Asymptotic decomposition of substochastic semigroups and applications

Ryszard Rudnicki
Instytut of Mathematics
Polish Academy of Sciences

One-Parameter Semigroups of Operators
Nizhny Novgorod 6.4.2021
Outline:

1. Stochastic semigroups.
2. Asymptotic decomposition and corollaries.
4. Applications to a gene expression model, to immunology, and to (a kinetic equation).

References:

Stochastic semigroups

\((X, \Sigma, m) \rightarrow \sigma\)-finite measure space.

\(D = \{f \in L^1 : f \geq 0, \|f\| = 1\} \) — densities.

**Stochastic operator (Markov operator):**

\(P : L^1 \rightarrow L^1\) linear, \(P(D) \subset D\).

**Substochastic operator:**

\(P : L^1 \rightarrow L^1\) linear, positive and \(\|P\| \leq 1\).

**Stochastic (substochastic) semigroup:** \(\{P(t)\}_{t \geq 0}\),

\(P(t)\) - stochastic (substochastic) operators,

\(P(0) = Id,\ P(t+s) = P(t)P(s), \ s, t \geq 0,\)

(c) for each \(f \in L^1\), the function \(t \mapsto P(t)f\) is continuous.
Asymptotic stability

\( f_* \in D \) — invariant if \( P(t)f_* = f_* \) for \( t \geq 0 \).

\( \{P(t)\} \) — asymptotically stable if there is an invariant density \( f_* \) such that

\[
\lim_{t \to \infty} \|P(t)f - f_*\| = 0 \quad \text{for} \quad f \in D.
\]

Sweeping (zero-type property)

\( \{P(t)\} \) — sweeping with respect to a family of sets \( \mathcal{F} \) if for \( B \in \mathcal{F} \) and for \( f \in D \)

\[
\lim_{t \to \infty} \int_B P(t)f(x) \, m(dx) = 0.
\]
\{P(t)\} – partially integral if there exist \( t > 0, \)
\( k(t, x, y) \geq 0 \)
\[ \int_X \int_X k(t, x, y) m(dx) m(dy) > 0 \]

\( P(t)f(x) \geq \int k(t, x, y) f(y) m(dy) \) for \( f \in D. \)

**Theorem 1** If a partially integral stochastic semigroup \( \{P(t)\}_{t \geq 0} \) has a unique invariant density \( f_* \) and \( f_* > 0 \) then it is asymptotically stable.
$X$ separable metric space, $\Sigma = B(X)$, 
$\{P(t)\}_{t \geq 0}$ substochastic semigroup with the kernel part $k(t, x, y)$,
(K) for every $y_0 \in X$ there exist $r > 0$, $t > 0$, and a function $\eta \geq 0$ s.t. $\int \eta \, dm > 0$ and 

$$k(t, x, y) \geq \eta(x) 1_{B(y_0, r)}(y).$$
Theorem 2 If (K) holds then:
there are a countable (possible empty) set $I$, 
continuous positive functionals $\alpha_i, \ i \in I$, 
and invariant densities $f_i^*, \ i \in I$, 
with pairwise disjoint supports $A_i$, 
such that for every density $f$ and every compact set $F$ we have

$$\lim_{t \to \infty} \|1_{A_i}P(t)f - \alpha_i(f)f_i^*\| = 0,$$

$$\lim_{t \to \infty} \int_{F \cap Y} P(t)f(x) \, m(dx) = 0, \ Y = X \setminus \bigcup_{i \in I} A_i.$$
Remark. Theorem 2 has a version for stochastic operators, but we replace the condition
\[
\lim_{t \to \infty} \|1_{A_i} P(t) f - \alpha_i(f) f_i^* \| = 0,
\]
by asymptotic periodicity.
Idea of the proof of Theorems 2

First we prove a version for operators:
The space $X$ can be divided into two parts: $C$ – conservative part and $\tilde{D}$ – dissipative part:

$$C = \{ x \in X : \sum_{n=0}^{\infty} P^n f(x) = \infty \}, \quad \tilde{D} = X \setminus C,$$

where $f$ is a density with $f > 0$ a.e.

