Optimal control problem for linear Sobolev type mathematical models

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Keywords: Sobolev type equations; strong solutions; optimal control; phase space.

MSC2010 codes: 35K70, 49K20

The report presents a review of the work of the Chelyabinsk mathematical school on Sobolev type equations in studying the optimal control problems for linear Sobolev type models with initial Cauchy (Showalter–Sidorov) conditions or initial-final conditions. To identify the nonemptiness of the set of solutions to the control problem we use the phase space method, which has already proved itself in solving Sobolev type equations. The method reduces the singular equation to a regular one defined on some subspace of the original space and applies the theory of degenerate (semi)groups of operators to the case of relatively bounded, sectorial and radial operators. Here mathematical models are reduced to initial (initial-final) problems for an abstract Sobolev type equation. Abstract results are applied to the study of control problems for the Barenblatt–Zheltov–Kochina mathematical model, which describes fluid filtration in a fractured-porous medium, the Hoff model on a graph simulating the dynamics of I-beam bulging in a construction, and the Boussinesq–Löve model describing longitudinal vibrations in a thin elastic rod, taking into account inertia and under external load, or the propagation of waves in shallow water. The mathematical models under consideration belong to a wide class of Sobolev type models (i.e., models based on Sobolev type equations).

The mentioned mathematical models with one or another initial (initial-final) conditions in suitable Hilbert spaces can be reduced to the corresponding problems for a linear Sobolev type equation

$$Ax^{(n)} = Bx + y + Cu, \tag{1}$$

where operators $A \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $B \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, $C \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$ functions $u : \mathfrak{I} \to \mathfrak{U}$, $y : \mathfrak{I} \to \mathfrak{Y}$ ($\mathfrak{I} \subset \mathbb{R}$), and $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ are Hilbert spaces. To select the only process under study, the mathematical models under consideration and their abstract interpretation (1) are supplemented by one of the following conditions:

– the Cauchy conditions

$$x^{(m)}(0) = x_m,\ m = 0, \ldots n-1, \tag{2}$$

– the Showalter–Sidorov conditions

$$P\left(x^{(m)}(0) - x_m\right) = 0,\ m = 0, ..., n-1, \tag{3}$$

– the initial-final conditions

$$P_{in}\left(x^{(m)}(0) - x_m^0\right) = 0,\ P_{fin}\left(x^{(m)}(\tau) - x_m^\tau\right) = 0,\ m = 0, ..., n-1, \tag{4}$$

where P, P_{in}, P_{fin} are some spectral projectors in the space $\mathfrak{X}$. Condition (4) differs from the initial conditions in that one projection of the solution is specified at the initial moment of time, and the other at the final moment of the considered time interval. The initial-final condition is a generalization of the Showalter–Sidorov condition, which in turn is a generalization of the classical Cauchy condition. As it is well known, the Cauchy problem for the Sobolev type

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equation (1) (in case $\ker A \neq \{0\}$) is not solvable for arbitrary initial values x_m. To overcome this difficulty, G.A. Sviridyuk proposed the phase space method. The foundations of this concept were laid down in [1]. Another approach to overcome the difficulties associated with non-existence of the solution to (1), (2) is to consider the initial Showalter–Sidorov condition (3) and a more general initial-final condition (4) instead of the initial Cauchy condition (2). We are interested in solving the optimal control problem, which consists in finding a pair $(\hat{x}, \hat{u})$, for which the relation

$$J(\hat{x}, \hat{u}) = \min_{(x,u) \in \mathfrak{X} \times \mathfrak{U}_{ad}} J(x, u), \tag{5}$$

holds. Here the pairs (x, u) satisfy the Cauchy problem (1), (2) or the Showalter–Sidorov problem (1), (3), or the initial-final problem (1), (4) and $J(x, u)$ is some specially constructed quality functional, $\mathfrak{U}_{ad}$ is some closed and convex set in the control space $\mathfrak{U}$.

The report provides an overview of the results developed in the framework of the direction headed by G.A. Sviridyuk on the optimal control of the solutions to the initial-final problem and, in particular, the Showalter–Sidorov and Cauchy problems for linear Sobolev type equations. The first who began to study the controllability problems and the optimal control problem for linear Sobolev type equations with the Cauchy condition were G.A. Sviridyuk and A.A. Efremov [2]. In these papers, the optimal control problem with a quadratic quality functional was studied in case $n = 1$ with (A, p)-bounded or (A, p)-sectorial operator B and the Cauchy condition, the necessary and sufficient conditions for the existence and uniqueness of a solution were obtained. G.A. Sviridyuk suggested moving from considering the classical solution $x \in C^1(\mathfrak{J}; \mathfrak{X})$ of (1), (2) to the strong solution $x \in H^{p+1}(\mathfrak{X})$ of this problem, which allowed to set the optimal control problem (1), (2), (5) and to use the technique of Hilbert spaces for its research. These studies formed the basis of a number of works by G.A. Sviridyuk's disciples and followers on the study of optimal control problems for linear Sobolev type equations based on the theory of degenerate resolving (semi)groups of operators. When considering the classical Cauchy condition, due to the degeneracy of the equation, it was necessary [1-2] to reconcile the initial data with the control action. Then G.A. Sviridyuk suggested an idea to use more general initial Showalter–Sidorov condition (initial-final condition), which made it possible to remove the restriction on the set of optimal controls and opened the way to a whole class of problems on this subject. In [3] the necessary and sufficient conditions for the existence and uniqueness of the solution of optimal control problems for high-order Sobolev type equations with an initial-finite condition were obtained. The ideas and methods developed by G.A. Sviridyuk and A.A. Efremov on controllability of linear abstract Sobolev type equation opened the way to the study of more general controllability problems.

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