Asymptotic decomposition of substochastic semigroups and applications

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One-Parameter Semigroups of Operators
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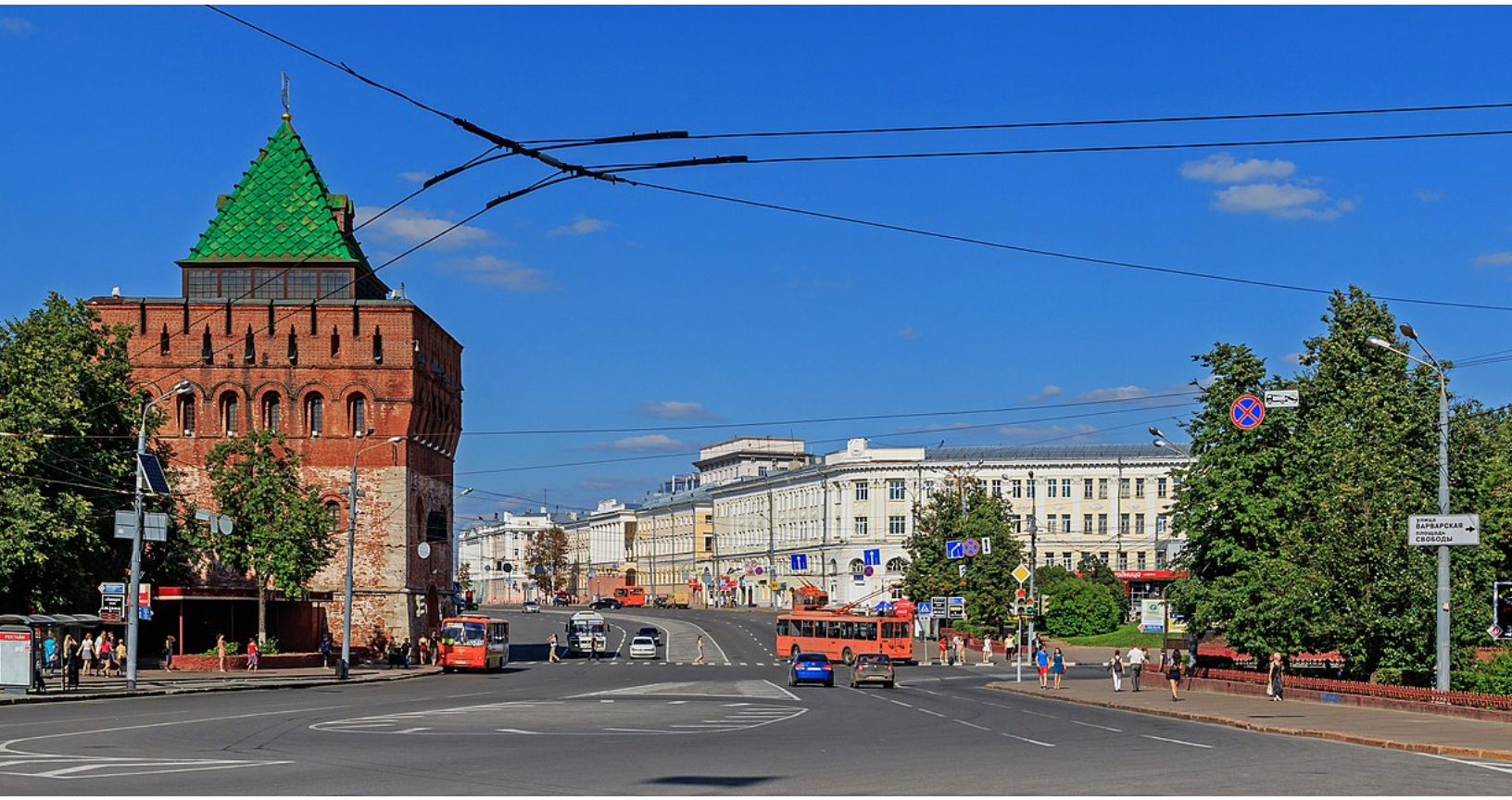

ВАРВАРСКАЯ
ПЛОЩАДЬ
СВОБОДЫ

Outline:

1. Stochastic semigroups.
2. Asymptotic decomposition and corollaries.
3. Piecewise deterministic Markov processes.
3. Applications to a gene expression model, to immunology, and to (a kinetic equation).

