solutions $u(t) \in \mathcal{B}$ for each $t \in \mathbb{R}$; and for any $u_0 \in \mathcal{B}$ there exsists a unique solution $u \in C^1(\mathbb{R}, \mathcal{U})$ to problem (2) for equation (1). The following theorem is true.

Theorem 2. Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then

(i) the phase space of equation (1) is the subspace $\mathcal{U}^1$;

(ii) for any $u_0 \in \mathcal{U}$ there exists a unique solution $u = u(t)$ to problem (1), (3) of the form $u(t) = U^t u_0$.

By virtue of statement (i) of this theorem, any solution to equation (1) belongs to the space $\mathcal{U}^1$ pointwise, that is $u(t) \in \mathcal{U}^1$ for all $t \in \mathbb{R}$. This means that if we represent an arbitrary initial vector u_0 in the form $u_0 = u_0^0 + u_0^1$, where $u_0^k \in \mathcal{U}^k, k = 0, 1$ (according to the splitting theorem [1]), then the solution $u(t) = U^t u_0$ to problem (1), (3) is also the unique solution to this problem under the initial condition $v_0 = v_0^0 + u_0^1$, where the vector $v_0^0 \in \mathcal{U}^0$ is arbitrary.

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