Problem: The operator (semigroup) can be neither conservative nor dissipative.

1. Construct an operator $\tilde{P}$ such that $P$ and $\tilde{P}$ have the same conservative part $C$,

$$\tilde{P}|_{L^1(C)} = P|_{L^1(C)}, \quad \tilde{P}|_{L^1(\tilde{D})} \geq P|_{L^1(\tilde{D})},$$

$\tilde{P}(L^1(\tilde{D})) \subset P(L^1(\tilde{D}))$ and $\tilde{P}$ satisfies (K).
2. Check that $\tilde{P}|_{L^1(C)}$ is a Harris operator.

The operator $P$ is called a **Harris operator** if $P$ is conservative, $m(X) = 1$, and

$$\int_X \sum_{n=1}^{\infty} k_n(x, y) m(dy) > 0 \quad x - a.e.,$$

where $k_n$ is the kernel part of $P^n$. 

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3. Find a measurable function $h > 0$ such that $\tilde{P} h \leq h$ and $\int_{F} h < \infty$ for compact sets $F$.

Remark: If $P$ is dissipative, $f_{*} = \sum_{n=0}^{\infty} P^{n} f$, then $f_{*} < \infty$ and $P f_{*} \leq f_{*}$.

If $P$ is a Harris operator then there exists a measurable function $h$ such that $0 < h < \infty$ and $P h = h$. 
4. Apply results concerning Harris operators to prove asymptotic periodicity and sweeping.

5. In case of a semigroup check that asymptotic periodicity implies asymptotic stability.

Problem: A substochastic semigroup \( \{P(t)\} \) can satisfy condition (K) but neither one of the operator \( P(t) \) can satisfy (K).
Theorem 2 If \((K)\) holds then:
there are a countable (possible empty) set \(I\),
continuous positive functionals \(\alpha_i, \ i \in I\),
and invariant densities \(f^*_i, \ i \in I\),
with pairwise disjoint supports \(A_i\),
such that for every density \(f\) and every compact set \(F\) we have
\[
\lim_{t \to \infty} \|1_{A_i} P(t)f - \alpha_i(f)f^*_i\| = 0,
\]
\[
\lim_{t \to \infty} \int_{F \cap Y} P(t)f(x) \, m(dx) = 0, \ Y = X \setminus \bigcup_{i \in I} A_i.
\]
**Corollary 1** Assume (K) and that \( \{P(t)\}_{t \geq 0} \) has no invariant density. Then \( \{P(t)\}_{t \geq 0} \) is sweeping with respect to compact sets.

**Corollary 2** Assume (K), and \( \int_0^\infty P(t) f \, dt > 0 \) a.e. for \( f \in D \). Then \( \{P(t)\}_{t \geq 0} \) is asymptotically stable or sweeping from compact sets.

**Corollary 3** Let \( X \) be a compact space. Assume (K), and that \( \int_0^\infty P(t) f \, dt > 0 \) a.e. for \( f \in D \). Then \( \{P(t)\}_{t \geq 0} \) is asymptotically stable.
Corollary 4 If (K) holds and there exists a point $x_0$ such that for each $\varepsilon > 0$ and each density $f$ we have

$$\int_{B(x_0,\varepsilon)} P(t)f\,dt > 0 \quad \text{for some } t \geq 0.$$ (2)

Then there is at most one invariant density for this semigroup.  
In particular, if $X$ is compact then the stochastic semigroup is asymptotically stable.

Ryszard Rudnicki
Marta Tyran-Kamińska

Piecewise Deterministic Processes in Biological Models

Springer
Davis (1984): "PDMPs is a general family of stochastic models covering virtually all non-diffusion applications."

