Graded semigroup C^*-algebras and non-commutative Fourier coefficients

R. N. Gumerov[1], E. V. Lipacheva[2]

Keywords: contractive operator; grading; semigroup of operators; reduced semigroup C^*-algebra; σ-index of a monomial; topologically graded C^*-algebra.

MSC2010 codes: 46L05, 47L40

Introduction. This report presents a construction that allows us to set a topological grading on the reduced semigroup C^*-algebra over an arbitrary group.

The reduced semigroup C^*-algebra is the algebra which is generated by the left regular representation of a semigroup with the cancellation property. The study of such C^*-algebras was started by Coburn in 1960s. It has been further developed in the papers by a number of authors (see, for example, [1]).

A grading for an object of a category allows us to investigate the structure of this object. In the category of C^*-algebras, one considers the gradings that are also called the C^*-bundles, or the Fell bundles [2]. The notion of the topologically graded C^*-algebra was introduced in [3] with the aim to extend the concepts of harmonic analysis to the non-commutative case. An important property of a such grading is the existence of special operators, namely, the conditional expectation and the Fourier coefficients.

We studied the semigroup C^*-algebras and their gradings in [4–9].

The report presents the results of [7, 9]. In [7], we considered the topological gradings of the semigroup C^*-algebras over the group $\mathbb{Z}_n$ of integers modulo n.

Fell bundle for semigroup C^*-algebra. Throughout S is a discrete semigroup with the cancellation property and the unit e.

Let us consider the Hilbert space $l^2(S)$ of all square summable complex-valued functions defined on the semigroup S. The reduced semigroup C^*-algebra $C_r^*(S)$ is the C^*-subalgebra generated by the set of isometries $\{T_a \mid a \in S\}$ in the algebra of all bounded operators on $l^2(S)$. Here the operator T_a is defined as follows: $T_a(e_b) = e_{ab},\ a, b \in S$, where $\{e_a \mid a \in S\}$ is the canonical orthonormal basis in the space $l^2(S)$.

Further, in the C^*-algebra $C_r^*(S)$, we consider the involutive subsemigroup Mon consisting of operators of the form

$$V = T_{a_1}^{i_1} T_{a_2}^{i_2} \ldots T_{a_k}^{i_k},$$

where $a_j \in S,\ i_j \in \{-1, 1\},\ j = 1, \ldots, k,\ k \in \mathbb{N}$, and $T_{a_j}^1 := T_{a_j}$, $T_{a_j}^{-1} := T_{a_j}^*$. These operators are called the monomials.

Let G be an arbitrary group. We assume that there is a surjective semigroup homomorphism

$$\sigma : S \longrightarrow G.$$

It was shown in [9] that the formula

$$\operatorname{ind} V = \sigma(a_1)^{i_1} \sigma(a_2)^{i_2} \ldots \sigma(a_k)^{i_k}$$

defines an involutive surjective homomorphism of semigroups $\operatorname{ind} : Mon \longrightarrow G$.

The value $\operatorname{ind} V$ is called the σ-index of the monomial V. This concept was introduced in [7] for the case $G = \mathbb{Z}_n$.

Monomials with the σ-index e constitute an involutive subsemigroup of operators in the monomial semigroup Mon. In the C^*-algebra $C_r^*(S)$ we consider the C^*-subalgebra $\mathfrak{A}_e$ generated by this subsemigroup of operators.

[1]Kazan (Volga region) Federal University, N.I. Lobachevskii Institute of Mathematics and Mechanics, Chair of Mathematical Analysis, Russian Federation, Kazan. Email: Renat.Gumerov@kpfu.ru

[2]Kazan Power Engineering University, Chair of Higher Mathematics, Russian Federation, Kazan. Email: elipacheva@gmail.com

For each $g \in G$, let $\mathfrak{A}_g$ be the closure of the linear span for all operators of the σ-index g.

The semigroup C^*-algebra $C_r^*(S)$ is said to be G-graded if it contains a family of Banach subspaces satisfying the conditions listed in the following theorem.

Theorem 1. For the family of Banach spaces $\{\mathfrak{A}_g \mid g \in G\}$ the following statements hold:
1) $\mathfrak{A}_g\mathfrak{A}_h \subset \mathfrak{A}_{gh}$ for all $g, h \in G$;
2) $\mathfrak{A}_g^* = \mathfrak{A}_{g^{-1}}$ for every $g \in G$;
3) the family $\{\mathfrak{A}_g \mid g \in G\}$ consists of the linearly independent subspaces of $C_r^*(S)$;
4) $C_r^*(S) = \overline{\bigoplus_{g\in G} \mathfrak{A}_g}$.

So the family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ constitutes a Fell bundle for the semigroup C^*-algebra $C_r^*(S)$ over the group G.

Topological grading of semigroup C^*-algebra. The G-graded C^*-algebra $C_r^*(S)$ is said to be topologically graded if there exists an operator with the properties formulated in the following theorem.

Theorem 2. There exists a contractive linear operator

$$F : C_r^*(S) \longrightarrow \mathfrak{A}_e$$

which is the identity mapping on $\mathfrak{A}_e$ and vanishes on each subspace $\mathfrak{A}_g$, $g \in G$, $g \neq e$.

Thus, in addition, the family $\{\mathfrak{A}_g \mid g \in G\}$ forms the topological G-grading of $C_r^*(S)$.

As a corollary, for each $g \in G$, there is a contractive linear operator

$$F_g : C_r^*(S) \longrightarrow \mathfrak{A}_g$$

which is the identity mapping on $\mathfrak{A}_g$ and vanishes on every subspace $\mathfrak{A}_h$, $h \in G$, $h \neq g$.

As is known, the operators F_g are non-commutative analogs of the Fourier coefficients.

References

[1] X. Li. Semigroup C^*-algebras. // In: T.M. Carlsen, N.S. Larsen, S. Neshveyev, C. Skau, eds. Operator Algebras and Applications. Abel Symposia 2015. V. 12. Cham, Springer, 2016. P. 191–202.

[2] J.M.G. Fell. An extension of Mackey's method to Banach $*$-algebraic bundles. // Mem. Amer. Math. Soc. 1969. V. 90.

[3] R. Exel. Amenability for Fell bundles. // J. Reine Angew. Math. 1997. V. 492. P. 41–73.

[4] E.V. Lipacheva. On a class of graded ideals of semigroup C^*-algebras. // Russ. Math. 2018. V. 62. No. 10. P. 37–46.

[5] R.N. Gumerov. Limit automorphisms of the C^*-algebras generated by isometric representations for semigroups of rationals. // Sib. Math. J. 2018. V. 59. No. 1. P. 73–84.

[6] S.A. Grigoryan, E.V. Lipacheva, A.S. Sitdikov. Nets of graded C^*-algebras over partially ordered sets. // St. Petersburg Math. J. 2019. V. 30. No. 6. P. 901–915.

[7] R.N. Gumerov, E.V. Lipacheva. Topological Grading of Semigroup C^*-Algebras. // Herald of the Bauman Moscow State Technical University. Series Natural Sciences. 2020. V. 90. No. 3. P. 44–55.

[8] E.V. Lipacheva. Extensions of semigroups and morphisms of semigroup $C*$-algebras. // Sib. Math. J. 2021. V. 62. No. 1. P. 66–76.

[9] E.V. Lipacheva. On graded semigroup C^*-algebras and Hilbertmodules. // Proceedings of the Steklov Institute of Mathematics. 2021. V. 313 (to appear).