References:

1. K. Pichór, R.R., JMAA 2016.
2. K. Pichór, R.R., Stochastic and Dynamics 2017.
3. A. Tomski, R.R., J. Theor. Biol. 2015.
4. K. Pichór, R.R., Math Meth Appl Sci. 2020.
5. M. Mokhtar-Kharroubi, R.R., Acta Appl. Math. 2016.

Stochastic semigroups

(X, Σ, m) — σ-finite measure space.
$D = \{f \in L^1 : \ f \geq 0, \ \|f\| = 1\}$ – densities.
Stochastic operator (Markov operator):
$P : L^1 \to L^1$ linear, $P(D) \subset D$.
Substochastic operator:
$P : L^1 \to L^1$ linear, positive and $\|P\| \leq 1$.

Stochastic (substochastic) semigroup : $\{P(t)\}_{t \geq 0}$,
$P(t)$ - stochastic (substochastic) operators,
$P(0) = Id$, $P(t+s) = P(t)P(s)$, $s, t \geq 0$,
(c) for each $f \in L^1$, the function $t \mapsto P(t)f$ is continuous.

Asymptotic stability

$f_* \in D$ – *invariant* if $P(t)f_* = f_*$ for $t \geq 0$.
$\{P(t)\}$ – *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for} \quad f \in D.$$

Sweeping (zero-type property)

$\{P(t)\}$ – *sweeping* with respect to a family of sets $\mathcal{F}$ if for $B \in \mathcal{F}$ and for $f \in D$

$$\lim_{t\to\infty} \int_B P(t)f(x)\, m(dx) = 0.$$

$\{P(t)\}$ – partially integral if there exist $t > 0$, $k(t,x,y) \geq 0$

$$\int_X \int_X k(t,x,y)\, m(dx) m(dy) > 0$$

$$P(t)f(x) \geq \int k(t,x,y) f(y)\, m(dy) \quad \text{for} \quad f \in D.$$

Theorem 1 *If a partially integral stochastic semigroup $\{P(t)\}_{t\geq 0}$ has a unique invariant density f_* and $f_* > 0$ then it is asymptotically stable.*

X separable metric space, $\Sigma = \mathcal{B}(X)$,
$\{P(t)\}_{t\geq 0}$ substochastic semigroup with the kernel part $k(t,x,y)$,
(K) for every $y_0 \in X$ there exist $r > 0$, $t > 0$, and a function $\eta \geq 0$ s.t. $\int \eta \, dm > 0$ and

$$k(t,x,y) \geq \eta(x) \mathbf{1}_{B(y_0,r)}(y).$$

Theorem 2 *If (K) holds then:*
there are a countable (possible empty) set I,
continuous positive functionals α_i, $i \in I$,
and invariant densities f_i^*, $i \in I$,
with pairwise disjoint supports A_i,
such that for every density f and every compact set F we have

$$\lim_{t\to\infty} \|\mathbf{1}_{A_i} P(t) f - \alpha_i(f) f_i^*\| = 0,$$

$$\lim_{t\to\infty} \int_{F\cap Y} P(t) f(x)\, m(dx) = 0,\ Y = X \setminus \bigcup_{i\in I} A_i.$$

Remark. Theorem 2 has a version for stochastic operators, but we replace the condition

$$\lim_{t\to\infty} \|\mathbf{1}_{A_i} P(t) f - \alpha_i(f) f_i^*\| = 0,$$

by asymptotic periodicity.

