High order heat-type equations and random walks on the complex plane

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The celebrated Feynman-Kac formula for the solution of the heat equation is the first (and most famous) example of an extensively developed theory connecting stochastic processes with the solution of parabolic equations associated to second order elliptic operators. However, this theory cannot be applied to higher order PDEs such as, for instance, high-order heat type equations of the form:

$$\frac{\partial}{\partial t}u(t,x) = a\frac{\partial^N}{\partial x^N}u(t,x), \tag{1}$$
$$u(0,x) = f(x),$$

where $t \in \mathbb{R}^+, x \in \mathbb{R}, a \in \mathbb{R}$ and $N \in \mathbb{N}$, $N > 2$. In fact, the lack of a maximum principle for Eq. (1) forbids a probabilistic representation of its solution of the form

$$u(t,x) = \mathbb{E}[f(x + X_t)],$$

in terms of the expectation with respect to the distribution of a *real valued* stochastic process $\{X_t\}_{t\in\mathbb{R}^+}$. This problem has been extensively studied, e.g. by Krylov (1960), Hochberg(1978), Funaki (1979), Burzdy (1995), Orsingher (1999), Levin and Lyons (2009).

Recently an alternative technique has been proposed in [1]. A sequence $\{W_n^N(t)\}$ of scaled random walks on the complex plane is constructed as

$$W_n^N(t) = \frac{1}{n^{1/N}}\sum_{j=1}^{\lfloor nt \rfloor}\xi_j, \tag{2}$$

where $\{\xi_j\}_{j\in\mathbb{N}}$ are independent identically distributed complex random variables, uniformly distributed on the set of $N-th$ roots of unit. If $N > 2$, the particular scaling exponent $1/N$ appearing in (2) does not allow the weak convergence of $W_n^N(t)$. Nevertheless, the expectation of particular functionals admit a limit for $n \to \infty$, in particular the following result holds

$$\lim_{n\to+\infty}\mathbb{E}[\exp(i\lambda W_n^N(t))] = \exp\left(\frac{i^n}{N!}\lambda^N t\right) \tag{3}$$

and allows to interpret in a weak sense the limit of $W_n^N(t)$ as an N-stable random variable. The convergence in Eq. (3) allows the proof of the following probabilistic representation formula for the solution of (1)

$$u(t,x) = \lim_{n\to+\infty}\mathbb{E}[f(x + W_n^N(t))],$$

for a suitable class of analytic initial data f.

The talk will provide an overview of these results.

References

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