Random linear operators and limit theorems for their compositions

V. Zh. Sakbaev[1], E. V. Shmidt.[2]

Keywords: random linear operators; random operator valued process; compositions of random operators; random semigroup; law of large numbers; generalized convergence in distribution

MSC2010 codes: 60B12, 60B20, 47B80, 47H40

Introduction. It is known (see [1]) that the limit properties of distribution of the sum of random variables with values in the topological vector spaces can be described by limit theorems. In particular, the law of large numbers describes the convergence in probability of the sequence of averaged sum of independent identically distributed (iid) random vector valued variables to the limit of the mean value of the sum. The central limit theorem gives the conditions of the convergence in distribution for the sequence of averaged sum of iid random vector valued variables to the Gaussian random vector.

We study the sequence of compositions of iid random variables with values in the Banach algebra of bounded linear operators $B(H)$ acting in the separable Hilbert space H. In the commutative case of operators of an argument shift on a random vector the limit distribution of averaged composition can be described by the limit theorems for the sum of vector valued variables. Some results on the LLN and CLT for the averaged composition of independent random matrices or linear operators was obtained in [2, 3, 4]. We obtain the analogs of LLN and CLT for the sequence of compositions of iid random semigroups or $B(H)$-valued random processes with non-commutative values.

Law of Large numbers for compositions of random semigroups. Let $\mathbf{A}_j$, $j \in \mathbb{N}$, be the sequence of independent identically distributed random variables with the values in Banach space of bounded linear operators $B(H)$ in some Hilbert space H. We are studying the asymptotic behavior of the probability distribution of the averaged random variables

$$\bar{\mathbf{A}}_n = (\mathbf{A}_n)^{\frac{1}{n}} \circ ... \circ (\mathbf{A}_1)^{\frac{1}{n}},$$

when $n \to \infty$. Here the fractional power of the operator is defined by means of spectral decomposition for unitary or self-adjoin operator. The fractional power for the operator $\mathbf{U}(t)$ belonging to the set of semigroup values $\{\mathbf{U}(t),\ t \geq 0\}$ is defined according to its dynamical sense: $(\mathbf{U}(t))^{\frac{1}{n}} = \mathbf{U}(\frac{t}{n})$.

Let $Y_s(H)$ be the topological vector space of the maps $[0, +\infty) \to B(H)$ which is continuous in the strong operator topology. The topology τ_s of the space $Y_s(H)$ is generated by the family of seminorms $\Phi_{T,v}$, $T \geq 0$, $v \in H:\ \Phi_{T,v}(\mathbf{U}) = \sup\limits_{t \in [0,T]} \|\mathbf{U}(t)v\|_H$, $\mathbf{U} \in Y_s$.

The random semigroup is defined as the measurable mapping $\mathbf{U}:\ \Omega \to Y_s(H)$ of the probability space $(\Omega, \mathcal{A}, \mu)$ into the measurable space $(Y_s, \mathcal{B}_s)$ such that the values of this map are C_0-semigroups. Here $\mathcal{B}_s$ is the Borel σ-algebra of subsets of the topological space $(Y_s(H), \tau_s)$.

Theorem 1. Let $\mathbf{A}$ be the random variable with the values in the set of self-adjoint operators in the space H and let $\mathbf{U}(t) = \exp(i\mathbf{A}t)$, $t \geq 0$, be the corresponding random semigroup. Let $\mathcal{D}$ be the dense linear manifold of the space H such that $\int\limits_\Omega \|\mathbf{A}(\omega)u\|_H d\mu(\omega) < \infty\ \forall\ u \in \mathcal{D}$. Let operator $\bar{\mathbf{A}}u = \int\limits_\Omega \mathbf{A}(\omega)u\, d\mu(\omega),\ u \in \mathcal{D}$ be essentially self-adjoint. Let $\{\mathbf{U}_n\}$ be the sequence of independent identically distributed random semigroups such that any of them has the same

[1]Moscow Institute of Physics and Technology, Department of General Mathematics, Russia, Dolgoprudny. Email: sakbaev.vzh@phystech.edu