Idea of the proof of Theorems 2

First we prove a version for operators:
The space X can be divided into two parts: C – *conservative part* and $\tilde{D}$ – *dissipative part*:

$$C = \{x \in X : \sum_{n=0}^{\infty} P^n f(x) = \infty\}, \quad \tilde{D} = X \setminus C,$$

where f is a density with $f > 0$ a.e.
Problem: The operator (semigroup) can be neither conservative nor dissipative.
1. Construct an operator $\tilde{P}$ such that P and $\tilde{P}$ have the same conservative part C,
$\tilde{P}|_{L^1(C)} = P|_{L^1(C)}$, $\tilde{P}|_{L^1(\tilde{D})} \geq P|_{L^1(\tilde{D})}$,
$\tilde{P}(L^1(\tilde{D})) \subset \tilde{P}(L^1(\tilde{D}))$ and $\tilde{P}$ satisfies (K).

2. Check that $\tilde{P}|_{L^1(C)}$ is a Harris operator.

The operator P is called a *Harris operator* if P is conservative, $m(X) = 1$, and

$$\int_X \sum_{n=1}^{\infty} k_n(x, y)\, m(dy) > 0 \quad x - a.e., \tag{1}$$

where k_n is the kernel part of P^n.

3. Find a measurable function $h > 0$ such that $\tilde{P}h \leq h$ and $\int_F h < \infty$ for compact sets F.

Remark: If P is dissipative, $f_* = \sum_{n=0}^{\infty} P^n f$, then $f_* < \infty$ and $Pf_* \leq f_*$.

If P is a Harris operator then there exists a measurable function h such that $0 < h < \infty$ and $Ph = h$.

4. Apply results concerning Harris operators to prove asymptotic periodicity and sweeping.
5. In case of a semigroup check that asymptotic periodicity implies asymptotic stability.

Problem: A substochastic semigroup $\{P(t)\}$ can satisfy condition (K) but neither one of the operator $P(t)$ can satisfy (K).

Theorem 2 *If (K) holds then:*
there are a countable (possible empty) set I,
continuous positive functionals α_i, $i \in I$,
and invariant densities f_i^*, $i \in I$,
with pairwise disjoint supports A_i,
such that for every density f and every compact set F we have

$$\lim_{t\to\infty} \|\mathbf{1}_{A_i} P(t) f - \alpha_i(f) f_i^*\| = 0,$$

$$\lim_{t\to\infty} \int_{F\cap Y} P(t) f(x)\, m(dx) = 0,\ Y = X \setminus \bigcup_{i\in I} A_i.$$

Corollary 1 *Assume (K) and that* $\{P(t)\}_{t\geq 0}$ *has no invariant density. Then* $\{P(t)\}_{t\geq 0}$ *is sweeping with respect to compact sets.*

Corollary 2 *Assume (K), and* $\int_0^\infty P(t)f\,dt > 0$ *a.e. for* $f \in D$*. Then* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable or sweeping from compact sets.*