A continuous time (homogeneous) Markov process $X(t)$ is a PDMP if there is an increasing sequence of random times $(t_n)$, called jumps, such that sample paths of $X(t)$ are defined in a deterministic way in each interval $(t_n, t_{n+1})$.

Two types of jumps: the process can jump to a new point or can change the dynamics which defines its trajectories.
Dynamical system with random jumps

Process with switching dynamics
Examples:
3. Semiflows with jumps: cell cycle models, immune systems.
6. Individual-based models (agent-based m.): structured population models, coagulation-fragmentation process.
Gene expression model

\[ x_1, x_2, x_3 \] — the number of pre-mRNA, mRNA, protein molecules,
\[ d_{pmR}, d_m, d_P \] — degradation coefficients,
\[ A, Rx_1, Px_2 \] - velocities of transcription; conversion of pre-mRNA to mRNA; translation.
\begin{align*}
x'_1 &= A\gamma(t) - (d_{pmR} + R)x_1 \\
x'_2 &= Rx_1 - d_{mR}x_2 \\
x'_3 &= Px_2 - d_px_3, \\
\end{align*}
\hspace{1cm} (3)

where \( \gamma(t) = 1 \) if a gene is active or \( \gamma(t) = 0 \) if it is inactive.

We assume that the gene is activated with rate \( q_0(x) \) and inactivated with rate \( q_1(x) \).

\[ x'_1 = \gamma(t) - x_1; \quad x'_2 = \alpha(x_1 - x_2); \quad x'_3 = \beta(x_2 - x_3) \]

Processes with switching dynamics (PSD)
Markov process

\((x_1(t), x_2(t), x_3(t))\) is not a Markov process.

\[\xi_t = (x_1(t), x_2(t), x_3(t), \gamma(t)), \ t \geq 0.\]

The state-space

\[E = \mathbb{R}^3_+ \times \{0, 1\}.\]

Partial density functions \(f_i(x_1, x_2, x_3, t)\):

\[\Pr(\xi_t \in B \times \{i\}) = \int \int \int_B f_i(x_1, x_2, x_3, t) \, dx_1 \, dx_2 \, dx_3,\]

where \(B\) is a Borel subset of \(\mathbb{R}^3_+, i = 0, 1.\)

\(f\) – density of \(\xi(0)\), \(P(t)f\) – density of \(\xi_t\).
\( x'(t) = b_i(x(t)), \) 
\( i = 1, \ldots, N \) and at point \( x \in G \subset \mathbb{R}^d \) it can jump from \( j \) to \( i \) state with intensity \( q_{ij}(x) \). 
\( \xi_t = (x(t), i(t)), t \geq 0, \) is a PDMP.

\[
\Pr(\xi_t \in E \times \{i\}) = \int_E u(x, i, t) \, dx.
\]

\[
A_i f = - \sum_{k=1}^{d} \frac{\partial (b_i^k(x, i) f)}{\partial x_k}.
\]

\[
\frac{\partial u}{\partial t} = Mu + Au
\]
where \( Au = (A_1 u_1, \ldots, A_N u_N), u_i(x, t) = u(x, i, t) \)
\( M = [m_{ij}(x)], m_{ij}(x) = q_{ij}(x) \) for \( i \neq j \)
and \( m_{ii}(x) = -\sum_{k \neq i} q_{ki}(x) \).
Applications to PSD – how to check (K)?

The system is governed by $k$ flows $\pi^i_t$ and each flow $\pi^i_t$ is the solution of a differential equation $x' = b^i(x)$ on $G \subset \mathbb{R}^d$. All transition intensities $q_{ij}(x)$ are continuous and positive in a neighbourhood of $y_0$.

(Hörmander condition) If vectors

$$b^2(y_0) - b^1(y_0), \ldots, b^k(y_0) - b^1(y_0),$$

$$[b^i, b^j](y_0)_{1 \leq i, j \leq k}, [b^i, [b^j, b^l]](y_0)_{1 \leq i, j, l \leq k}, \ldots$$

span the space $\mathbb{R}^d$ then (K) holds for any point $x$ which is connected with $y_0$. 

