Semigroup-theoretic approach to thin layers: the role of transmission conditions

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Motivated by models of signaling pathways in B lymphocytes, which have extremely large nuclei, we study the question of how reaction-diffusion equations in thin domains may be approximated by diffusion equations in regions of smaller dimensions. In particular, we study how transmission conditions featuring in the approximating equations become integral parts of the limit master equation. We device a scheme which, by appropriate rescaling of coefficients and finding a common reference space for all Feller semigroups involved, allows deriving the form of the limit equation formally. The results obtained, are expressed as convergence theorems for the Feller semigroups.

In the first part of the talk we take a look at the question of approximating solutions to equations governing diffusion in thin $3D$ layers separated by a semi-permeable membrane (see Figure 1). We show that as thickness of the layers converges to 0, the solutions, which by nature are functions of 3 variables, gradually lose dependence on the vertical variable and thus may be regarded as functions of 2 variables. The limit equation describes diffusion on the lower and upper sides of a two-dimensional surface (the membrane) with jumps from one side to the other. The latter possibility is expressed as an additional term in the generator of the limit semigroup, and this term is build from permeability coefficients of the membrane featuring in the transmission conditions of the approximating equations (i.e., in the description of the domains of the generators of the approximating semigroups).

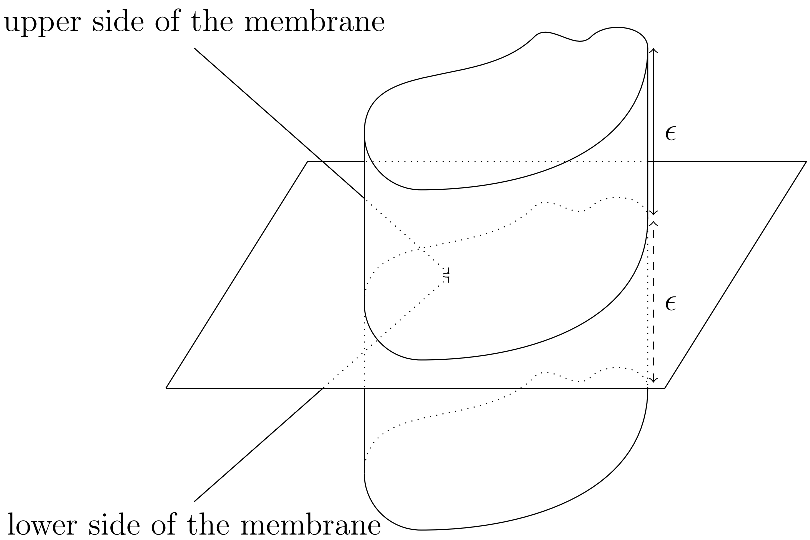

Figure 1: Two thin layers separated by a semi-permeable flat membrane

In the second part we work in two dimensions and deal with different geometry: we choose the example of a circular membrane as a case study. Again, the goal of this study is to establish the fact that in the thin layer approximation transmission conditions become integral parts of the limit equation in three natural scenarios.

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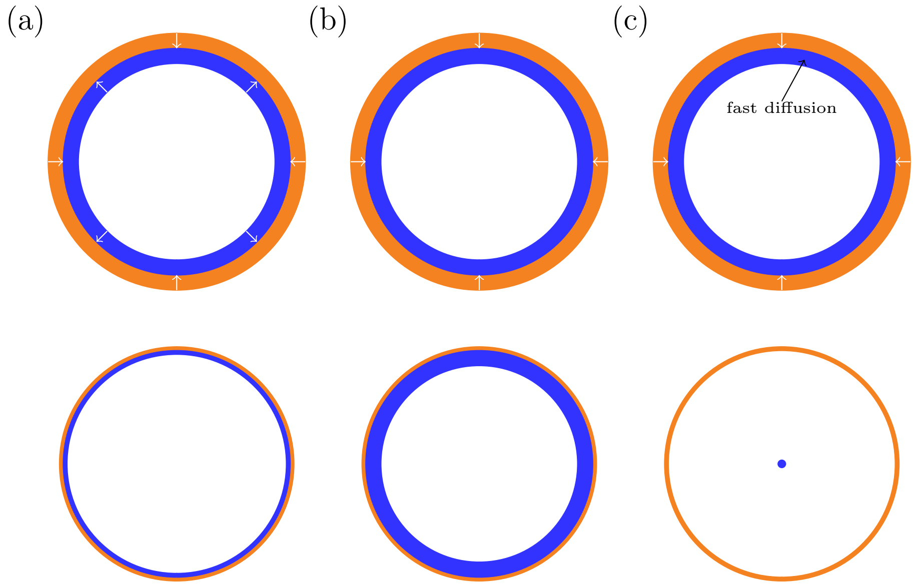

Figure 2: Different limit state-spaces resulting from different approximations: In the case (a) both layers are thin, and the limit state-space is composed of the upper and the lower sides of the unit circle. In the case (b) only the upper layer is thin, and thus the limit state-space is composed of the union of the unit circle and the lower layer/annulus. In the case (c), even though the lower layer is thick, diffusion there is fast and thus the limit state-space is the union of the unit circle and the point formed by compounding all elements of the lower annulus.

In the first scenario we assume that both the upper and the lower layers are thin, and in the limit obtain two coupled PDEs on a surface of dimension one (see Figure 2 (a)). Next, we turn to a 'thick' two-dimensional region bordering a 'thin' two-dimensional region (see Figure 2 (b)). In this case, in the limit we face a PDE on a two-dimensional region coupled with a PDE on a one-dimensional surface. Interestingly, here also one of permeability coefficients becomes an integral part of the main equation and describes jumps from the one-dimensional surface to the two-dimensional region.

In the third scenario, a diffusion in a thin layer is accompanied by a very fast diffusion in the bordering 'thick' two-dimensional region (see Figure 2 (c)). Then the limit equation involves a PDE on a surface coupled with an ODE: the fast diffusion averages out the concentration in the two-dimensional region, and thus, at any particular time, the concentration may be described by a single real number. Again, permeability coefficients become jump intensities between points of the surface and the isolated point formed by lumping all the points of the 'thick' two-dimensional region.

References

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