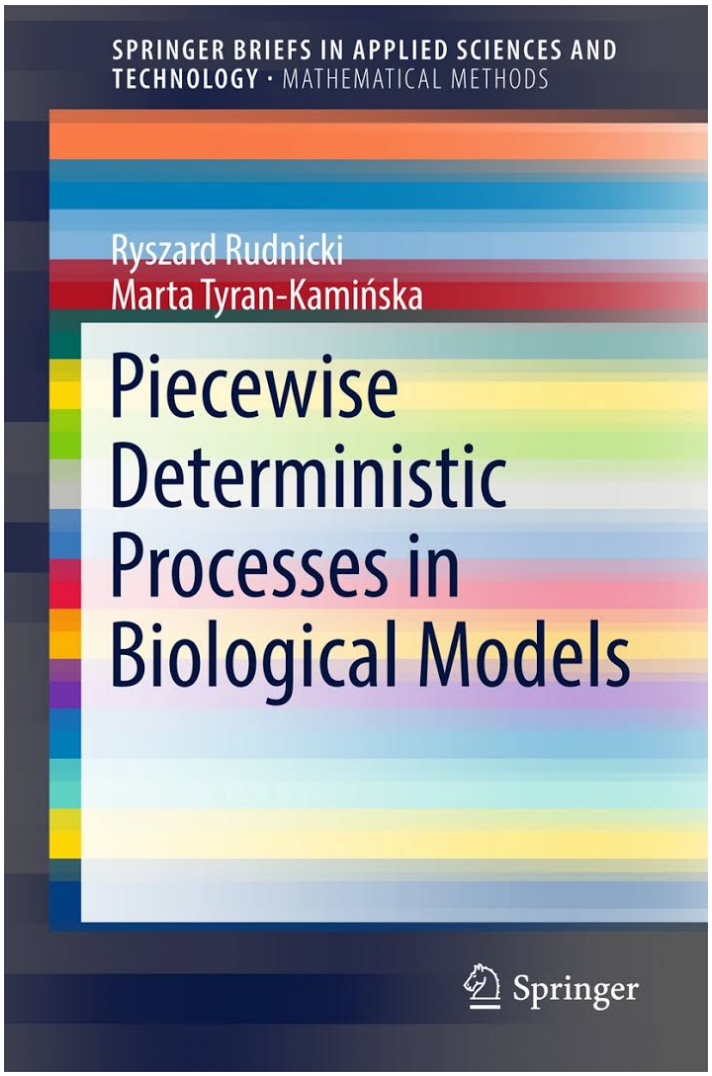
Corollary 3 *Let* X *be a compact space. Assume (K), and that* $\int_0^\infty P(t)f\,dt > 0$ *a.e. for* $f \in D$*. Then* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable.*

Corollary 4 *If (K) holds and there exists a point x_0 such that for each $\varepsilon > 0$ and each density f we have*

$$\int_{B(x_0,\varepsilon)} P(t) f \, dt > 0 \quad \text{for some } t \geq 0. \qquad (2)$$

Then there is at most one invariant density for this semigroup.
In particular, if X is compact then the stochastic semigroup is asymptotically stable.

Remark. New and a review of the old results: M. Gerlach, J. Glück, Trans. AMS 372 (2019), 6603–6627.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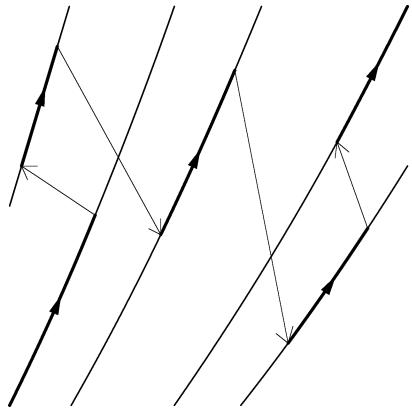
Piecewise deterministic Markov processes

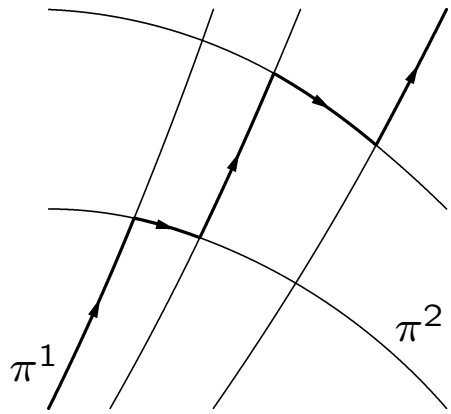
Davis (1984): "PDMPs is a general family of stochastic models covering virtually all non-difussion applications."

A continuous time (homogeneous) Markov process $X(t)$ is a PDMP if there is an increasing sequence of random times (t_n), called jumps, such that sample paths of $X(t)$ are defined in a deterministic way in each interval (t_n, t_{n+1}).

Two types of jumps: the process can jump to a new point or can change the dynamics which defines its trajectories.