Assume that $G$ is a bounded set and there exist $x_0 \in G$ and $i_0 \in I$ such that starting from any state $(x, i) \in X$ we are able to go arbitrarily close to $(x_0, i_0)$ by a cumulative flow and that the Hörmander’s condition holds. Then the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable.

**Corollary 5** The semigroup generated by the gene expression model is asymptotically stable.
Dynamics of antibody levels


The immune status $x$ is the concentration of specific antibodies, which appear after infection with a pathogen and remain in serum, providing protection against future attacks of that same pathogen.

$x(t)$ is a stochastic process whose trajectories are decreasing functions $x(t)$ between subsequent infections:

$$x'(t) = g(x(t)), \quad g < 0,$$

If $x$ is the concentration of antibodies at the moment of infection, then $Q(x) > x$ is the concentration of antibodies just after clearance of infection.

The moments of infections are independent of the state of the immune system and they are distributed according to a Poisson process $(N_t)_{t \geq 0}$ with rate $\Lambda > 0$. 

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\( \xi_{t_n} = Q(\xi_{t_n^-}) \), \( \xi'_t = g(\xi_t) \) for \( t \in [t_{n-1}, t_n) \),
\[
N_{t_n} = N_{t_n^-} + 1 = n.
\]
The process \((\xi_t)_{t \geq 0}\) satisfies the stochastic differential equation
\[
d\xi_t = g(\xi_t) \, dt + (Q(\xi_t) - \xi_t) \, dN_t.
\]
If $f$ is the density distribution of $x$ before an infection then $P_Q f$ is the density distribution of $x$ just after clearance of infection:

$$P_Q f(x) = \sum_{i \in I_x} f(\varphi_i(x)) \varphi'_i(x)|,$$

where $\varphi_i$ are the right-inverse functions of $Q\big|_{(a_i,b_i)}$.

$$P^*_Q f(x) = f(Q(x)).$$

The density of $\xi_t$ is given by $u(t)(x)$ and

$$u'(t) = \mathcal{A} u(t),$$

$$\mathcal{A} f(x) = -\frac{d}{dx}(g(x)f(x)) + \Lambda P_Q f(x) - \Lambda f(x).$$

$\mathcal{A}$ is a generator of a semigroup $\{U(t)\}_{t \geq 0}$. 

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Results: Case $X = [0, \infty)$.

**Theorem 3** The semigroup $\{U(t)\}_{t \geq 0}$ satisfies the Foguel alternative, i.e. it is asymptotically stable or for every $f \in L^1[0, \infty)$ and $M > 0$

$$\lim_{t \to \infty} \int_0^M U(t)f(x)\,dx = 0.$$ 

The proof of (K) is based on the Dyson-Phillips expansion.

Asymptotic stability — if we have $V$ such that:

$$\limsup_{x \to \infty} [g(x)V'(x) + \Lambda V(Q(x)) - \Lambda V(x)] < 0.$$ 

Example: if $\lim_{x \to \infty} g(x) = -\infty$ and $Q(x) \leq x + L$, then $V(x) = x$ is OK.
Example: \( g(x) \leq -ax \) and \( Q(x) \leq bx \) for a sufficiently large \( x \).

1) If \( a > \Lambda \log b \), then \( \{U(t)\}_{t \geq 0} \) is asymptotically stable.

2) If \( a < \Lambda \log b \), \( g(x) \leq -ax \) and \( Q(x) \geq bx \) then the semigroup is sweeping from compact sets.

Sweeping can be interpreted as asymptotic permanent immunity of the population.
Case $X = [0, M]$. 

