Inverse problem for Sobolev type equation of the second order

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Introduction. Let $\mathcal{U}, \mathcal{F}, \mathcal{Y}$ be Banach spaces, operators $B_1, B_0 \in \mathcal{C}l(\mathcal{U};\mathcal{F})$, $A \in \mathcal{L}(\mathcal{U};\mathcal{F})$, $ker A \neq \{0\}$, $C \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, the functions $\chi : [0,T] \to \mathcal{L}(\mathcal{Y};\mathcal{F})$, $f : [0,T] \to \mathcal{F}$, $\Psi : [0,T] \to \mathcal{Y}$. Consider the following problem

$$Av''(t) = B_1 v'(t) + B_0 v(t) + \chi(t)q(t) + f(t), \quad t \in [0,T], \tag{1}$$

$$v(0) = v_0, \quad v'(0) = v_1, \tag{2}$$

$$Cv(t) = \Psi(t). \tag{3}$$

The problem of finding a pair of functions $v(t) \in C^2([0,T];\mathcal{U})$ and $q(t) \in C^1([0,T];\mathcal{Y})$ from relations (1) – (3) is called the inverse problem.

Existence of solutions. Let the pencil $\vec{B} = (B_0, B_1)$ be polynomially A-bounded and condition

$$\int_\gamma R^A_\mu(\overrightarrow{B})d\mu \equiv \mathbb{O}, \tag{A}$$

where $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$, be fulfilled, then $v(t) = Pv(t) + (\mathbb{I} - P)v(t) = u(t) + \omega(t)$. Here P is the relatively spectral projector in $\mathcal{U}$. Put $\mathcal{U}^0 = ker\ P$, $\mathcal{U}^1 = im\ P$. Suppose that $\mathcal{U}^0 \subset ker\ C$. Then, by virtue of [3], problem (1) – (3) is equivalent to the problem of finding the functions $u \in C^2([0,T];\mathcal{U}^1)$, $\omega \in C^2([0,T];\mathcal{U}^0)$, $q \in C^1([0,T];\mathcal{Y})$ from the relations

$$u''(t) = S_1 u'(t) + S_0 u(t) + (A^1)^{-1}Q\chi(t)q(t) + (A^1)^{-1}Qf(t), \tag{4}$$

$$u(0) = u_0, \ u'(0) = u_1, \tag{5}$$

$$Cu(t) = \Psi(t) \equiv Cv(t), \tag{6}$$

$$H_0\omega''(t) = H_1\omega'(t) + \omega(t) + (B^0_0)^{-1}(\mathbb{I} - Q)\chi(t)q(t) + (B^0_0)^{-1}(\mathbb{I} - Q)f(t), \tag{7}$$

$$\omega(0) = \omega_0, \ \omega'(0) = \omega_1, \tag{8}$$

where $S_1 = (A^1)^{-1}B^1_1$, $S_0 = (A^1)^{-1}B^1_0$, $u_0 = Pv_0, u_1 = Pv_1$, $\omega_0 = (\mathbb{I} - P)v_0$, $\omega_1 = (\mathbb{I} - P)v_1$, $H_0 = (B^0_0)^{-1}A^0$, $H_1 = (B^0_0)^{-1}B^0_1$, $t \in [0,T]$.

Theorem 1. Let the pencil $\vec{B}$ be polynomially A-bounded and condition (A) be fulfilled, moreover, the ∞ be a pole of order $p \in \mathbb{N}_0$ of the A-resolvent of the pencil $\vec{B}$, $\mathcal{U}^0 \subset ker\ C$, $\chi \in C^{p+2}([0,T];\mathcal{L}(\mathcal{Y};\mathcal{F}))$, $f \in C^{p+2}([0,T];\mathcal{F})$, $\Psi \in C^{p+4}([0,T];\mathcal{Y})$, for any $t \in [0,T]$ operator $C(A^1)^{-1}Q\chi$ be invertible, with $\left(C(A^1)^{-1}Q\chi\right)^{-1} \in C^{p+2}([0,T];\mathcal{L}(\mathcal{Y}))$ the condition $Cu_1 = \Psi'(0)$ be satisfied at some initial value $v_1 \in \mathcal{U}^1$, and the initial values $w_k = (\mathbb{I} - P)v_k \in \mathcal{U}^0$ satisfy

$$w_k = -\sum_{j=0}^{p} K^2_j (B^0_0)^{-1} \frac{d^{j+k}}{dt^{j+k}}\Big[(\mathbb{I} - Q)(\chi(0)q(0) + f(0))\Big], \quad k = 0, 1.$$

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Then there exists a unique solution (v, q) of inverse problem (1) – (3), where $q \in C^{p+2}([0,T];\mathcal{Y})$, $v = u + w$, whence $u \in C^2([0,T];\mathcal{U}^1)$ is a solution of (4) – (6) and the function $w \in C^2([0,T];\mathcal{U}^0)$ is a solution of (7) – (8) given by

$$w(t) = -\sum_{j=0}^{p} K_j^2 (B_0^0)^{-1} \frac{d^j}{dt^j}\Big[(\mathbb{I} - Q)(\chi(t)q(t) + f(t))\Big].$$

Applications. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary $\partial\Omega$ of class C^∞. In the cylinder $\Omega \times [0;T]$ consider the Boussinesq – Love equation

$$(\lambda - \Delta)v_{tt} = \alpha(\Delta - \lambda')v_t + \beta(\Delta - \lambda'')v + fq, \tag{9}$$

with initial conditions

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \tag{10}$$

boundary condition

$$v(x,t)|_{\partial\Omega} = 0 \tag{11}$$

and overdetermination condition

$$\int\limits_{\Omega} v(x,t)K(x)dx = \Phi(t), \tag{12}$$

where $K(x)$ is a given function in $L_2(\Omega)$ and $f(x,t) = \chi(t)$.