Dynamical system with random jumps

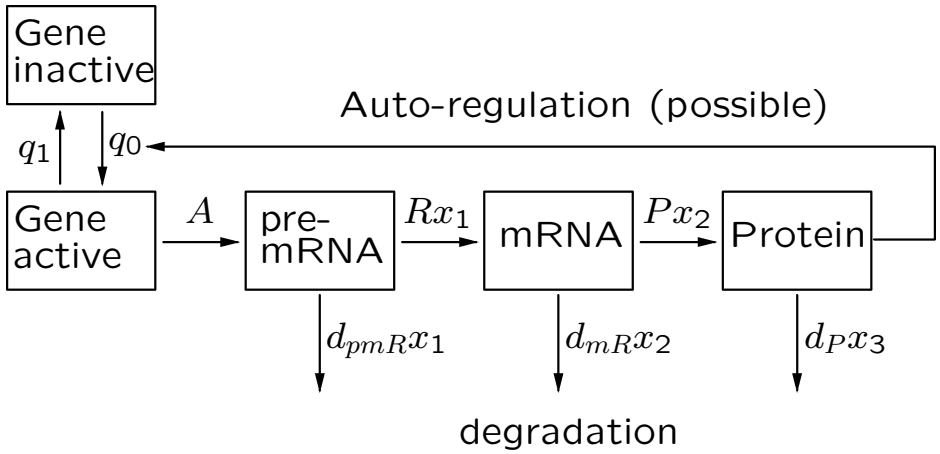


Process with switching dynamics

Examples:

1. Pure jump processes: Markov chains, kangaroo movement.
2. Velocity jump processes: dispersal of cells and insects, stochastic billiards.
3. Semiflows with jumps: cell cycle models, immune systems.
4. Processes with switching dynamics: stochastic gene expression.
5. Mixture of 3 and 4: neural activity model, production of subtilin.
6. Individual-based models (agent-based m.): structured population models, coagulation-fragmentation process.

Gene expression model



x_1, x_2, x_3 — the number of pre-mRNA, mRNA, protein molecules,

d_{pmR}, d_{mR}, d_P — degradation coefficients,

A, Rx_1, Px_2 - velocities of transcription; conversion of pre-mRNA to mRNA; translation.

$$\begin{cases} x_1' = A\gamma(t) - (d_{pmR} + R)x_1 \\ x_2' = Rx_1 - d_{mR}x_2 \\ x_3' = Px_2 - d_P x_3, \end{cases} \tag{3}$$

where $\gamma(t) = 1$ if a gene is active or

$\gamma(t) = 0$ if it is inactive.

We assume that the gene is activated with rate $q_0(\mathbf{x})$ and inactivated with rate $q_1(\mathbf{x})$.

$$x_1' = \gamma(t) - x_1;\ x_2' = \alpha(x_1 - x_2);\ x_3' = \beta(x_2 - x_3)$$

Processes with switching dynamics (PSD)

Markov process

$(x_1(t), x_2(t), x_3(t))$ is not a Markov process.

$$\xi_t = (x_1(t), x_2(t), x_3(t), \gamma(t)),\ t \geq 0.$$

The state-space

$$\mathsf{E} = \mathbb{R}^3_+ \times \{0, 1\}.$$

Partial density functions $f_i(x_1, x_2, x_3, t)$:

$$\Pr(\xi_t \in B \times \{i\}) = \iint\limits_B f_i(x_1, x_2, x_3, t)\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3,$$

where B is a Borel subset of $\mathbb{R}^3_+$, $i = 0, 1$.

f – density of $\xi(0)$, $P(t)f$ – density of ξ_t.

$$x'(t) = b_i(x(t)), \tag{4}$$

$i = 1, \ldots, N$ and at point $x \in G \subset \mathbb{R}^d$ it can jump from j to i state with intensity $q_{ij}(x)$. $\xi_t = (x(t), i(t))$, $t \geq 0$, is a PDMP.

$$\text{prob}(\xi_t \in E \times \{i\}) = \int_E u(x, i, t)\, dx.$$

$$A_i f = -\sum_{k=1}^{d} \frac{\partial(b_i^k(x, i) f)}{\partial x_k}.$$

$$\frac{\partial u}{\partial t} = Mu + Au$$

where $Au = (A_1 u_1, \ldots, A_N u_N)$, $u_i(x, t) = u(x, i, t)$
$M = [m_{ij}(x)]$, $m_{ij}(x) = q_{ij}(x)$ for $i \neq j$
and $m_{ii}(x) = -\sum_{k \neq i} q_{ki}(x)$.