Examples of graphs of $y = Q(x)$

If $\tilde{Q} = Q \bigg|_{[0,K]}$, then

$$Af = -(gf)' + \Lambda P_{\tilde{Q}}f - \Lambda f,$$

$$\Omega(A) = \left\{ f \in L^1[0, M]: f' \in L^1[0, M], g(M)f(M) = -\Lambda \int_K^M f(x) \, dx \right\}.$$
Collisionless kinetic equations

$$\|v\| = \|v'\| = 1, \quad v = v_z = \cos \varphi, \quad v' = v'_z = \cos \varphi'$$
One dimensional stochastic billiard

\[ d = 2a \] - distance between slabs

\[ -a \leq x \leq a, \quad v \in [-1, 1] \setminus \{0\}. \]

Stochastic process \( \xi(t) = (x(t), v(t)) \) such that:

1) \( x'(t) = v(t) \), \( v'(t) = 0 \) if \( x(t) \in (-a, a) \),

2) if \( x(t^-) = -a \) and \( v(t^-) = v \), then \( x(t) = -a \) and \( v(t) = v' \), where \( v' \) has a distribution given by the density \( v' \mapsto h_1(v', v) \).

3) if \( x(t^-) = a \) and \( v(t^-) = v \), then \( x(t) = a \) and \( v(t) = v' \), where \( v' \) has a distribution given by the density \( v' \mapsto h_2(v', v) \).
Let $f(t, x, v)$ be the density of $\xi(t)$. Then
\[
\frac{\partial f}{\partial t}(t, x, v) + v \frac{\partial f}{\partial x} = 0 \quad (7)
\]
for $t \geq 0$, $x \in (-a, a)$, $v \in [-1, 0) \cup (0, 1]$, with boundary conditions
\[
v' f(t, -a, v') = \int_{-1}^{0} |v| h_1(v', v) f(t, -a, v) \, dv \quad (8)
\]
for $v' > 0$ and
\[
|v'| f(t, a, v') = \int_{0}^{1} vh_2(v', v) f(t, a, v) \, dv \quad (9)
\]
for $v' < 0$.

**Invariant density:** $f(t, x, v) = f(v)$.

If $h(v) = vf(v)$ for $v > 0$, $\tilde{h}(v) = |v| f(v)$ for $v < 0$. Then $h$ and $\tilde{h}$ are fixed point of integral operators.
System (7)–(9) generates a stochastic semi-group \( \{P(t)\} \) on the space \( L^1(X, \mathcal{B}(X), m) \), where

\[
X = [-a, a] \times ([-1, 0) \cup (0, 1])
\]

with the Lebesgue measure \( m \) on \( X \).

It is not difficult to check that the semigroup generated by (7)–(9) satisfies condition (K).
Irreducibility of \( \{P(t)\} \):
Let \( B_1 h(v') := \int_{-1}^{0} h_1(v',v)h(v) \, dv \) for \( v' > 0 \) and \( B_2 h(v') := \int_{0}^{1} h_2(v',v)h(v) \, dv \) for \( v' < 0 \).
If the operators \( G h := B_1 B_2 h \) and \( \tilde{G} h := B_2 B_1 h \) are irreducible, then the semigroup \( \{P(t)\} \) is irreducible.

**Corollary 6** If the operators \( G_1 \) and \( G_2 \) are irreducible, then the semigroup \( \{P(t)\} \) is asymptotically stable or is sweeping from compact sets, i.e. for each \( \varepsilon > 0 \) we have
\[
\lim_{t \to \infty} \int_{-\varepsilon}^{\varepsilon} \int_{-a}^{a} f(t,x,v) \, dx \, dv = 1.
\]
Special example: $h_1 \equiv 1$, $h_2 \equiv 1$. Then

$$h_1 \equiv \text{const} \quad \text{and} \quad h_2 \equiv \text{const}$$

then the semigroup is sweeping from compact sets and, consequently, the distribution of velocity converges to $\delta_0$.

It is interesting that

$$f(t, x, v) \sim \frac{c}{|v|} (\log t)^{-1} \quad \text{when} \ t \to \infty$$

for $|v| \geq \varepsilon \ i \ x \in [-a, a]$, where $c$ is some constant.
Thank You!