Long-term Behaviour of Flows in Infinite Networks

A. Dobrick[1]

Keywords: transport equations; infinite metric graphs; long-term behaviour; operator semigroups.

MSC2010 codes: 35B40; 47D06; 35R02; 35F46

Abstract. We study transport processes on infinite networks which can be modeled by an operator semigroup on a Banach space. Classically, such semigroups are strongly continuous and therefore their asymptotic behaviour is quite well understood. However, recently new examples of transport processes emerged in which the corresponding semigroup is not strongly continuous. Due to this lack of strong continuity, there are currently no results on the long-term behaviour of these semigroups. In this paper, we close this gap for a certain class of transport processes. In particular, it is proven that the solution semigroups behave asymptotically periodic with respect to the operator norm as a consequence of a more general result on the long-term behaviour by positive semigroups that contain a multiplication operator. Furthermore, we revisit known results on the asymptotic behaviour of transport processes on infinite networks and prove the asymptotically periodicity of the extensions of those semigroups to the space of bounded measures.

Introduction. Consider a transport process on an infinite network, modeled by an infinite, directed graph $G = (V, E)$ which is assumed to be simple, locally finite and non-degenerate and consider G as a metric graph by identifying each edge with the unit interval $[0, 1]$ and parametrizing it contrarily to its direction.

The distribution of mass transported along one edge e_j, $j \in J$, at some time $t \geq 0$ is described by a function $u_j(t, x)$ for $x \in [0, 1]$ and the material is transported along e_j with a constant velocity $c_j > 0$ that suffices

$$0 < c_{\min} \leq c_j \leq c_{\max} < \infty.$$

Define $\mathbb{B}^C := C^{-1}\mathbb{B}C$, where $\mathbb{B}$ denotes the weighted (transposed) adjacency matrix and $C := \operatorname{diag}(c_j)$ denotes the diagonal velocity matrix. Moreover, suppose that the functions u_j satisfy the generalized Kirchhoff law

$$\sum_{j \in J} \phi_{ij}^- c_j u_j(1, t) = \sum_{j \in J} \phi_{ij}^+ c_j u_j(0, t)$$

for all $i \in I$ and $t > 0$. Then the transport process can be modeled by the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t} u_j(t, x) = c_j \dfrac{\partial}{\partial x} u_j(t, x), & x \in (0, 1),\ t \geq 0, \\ u_j(0, x) = f_j(x), & x \in (0, 1), \\ u_j(1, t) = \displaystyle\sum_{j \in J} \mathbb{B}_{jk}^C u_k(0, t), & t \geq 0, \end{cases}$$

where f_j, $j \in J$, are the initial distributions of mass along the edges of G.

The investigation of equation systems of the above form on metric graphs by employing the theory of strongly continuous semigroups has quite some history. The semigroup approach to this kind of transport problems was introduced first in [1]. This paper was followed by a series of papers [2,3,4,5] from several different authors using the semigroup approach to discuss transport processes on metric graphs. In all these papers, transport equations are considered on the state

[1]Christian-Albrechts-Universität zu Kiel, Mathematisches Seminar, Germany, Kiel. Email: dobrick@math.uni-kiel.de

space $L^1([0,1];\ell^1)$, where the solution semigroups turn out to be strongly continuous, and a major point is the asymptotic behaviour of the solution semigroup. However, the investigation of the long-term behaviour relied on the theory of strongly continuous semigroups, the Jacobs-de Leeuw-Glicksberg decomposition and classical results from Perron-Frobenius theory.

Motivated by results from [6,7], the authors in [8] discuss transport processes on the state space $L^\infty([0,1];\ell^1)$, where the solution semigroup is only bi-continuous with respect to the weak*-topology on $L^\infty([0,1];\ell^1)$ (see [9] for a definition) but not strongly continuous. Although the same kind of equation is investigated (however in different state spaces), in contrast to the papers from the L^1-case, [8] does not discuss the asymptotic behaviour of the solutions in the bi-continuous case. In this paper, we close this gap by combining spectral theoretic observations, the concept of the semigroup at infinity and classical Perron-Frobenius theory. In particular, the following theorem holds without any regularity assumptions on the semigroup:

Theorem 1. Let $\Omega = (\Omega, \Sigma, \mu)$ be a measure space, E a Banach lattice and $(T(t))_{t\geq 0}$ a bounded, positive semigroup on $X := L^p(\Omega; E)$, $1 \leq p \leq \infty$. Suppose that there is $t_0 \in [0,\infty)$ such that $T(t_0) = \mathcal{M}_B$ for some irreducible operator B on E with $r(B) = 1$. If $r(B)$ is a pole of the resolvent of B, then there is a strictly positive projection P commuting with the semigroup $(T(t))_{t\geq 0}$ with the following properties:

(i) $(T(t)|_{PX})_{t\geq 0}$ can be extended to a positive, periodic group on the Banach lattice PX.

(ii) $(T(t)|_{\ker P})_{t\geq 0}$ is uniformly exponentially stable, i.e., there are $M \geq 1$ and $\varepsilon > 0$ such that $\|T(t) - T(t)P\| \leq M\mathrm{e}^{-\varepsilon t}$ for all $t \geq 0$.

Using our new methods we also revisit a result on the asymptotic behaviour of solutions from [2] and improve slightly upon the statement as well as the proof. In particular, our argument shows that the asymptotical periodicity of the solution, obtained there, does not depend on the strong continuity of the semigroup.

Acknowledgments. The author thanks Abdelaziz Rhandi for his very kind hospitality at the university of Salerno in Spring 2020 at the begin of the extraordinary Covid-19 situation in Italy. At this time major parts of this paper have been written. Moreover, I am indebted to my dear colleagues Marco Peruzzetto for several helpful discussions and Jochen Glück for helpful feedback at an early stage of the paper and bringing reference [1] to my attention.

References

[1] M. Kramar, E. Sikolya. Spectral properties and asymptotic periodicity of flows in networks. — Math. Z., 2005.

[2] B. Dorn. Semigroups for flows in infinite networks. — Semigroup Forum, 2008.

[3] B. Dorn, V. Keicher, E. Sikolya. Asymptotic periodicity of recurrent flows in infinite networks. — Math. Z., 2009.

[4] B. Dorn, M. Kramar Fijavž, R. Nagel, and A. Radl. The semigroup approach to transport processes in networks. — Phys. D, 2010.

[5] T. Matrai, E. Sikolya. Asymptotic behavior of flows in networks. — Forum Mathematicum, 2007.

[6] J. von Below and J. Lubary. The eigenvalues of the laplacian on locally finite networks. — Results in Mathematics, 2005.

[7] J. von Below and J. Lubary. The eigenvalues of the laplacian on locally finite networks under generalized node transition. — Results in Mathematics, 2009.

[8] C. Budde, M. Kramar Fijavž. Bi-continuous semigroups for flows in infinite networks. — *Preprint, available online from arxiv.org/abs/1901.10292.*

[9] F. Kühnemund. A Hille-Yosida theorem for bi-continuous semigroups. — Semigroup Forum, 2003.