Applications to PSD – how to check (K)?

The system is governed by k flows π_t^i and each flow π_t^i is the solution of a differential equation $x' = b^i(x)$ on $G \subset \mathbb{R}^d$. All transition intensities $q_{ij}(x)$ are continuous and positive in a neighbourhood of y_0.

(**Hörmander condition**) If vectors

$$b^2(y_0) - b^1(y_0), \ldots, b^k(y_0) - b^1(y_0),$$
$$[b^i, b^j](y_0)_{1 \le i,j \le k},\ [b^i, [b^j, b^l]](y_0)_{1 \le i,j,l \le k}, \ldots$$

span the space $\mathbb{R}^d$ then (K) holds for any point x which is connected with y_0.

Assume that G is a bounded set and there exist $x_0 \in G$ and $i_0 \in I$ such that starting from any state $(x, i) \in X$ we are able to go arbitrarily close to (x_0, i_0) by a cumulative flow and that the Hörmander's condition holds. Then the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable.

Corollary 5 *The semigroup generated by the gene expression model is asymptotically stable.*

Dynamics of antibody levels

O. Diekmann O, WF de Graaf, MEE Kretzschmar, PFM Teunis, Waning and boosting: on the dynamics of immune status. J. Math. Biol. 2018.

K. Pichór, R.R., Dynamics of antibody levels: Asymptotic properties, Math. Meth. Appl. Sci. 2020.

The immune status x is the concentration of specific antibodies, which appear after infection with a pathogen and remain in serum, providing protection against future attacks of that same pathogen.

$x(t)$ is a stochastic process whose trajectories are decreasing functions $x(t)$ between subsequent infections:

$$x'(t) = g(x(t)), \quad g < 0, \tag{5}$$

If x is the concentration of antibodies at the moment of infection, then $Q(x) > x$ is the concentration of antibodies just after clearance of infection.
The moments of infections are independent of the state of the immune system and they are distributed according to a Poisson process $(N_t)_{t \geq 0}$ with rate $\Lambda > 0$.

$$\xi_{t_n} = Q(\xi_{t_n^-}), \quad \xi_t' = g(\xi_t) \text{ for } t \in [t_{n-1}, t_n),$$

$$N_{t_n} = N_{t_n^-} + 1 = n.$$

The process $(\xi_t)_{t \geq 0}$ satisfies the stochastic differential equation

$$d\xi_t = g(\xi_t)\, dt + (Q(\xi_t) - \xi_t)\, dN_t.$$

If f is the density distribution of x before an infection then $P_Q f$ is the density distribution of x just after clearance of infection:

$$P_Q f(x) = \sum_{i \in I_x} f(\varphi_i(x))|\varphi_i'(x)|, \qquad (6)$$

where φ_i are the right-inverse functions of $Q\Big|_{(a_i,b_i)}$.

$$P_Q^* f(x) = f(Q(x)).$$

The density of ξ_t is given by $u(t)(x)$ and

$$u'(t) = \mathcal{A}u(t),$$

$$\mathcal{A}f(x) = -\frac{d}{dx}(g(x)f(x)) + \Lambda P_Q f(x) - \Lambda f(x).$$

$\mathcal{A}$ is a generator of a semigroup $\{U(t)\}_{t \geq 0}$.

Results: Case $X = [0, \infty)$.

Theorem 3 *The semigroup* $\{U(t)\}_{t \geq 0}$ *satisfies the Foguel alternative, i.e. it is asymptotically stable or for every* $f \in L^1[0, \infty)$ *and* $M > 0$

$$\lim_{t \to \infty} \int_0^M U(t) f(x)\, dx = 0.$$

The proof of (K) is based on the Dyson-Phillips expansion.

Asymptotic stability — if we have V such that:

$$\limsup_{x \to \infty} [g(x) V'(x) + \Lambda V(Q(x)) - \Lambda V(x)] < 0.$$

Example: if $\lim_{x \to \infty} g(x) = -\infty$ and $Q(x) \leq x + L$, then $V(x) = x$ is OK.