Equation (9) describes the longitudinal vibration in the elastic rod, taking into account the inertia and under external load. Conditions (10) set the initial displacement and initial speed, respectively, and (11) sets the value at the boundaries. The overdetermination condition (12) arises when, in addition to finding the function u, one needs to restore part of external load q. Problem (9) – (12) can be reduced to the second-order Sobolev type equation (1) with conditions (2), (3).

Theorem 2. Let one of the conditions

$$\lambda \notin \sigma(\Delta)\ or\ (\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$$

be fulfilled. Moreover, $K,\ u_0,\ u_1 \in \mathcal{U}^1$, $f \in C^2([0,T];\mathcal{L}(\mathcal{Y};\mathcal{F}))$, $\Phi \in C^4([0,T];\mathcal{Y})$, $\sum\limits_{\lambda \neq \lambda_k} \frac{<f(\cdot,t),K>}{\lambda - \lambda_k} \neq 0$, the condition $\int\limits_{\Omega} v_1(x)K(x)dx = \Phi'(0)$ be satisfied at some initial value $v_1 \in \mathcal{U}^1$, and the initial values $w_k = (\mathbb{I} - P)v_k \in \mathcal{U}^0$ satisfy

$$< v_0 + \frac{f(\cdot,0)q(0)}{\beta(\lambda_k - \lambda'')}, \varphi_k >= 0\ for\ k : \lambda_k = \lambda,$$

$$< v_1 + \frac{f_t(\cdot,0)q(0) + f(\cdot,0)q'(0)}{\beta(\lambda_k - \lambda'')}, \varphi_k >= 0\ for\ k : \lambda_k = \lambda.$$

Then there exists a unique solution (v, q) of inverse problem (9) – (12), where $q \in C^2([0,T];\mathcal{Y})$, $v = u + w$, whence $u \in C^2([0,T];\mathcal{U}^1)$ is the solution of (4) – (6) and the function $w \in C^2([0,T];\mathcal{U}^0)$ is a solution of (7), (8) given by

$$w(t) = -\sum_{\lambda = \lambda_k} < \frac{f(\cdot,t)q(t)}{\beta(\lambda_k - \lambda'')}, \varphi_k > \varphi_k.$$

Computational experiment. Let

$$\lambda = -1,\ \lambda' = -1,\ \lambda'' = -2,\ \alpha = 2,\ \beta = -2,\ \varepsilon = 4,\ T = 1,\ l = \pi,$$

$$f(x,t)=\cos(x),\quad v_0(x)=\sin(2x),\quad v_1(x)=\sin(2x),\ K(x)=\cos(x),\quad F(t)=\frac{4}{3}\sin(t).$$

Consequently, the Boussinesq – Love equation (9) takes the form

$$(-1-\Delta)v_{tt}=2(\Delta+1)v_t-2(\Delta+2)v+\cos(x)q(t),$$

conditions (10) have the form

$$v(x,0)=\sin(2x),\quad v_t(x,0)=\sin(2x),$$

and the overdetermination condition

$$\int\limits_0^1 v(x,t)\cos(x)dx=\frac{4}{3}\sin(t).$$

Therefore, all conditions of Theorem 2 are satisfied. The function q was obtained by the method of successive approximations.

$$q(t)=\frac{(-24\sqrt{21}+56)e^{\frac{(-3+\sqrt{21})t}{3}}+(24\sqrt{21}+56)e^{-\frac{(3+\sqrt{21})t}{3}}-168\sin(t)}{21\pi}.$$

It is an approximate solution of the problem posed, reaching admissible error $1.944964447<\varepsilon$ at the first approximation step.

Further, the required function $v(x,t)$ was found using the algorithms developed for the direct problem [2]

$$v(x,t)=\frac{\sqrt{2}\sin(\sqrt{4}x)}{\sqrt{\pi}}\left(\frac{64\sqrt{2}\cos(t)}{85\pi^{\frac{3}{2}}}+\frac{224\sqrt{2}\sin(t)}{255\pi^{\frac{3}{2}}}+\right.$$

$$+\frac{\sqrt{2}e^{-t}}{224910\pi^{\frac{3}{2}}}\left(-63\cosh\left(\frac{t\sqrt{21}}{3}\right)(-1785\pi^2+5440t+2688)+\right.$$

$$\left.\left.+2\sinh\left(\frac{t\sqrt{21}}{3}\right)\sqrt{21}(16065\pi^2+19040t-1728)\right)\right).$$

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