# Degenerate Nonlinear Semigroups for General Mathematical Filtration Boussinesq Model

**N. A. Manakova,[1] K. V. Perevozchikova.[2]**

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Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary of class $C^\infty$. In the cylinder $\Omega \times \mathbf{R}_+$ consider the Cauchy initial condition

$$u(0) = u_0 \tag{1}$$

and Dirichlet boundary condition

$$u(s,t) = 0, \ (s,t) \in \partial\Omega \times \mathbf{R}_+ \tag{2}$$

for the generalized filtration Boussinesq equation [1]

$$(\lambda - \Delta)u_t = \Delta(|u|^{p-2}u), \ p \geq 2. \tag{3}$$

Equation (3) is the most interesting particular case of the equation obtained by E.S. Dzektser [1]. Here the desired function $u = u(s,t)$ corresponds to the potential of speed of movement of the free surface of the filtered liquid; the parameter $\lambda \in \mathbf{R}$ characterizes the medium, and this parameter $\lambda$ can take negative values. In [2], a study of the phase space is given. The paper [3] was the first to obtain the conditions for the existence of a local solution to the Showalter–Sidorov–Dirichlet problem. Also, the paper [3] shows that the phase space of the generalized filtration Boussinesq equation is a smooth Banach manifold.

Let $\mathbf{H} = W_2^{-1}(\Omega)$, $\mathcal{H} = L_2(\Omega)$, $\mathcal{B} = L_p(\Omega)$ (all functional spaces are defined on domain $\Omega$). Note that there exists the dense and continuous embedding $\overset{\circ}{W}{}_2^1(\Omega) \hookrightarrow L_q(\Omega)$ for $p \geq \frac{2n}{n+2}$, therefore $L_p(\Omega) \hookrightarrow W_2^{-1}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In $\mathbf{H}$, define the scalar product by the formula

$$\langle x, y \rangle = \int_\Omega x\tilde{y}ds \ \ \forall x, y \in \mathbf{H},$$

where $\tilde{y}$ is the generalized solution to the homogeneous Dirichlet problem for Laplace operator $(-\Delta)$ in the domain $\Omega$. Let $\mathcal{B}^* = (L_p(\Omega))^*$ and $\mathcal{H}^* = (L_2(\Omega))^*$, where $(L_p(\Omega))^*$ is conjugate space with respect to duality

$$\mathcal{B} \hookrightarrow \mathcal{H} \hookrightarrow \mathbf{H} \hookrightarrow \mathcal{H}^* \hookrightarrow \mathcal{B}^*.$$

Define the operators $L$ and $M$ as follows:

$$\langle Lu, v \rangle = \int_\Omega (\lambda u\tilde{v} + uv)ds, \quad u, \, v \in \mathcal{H};$$

$$\langle M(u), v \rangle = -\int_\Omega |u|^{p-2}uvds, \quad u, \, v \in \mathcal{B}.$$

[1]Department of Mathematical Physics Equations, South Ural State University, 76, Lenin ave, Chelyabinsk, 454080, Russian Federation, manakovana@susu.ru

[2]Department of Mathematical Physics Equations, South Ural State University, 76, Lenin ave, Chelyabinsk, 454080, Russian Federation, perevozchikovakv@susu.ru

Let $\{\varphi_k\}$ be the sequence of eigenfunctions of the homogeneous Dirichlet problem for Laplace operator $(-\Delta)$ in the domain $\Omega$, and $\{\lambda_k\}$ be the corresponding sequence of eigenvalues numbered in non-increasing order taking into account the multiplicity.

*Lemma 1.* [4] (i) For all $\lambda \geq -\lambda_1$ the operator $L \in \mathcal{L}(\mathcal{H};\mathcal{H}^*)$ is self-adjoint, Fredholm, and non-negatively defined, and the orthonormal family $\{\varphi_k\}$ of its functions is total in the space $\mathcal{H}$.

(ii) Operator $M \in C^1(\mathcal{B};\mathcal{B}^*)$ is dissipative and $p$-coercive.

If $\lambda \geq -\lambda_1$

$$\ker L = \begin{cases} \{0\}, \ if \ \lambda > -\lambda_1; \\ \operatorname{span}\{\varphi_1\}, \ if \ \lambda = -\lambda_1. \end{cases}$$

Therefore

$$\operatorname{im} L = \begin{cases} \mathcal{H}^*, \ if \ \lambda > -\lambda_1; \\ \{u \in \mathcal{H}^* : \langle u, \varphi_1 \rangle = 0\}, \ if \ \lambda = -\lambda_1, \end{cases}$$

$$\operatorname{coim} L = \begin{cases} \mathcal{H}, \ if \ \lambda > -\lambda_1; \\ \{u \in \mathcal{H} : \langle u, \varphi_1 \rangle = 0\}, \ if \ \lambda = -\lambda_1. \end{cases}$$

Hence, the projectors

$$P = Q = \begin{cases} \mathbf{I}, \ \lambda > -\lambda_1; \\ \mathbf{I} - \langle \cdot, \varphi_1 \rangle, \ \lambda = -\lambda_1. \end{cases}$$

Construct the set

$$\mathbf{M} = \begin{cases} \mathcal{B}, \ if \ \lambda > -\lambda_1; \\ \{u \in \mathcal{B} : \int\limits_\Omega |u|^{p-2} u \varphi_1 \, ds = 0\}, \ if \ \lambda = -\lambda_1. \end{cases}$$

*Theorem 1.* [4] Suppose that $p \geq \frac{2n}{n+2}$, $\lambda \geq -\lambda_1$. Then

(i) the set $\mathbf{M}$ is a simple Banach $C^1$-manifold modelled by the space $\operatorname{coim} L \cap \mathcal{B}$;

(ii) $\forall \, u_0 \in \mathbf{M}$ there exists the unique solution $u \in C^k((0,+\infty);\mathbf{M})$ to problem (1) – (3).

Define the shift operator $U^t(u_0) \equiv u(t)$, where $u(t)$ is a solution to problem (1) – (3). Then $\{U^t : \ t \in \mathbf{R}_+\}$ forms a nonlinear semigroup of operators with domain $D(U) = \mathbf{M}$.

*Theorem 2.* [4] Suppose that $p \geq \frac{2n}{n+2}$, $\lambda \geq -\lambda_1$. Then there exists a resolving semigroup of contractive operators $\{U^t : \ t \in \mathbf{R}_+\}$ of equation (1) defined on the manifold $\mathbf{M}$.

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**References**

[1] E.S. Dzektser. The Generalization of the Equations of Motion of Groundwater. // Proceedings of the USSR Academy of Sciences. 1972. V. 5. P. 1031–1033. (In Russian)

[2] Sviridyuk G.A. A Problem of Generalized Boussinesq Filtration Equation. // Soviet Mathematics (Izvestiya VUZ. Matematika). 1989. V. 33. No. 2. P. 62–73. (In Russian)

[3] G.A. Sviridyuk, I.N. Semenova. Solvability of an Inhomogeneous Problem for the Generalized Boussinesq Filtration Equations. // Differential Equations 1988. V. 24. No. 9. P. 1065–1069.

[4] K. Vasiuchkova, N.A. Manakova. G.A. Sviridyuk. Degenerate Nonlinear Semigroups of Operators and Their Applications. // Springer Proceedings in Mathematics and Statistics. 2020. V. 325. P. 363–378.