Degenerate analytic resolving groups of operators for solutions of the Barenblatt – Zheltov – Kochina equation in "noise" spaces on a Riemannian manifold

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Keywords: Sobolev type equations; Showalter – Sidorov condition; Laplace – Beltrami operator; Nelson – Gliklikh derivative; stochastic process; differential forms.

MSC2010 codes: 35S11, 58A10, 60H15

Introduction. It is known (see [1]) that the linear Barenblatt – Zheltova – Kochina equation

$$(\lambda - \Delta)\dot{u} = \alpha\Delta u + f, \tag{1}$$

describing the dynamics of the pressure of the fluid filtered in a fractured porous medium. The coefficient λ corresponds to the ratio of cracks and pores in rock, and the coefficient α corresponds for the visco-elastic properties of the liquid. The beginning of the investigation of equation (1) should be related to [2], where this equation was considered for the first time as a linear inhomogeneous Sobolev type equation

$$L\dot{u} = Mu + f. \tag{2}$$

Here the operators L and M are the operators $\lambda - \Delta$ and $\alpha\Delta$, given in some functional spaces $L, M : \mathfrak{U} \to \mathfrak{F}$. Equation (2) is equipped with the initial Showalter – Sidorov condition [3]

$$P(u(0) - u_0) = 0, \tag{3}$$

where the projector P is constructed with the use of operators L and M. Note that in the case of existence operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ (those linear and bounded operators), condition (3) becomes the Cauchy condition

$$u(0) = u_0. \tag{4}$$

We also note [4], where the Barenblatt–Zheltov–Kochina equation (1) is reduced to the form (2) defined in the spaces of differential forms on smooth Riemannian manifolds without boundary.

We will be interested in the stochastic interpretation of the deterministic of equation (2), namely:

$$L \overset{o}{\eta} = M\eta + N\Theta. \tag{5}$$

Here the operators L and M are the same as in (2), the operator $N \in \mathcal{L}(\mathfrak{U}; \mathfrak{F}), \eta = \eta(t)$ is required, and $\Theta = \Theta(t)$ is a given stochastic process with values in the Hilbert space $\mathfrak{U}$. Through $\overset{o}{\eta}$ we denote the Nelson – Gliklikh derivative of the stochastic process $\eta = \eta(t)$ (for details see [5]).

Preliminary information. Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $\mathbb{R}$ be a set of real numbers endowed with a Borel σ-algebra. Measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *a random variable.* A set of random variables with a zero mathematical expectation ($\mathbf{E}$) and finite variance ($\mathbf{D}$) forms a Hilbert space with scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. The resulting Hilbert space is denoted by symbol $\mathbf{L_2}$.

Let $\xi \in \mathbf{L_2}$ then $\Pi\xi$ is called *the conditional mathematical expectation* of a random variable ξ and denoted by the symbol $\mathbf{E}(\xi|\mathcal{A}_\mathbf{0})$. Recall also that the minimal σ-subalgebra $\mathcal{A}_0 \subset \mathcal{A}$,

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with respect to which the random variable ξ is measurable, is called the *σ-algebra generated by ξ*. Let $\mathfrak{I} \subset \mathbb{R}$ be a certain interval. Consider two mappings: $f : \mathfrak{I} \to \mathbf{L_2}$, which puts to each $t \in \mathfrak{I}$ a random variable $\xi \in \mathbf{L_2}$, and $g : L_2 \times \Omega \to \mathbb{R}$, which puts to each pair (ξ, ω) the point $\xi(\omega) \in \mathbb{R}$. The map $\eta : \mathfrak{I} \times \Omega \to \mathbb{R}$, having the form $\eta = \eta(f(t), \omega)$, is called *a stochastic process*. The stochastic process η is called *continuous*, if a.s. (almost surely) all its trajectories are continuous (for almost all $\omega \in \Omega$ the trajectories $\eta(\cdot, \omega)$ are continuous). By the symbol $\mathbf{CL_2}$ we denote the set of the continuous stochastic processes. Let's call *Gaussian* continuous stochastic process the process, if its (independent) random variables are Gaussian.

We use derivative was founded by E. Nelson, and the theory of such derivative was developed by Yu.E. Gliklikh (see [5]), then further, for brevity, the derivative of a stochastic process η will be called the *Nelson–Gliklikh derivative* and denote by $\overset{o}{\eta}$.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, the operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. Following [2] we introduce the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and the L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of the operator M. If the L-spectrum $\sigma^L(M)$ of the operator M is bounded, then the operator M is said to be (L, σ)-bounded. If the operator M is (L, σ)-bounded, then there exist projectors

$$P = \frac{1}{2\pi i} \int_\gamma R^L_\mu(M) d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i} \int_\gamma L^L_\mu(M) d\mu \in \mathcal{L}(\mathfrak{F}). \tag{6}$$

where expression $R^L_\mu(M) = (\mu L - M)^{-1} L$ is the *right*, and $L^L_\mu(M) = L(\mu L - M)^{-1}$ is the *left L-resolvent* of the operator M, and the closed contour $\gamma \subset \mathbb{C}$ bounds a domain containing $\sigma^L(M)$. We set $\mathfrak{U}^0(\mathfrak{U}^1) = \ker P(\mathrm{im} P)$, $\mathfrak{F}^0(\mathfrak{F}^1) = \ker Q(\mathrm{im} Q)$ and denote by $L_k(M_k)$ the restriction of the operator $L(M)$ on $\mathfrak{U}^k$, $k = 0, 1$.