Example: $g(x) \leq -ax$ and $Q(x) \leq bx$ for a sufficiently large x.

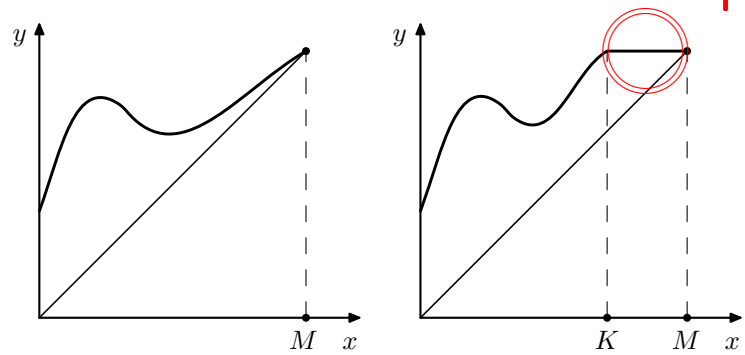
1) If $a > \Lambda \log b$, then $\{U(t)\}_{t\geq 0}$ is asymptotically stable.

2) If $a < \Lambda \log b$, $g(x) \leq -ax$ and $Q(x) \geq bx$ then the semigroup is sweeping from compact sets.

Sweeping can be interpreted as asymptotic permanent immunity of the population.

Case $X = [0, M]$.

F-P not defined



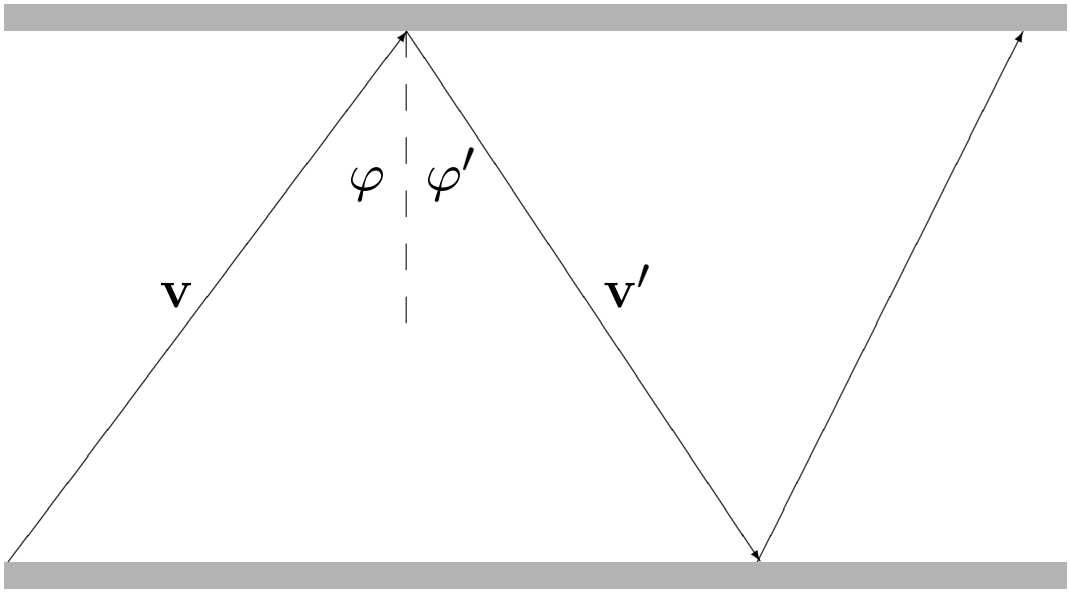
Examples of graphs of $y = Q(x)$

If $\widetilde{Q} = Q\Big|_{[0,K)}$, then

$$\mathcal{A}f = -(gf)' + \Lambda P_{\widetilde{Q}} f - \Lambda f,$$

$$\mathfrak{D}(\mathcal{A}) = \Big\{ f \in L^1[0, M] \colon f' \in L^1[0, M],$$
$$g(M)f(M) = -\Lambda \int_K^M f(x)\, dx \Big\}.$$