[2]Moscow Institute of Physics and Technology, Department of General Mathematics, Russia, Dolgoprudny. Email: eugenelighting@yandex.ru

distributions. Then the sequence $\{\mathbf{U}_n \circ ... \circ \mathbf{U}_1\}$ of its compositions satisfies the LLN in the strong operator topology:

$$\lim_{n\to\infty} [\sup_{t\in[0,T]} P(\|(\mathbf{U}_n^{\frac{1}{n}} \circ ... \circ \mathbf{U}_1^{\frac{1}{n}} - M[\mathbf{U}_n^{\frac{1}{n}} \circ ... \circ \mathbf{U}_1^{\frac{1}{n}}])x\|_H > \epsilon)] = 0 \quad \forall x \in H, \ \forall\, \epsilon > 0.$$

Generalization weak convergence and convergence in distribution for compositions of operator valued random processes. Let E be a Hilbert space. Let $B(E)$ be a Banach space of bounded linear operators in the space E endowed with some operator topology. Let $ca(B(E), \mathcal{B}(B(E)))$ be a Banach space of Borel measure with bounded variation oh the measurable space $(B(E), \mathcal{B}(B(E)))$. Let X be a locally convex space of complex valued functions on the space E which is invariant with respect to argument shift on any vector of the space E. Let $\mathcal{L}(X)$ be a locally convex space of linear operators acting in the space X.

Definition. A sequence of measures $\{\mu_n\} : \mathbb{N} \to ca(B(E), \mathcal{B}(B(E)))$ converges $\mathcal{L}(X)$-weakly to the measure $\mu \in ca(B(E), \mathcal{B}(B(E)))$ if the sequence of operators $\{\Psi_{\mu_n}\}$ where $\Psi_{\mu_n} u(x) = \int\limits_{B(E)} u(\mathbf{A}x) d\mu_n(\mathbf{A}), \ u \in X, \, x \in E$, converges in the space $\mathcal{L}(X)$ to the operator Ψ_μ: $\Psi_\mu u(x) = \int\limits_{B(E)} u(\mathbf{A}x) d\mu(\mathbf{A}), \ u \in X, \, x \in E$.

Definition. The sequence $\{\xi_n\}$ of random variables with values in the space $B(E)$ converges in the distribution $\mathcal{L}(X)$-weakly to the random variable ξ if the sequence of Borel measures $\{\mu_n\}$: $\mu_n(A) = P(\xi_n^{-1}(A)), \ A \in \mathcal{B}(B(E)), \, n \in \mathbb{N}$, converges $\mathcal{L}(X)$-weakly to the measure μ: $\mu(A) = P(\xi^{-1}(A)), \ A \in \mathcal{B}(B(E))$.

Theorem 2. Let $\xi(t), \ t \geq 0$, be a random process with values in the space $B(E)$. Let $\{\xi_n\}$ be a sequence of iid random processes such that any of them has the same distribution. Let X be a Banach space of functions $u : \ E \to \mathbb{C}$ such that for any $t \geq 0$ the linear operator $u \to \mathbf{F}(t)u = \mathrm{M}(u(\xi(t)\cdot))$ is satisfied on the space X. If the function $\mathbf{F}(t), \, t \geq 0$, satisfies the conditions of Chernoff theorem then the sequence of random processes $\{\eta_n(t), \, t \geq 0\}$, where $\eta_n(t) = \xi_n(\frac{t}{n}) \circ ... \circ \xi_1(\frac{t}{n})$, converges in distribution with respect to the space $(B(X), \tau_{sot})$ to the Markov random processes corresponding to the semigroup $\exp(\mathbf{F}'(0)t), \, t \geq 0$.

Remark 1. The CLT for the sum of iid random vectors in Euclidean space $\mathbb{R}^d$ gives the convergence in distribution with respect to topological vector space $C_b(\mathbb{R}^d)$ of bounded continuous function endowed with the pointwise convergence topology.

References

[1] W. Feller. An Introduction to Probability Theory and its Applications. Volume 2. — J. Wiley, New York, 1971.

[2] V. Tutubalin. A local limit theorem for products of random matrices. // Theory Probab. Appl. 1978. V. 22. No. 2. P. 203–214.

[3] A. Skorokhod. Products of independent random operators. // Russian Math. Surveys. 1983. V. 38. No. 4. P. 291–318.

[4] Yu.N. Orlov, V.Zh. Sakbaev, O.G. Smolyanov. Feynman Formulas and the Law of Large Numbers for Random One-Parameter Semigroups. // Proc. Steklov Inst. Math. 2019. V. 306. P. 196–211.