Lévy Laplacian and gauge fields

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The Lévy Laplacian Δ_L is an infinite dimensional Laplacian which was originally defined in the following way (see [1]). Let f be a twice Fréchet differentiable function on $L_2([0,1],\mathbb{R})$ and $\{e_n\}$ be an orthonormal basis in $L_2([0,1],\mathbb{R})$. Then

$$\Delta_L f(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \langle f''(x)e_k, e_k\rangle \text{ for all } x \in L_2([0,1],\mathbb{R}).$$

The definition of the Lévy Laplacian depends on the choice of an orthonormal basis.

Let M be a Riemannian manifold and Ω_m be a Hilbert manifold of H^1-curves in M with the fixed origin $m \in M$. The Lévy Laplacian Δ_L^{AGV} on the space of sections in a vector bundle over Ω_m was considered in [2,3,4,5,6]. It turns out that nonlinear differential equations of the Yang–Mills theory are equivalent to linear differential equations containing such Lévy Laplacian.

Le A be a connection in a vector bundle over M and F be the associated curvature. The parallel transport U^A generalized by the connection A can been considered as a section in a vector bundle over Ω_m. The following theorem was proved for $M = \mathbb{R}^4$ by Accardi, Gibilisco and Volovich in [2] and generalized for the Riemannian manifold by Léandre and Volovich in [3].

Theorem 1. A connection A is a solution of the Yang–Mills equations

$$D_A^* F = 0,$$

where D_A^* is the adjoint operator to the exterior covariant derivative generated by A, if and only if the parallel transport U^A is a solution of the Laplace equation for the Lévy Laplacian Δ_L^{AGV}

$$\Delta_L^{AGV} U^A = 0.$$

We show that the similar theorem holds for the Yang–Mills heat flow (see [5]).

Theorem 2. A time-depended connection $[0,T] \ni t \mapsto A(t,\cdot)$ is a solution of the Yang–Mills heat equations

$$\partial_t A = -D_A^* F$$

if and only if the flow of parallel transports $[0,T] \ni t \mapsto U^{A(t,\cdot)}$ is a solution of the heat equation for the Lévy Laplacian Δ_L^{AGV}

$$\partial_t U^A = \Delta_L^{AGV} U^A.$$

Let M be an orientable Riemannian 4-manifold. In this case, there is a connection between Lévy Laplacians and instantons. A connection A is called an instanton (antiinstanton) if it is a solution of the anti-self-duality (self-duality) Yang–Mills equations

$$F = - * F \ (F = *F),$$

where $*$ is the Hodge star on M. The Lévy Laplacian Δ_L^{AGV} is not rotation invariant. A modified Lévy Laplacian Δ_L^{W} can be obtained from the Lévy Laplacian by the action of an infinite

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dimensional rotation $W \in C^1([0,1], SO(4))$. We show (see [6]) that the Laplace equations for some modified Lévy Laplacians are equivalent to the anti-self-duality (self-duality) Yang–Mills equations, but not to the Yang–Mills equations.

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