Theorem 1. [2] Let the operator M be (L, σ)-bounded, then
(i) the operators $L_k(M_k) \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;
(ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$.

We construct the operators $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1} M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Corollary 1. [2] Suppose that the operator M is (L, σ)-bounded, then for all $\mu \in \rho^L(M)$

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty} \mu^k H^k M_0^{-1} (\mathbb{I} - Q) + \sum_{k=1}^{\infty} \mu^{-k} S^{k-1} L_1^{-1} Q. \tag{7}$$

The operator M is called *(L, p)-bounded,* $p \in \{0\} \cup \mathbb{N}$, if ∞ is a removable singular point (that is, $H \equiv \mathbb{O}, p = 0$) or a pole of order $p \in \mathbb{N}$ (that is $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of the *L-resolvent* $(\mu L - M)^{-1}$ of the operator M.

We consider a n-dimensional smooth compact oriented connected Riemannian manifold without boundary Ω and the space of differential q-forms on Ω we denote by $E^q = E^q(\Omega), 0 \leq q \leq n$. In particular $E^0(\mathbb{R}^n)$ is the space of functions of n variables. Note that there exists a linear Hodge operator $* : E^q \to E^{n-q}$, which associates the q-form with Ω $(n - q)$-form. In the double application of the Hodge operator, the equality $** = (-1)^{q(n-q)}$ holds. In addition, there is an operator for taking the external differential $d : E^q \to E^{q+1}$. We define the operator $\delta : E^q \to E^{q-1}$, setting $\delta = (-1)^{n(q+1)+1} * d *$. The Laplace-Beltrami operator $\Delta : E^q \to E^q$ is defined by the equality $\Delta = \delta d + d\delta$, and it is a linear operator on space $E^q, 0 \leq q \leq n$. We introduce the space of harmonic q-form $H^q = \{\omega \in E^q : \Delta\omega = 0\}$.

Theorem 2. [4] (Hodge's splitting theorem) For any integer $q, 0 \leq q \leq n$, space H^q is finite-dimensional and there is the following decomposition of the space of smooth q-forms on Ω into an orthogonal direct sum $E^q = \Delta(E^q) \oplus H^q = d\delta(E^q) \oplus \delta d(E^q) \oplus H^q$.

By the formula $(\xi, \eta)_0 = \int\limits_{\Omega_n} \xi \wedge *\eta$, $\xi, \eta \in E^q$ where $*$ is the Hodge operator, we define a scalar product in the space E^q, $q = 0, 1, \ldots, n$, and denote the corresponding norm by $|| \cdot ||_0$.

We continue this scalar product to a direct sum $\bigoplus\limits_{q=0}^{n} E^q$, requiring that different spaces E^q were orthogonal. Completion of space E^q in the norm $||\cdot||_0$ we denote by $\mathfrak{H}_0^q$. We denote by $P_{q\Delta}$ the projector on $\mathfrak{H}_\Delta^q$.

We similarly define $\mathfrak{H}_1^q$ and $\mathfrak{H}_2^q$ respectively. Actually upper index means how many times differentiable in the generalized sense of the q-form in the corresponding spaces.

The spaces $\mathfrak{H}_l^q$,$l=1,2$ are Hilbert spaces, and we have continuous and dense embedding $\mathfrak{H}_2^q \subset \mathfrak{H}_1^q \subset \mathfrak{H}_0^q$.

Corollary 2. For any $q=0,1,\dots,n$ there are splitting spaces

$$\mathfrak{H}_l^q = \mathfrak{H}_{l\Delta}^{q1} \oplus \mathfrak{H}_\Delta^q,$$

where $\mathfrak{H}_{l\Delta}^{q1} = (\mathbb{I} - P_\Delta)[\mathfrak{H}_l^q]$, $l=0,1,2$.

Main result. Let consider $\mathfrak{U} = \mathbf{UL_2\mathfrak{H}^q_{2\Delta}}, \mathfrak{F} = \mathbf{FL_2\mathfrak{H}^q_{0\Delta}}$ be a real separable Hilbert space by differential q-form with stochastic coefficient defined on a smooth compact oriented Riemannian manyfold without boundary two and zero times derivative respectively. Our operator $M=\lambda-\Delta$ be $(L,0)$-bounded($L=\alpha\Delta$).

Consider a linear stochastic equation of Sobolev type

$$L \overset{o}{\eta} = M\eta + N\Theta. \tag{8}$$

We supplement equation (8) with the initial Showalter-Sidorov condition

$$\left[R_\alpha^L(M)\right]^{p+1}(\eta(0) - \eta_0) = 0, \tag{9}$$

where

$$\eta_0 = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k, \tag{10}$$

$\{\varphi_k\}$ is an orthonormal basis of the space $\mathfrak{U}$, and pairwise independent random Gaussian variables $\xi_k \in \mathbf{L}_2$ are such that $D\xi_k \le C_0$, and $\{\lambda_k\}$ is the spectrum of some nuclear operator $(\mathbf{K} = \{\lambda_\mathbf{k}\}, \sum\limits_{k=0}^{\infty} \lambda_k^2 < \infty)$.

Theorem 3. For any $\lambda \in \mathbb{R}\setminus\{0\}$, $\alpha \in \mathbb{R}\setminus\{0\}$ and any operator $N \in \mathcal{L}(\mathfrak{F})$ and $\eta_0 \in \mathbf{L}_2$, that does not depend on Θ and satisfies (10) exists a unique classical solution $\eta = \eta(t)$ of problem (8), (9).

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