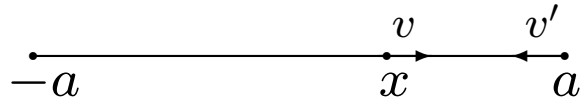
Collisionless kinetic equations



$$\|\mathbf{v}\| = \|\mathbf{v}'\| = 1, \quad v = \mathbf{v}_z = \cos\varphi, \quad v' = \mathbf{v}'_z = \cos\varphi'$$

One dimensional stochastic billiard

$d = 2a$ - distance between slabs



$-a \leq x \leq a, \quad v \in [-1, 1] \setminus \{0\}.$

Stochastic process $\xi(t) = (x(t), v(t))$ such that:
1) $x'(t) = v(t)$, $v'(t) = 0$ if $x(t) \in (-a, a)$,
2) if $x(t^-) = -a$ and $v(t^-) = v$, then $x(t) = -a$ and $v(t) = v'$, where v' has a distribution given by the density $v' \mapsto h_1(v', v)$.
3) if $x(t^-) = a$ and $v(t^-) = v$, then $x(t) = a$ and $v(t) = v'$, where v' has a distribution given by the density $v' \mapsto h_2(v', v)$.

Let $f(t,x,v)$ be the density of $\xi(t)$. Then

$$\frac{\partial f}{\partial t}(t,x,v) + v\frac{\partial f}{\partial x} = 0 \tag{7}$$

for $t \geq 0$, $x \in (-a,a)$, $v \in [-1,0) \cup (0,1]$, with boundary conditions

$$v' f(t,-a,v') = \int_{-1}^{0} |v| h_1(v',v) f(t,-a,v)\, dv \tag{8}$$

for $v' > 0$ and

$$|v'| f(t,a,v') = \int_{0}^{1} v h_2(v',v) f(t,a,v)\, dv \tag{9}$$

for $v' < 0$.

Invariant density: $f(t,x,v) = f(v)$.

If $h(v) = vf(v)$ for $v > 0$, $\tilde{h}(v) = |v| f(v)$ for $v < 0$. Then h and $\tilde{h}$ are fixed point of integral operators.

System (7)–(9) generates a stochastic semigroup $\{P(t)\}$ on the space $L^1(X, \mathcal{B}(X), m)$, where

$$X = [-a, a] \times ([-1, 0) \cup (0, 1])$$

with the Lebesgue measure m on X.
It is not difficult to check that the semigroup generated by (7)–(9) satisfies condition (K).

Irreducibility of $\{P(t)\}$:

Let $B_1 h(v') := \int_{-1}^{0} h_1(v', v) h(v)\, dv$ for $v' > 0$ and $B_2 h(v') := \int_{0}^{1} h_2(v', v) h(v)\, dv$ for $v' < 0$. If the operators $Gh := B_1 B_2 h$ and $\tilde{G}h := B_2 B_1 h$ are irreducible, then the semigroup $\{P(t)\}$ is irreducible.

Corollary 6 *If the operators G_1 and G_2 are irreducible, then the semigroup $\{P(t)\}$ is asymptotically stable or is sweeping from compact sets, i.e. for each $\varepsilon > 0$ we have*

$$\lim_{t \to \infty} \int_{-\varepsilon}^{\varepsilon} \int_{-a}^{a} f(t, x, v)\, dx\, dv = 1.$$

Special example: $h_1 \equiv 1$, $h_2 \equiv 1$. Then

$$h_1 \equiv \text{const} \quad \text{and} \quad h_2 \equiv \text{const}$$

then the semigroup is sweeping from compact sets and, consequently, the distribution of velocity converges to δ_0.
It is interesting that

$$f(t,x,v) \sim \frac{c}{|v|}(\log t)^{-1} \text{ when } t \to \infty$$

for $|v| \geq \varepsilon$ i $x \in [-a,a]$, where c is some constant.


Thank You!