Linear Dynamical Quantum Systems: New Directions and Opportunities

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Outline of talk

- Introduction to linear quantum (stochastic) systems
- System identification for linear quantum systems
- Infinite-dimensional linear quantum systems
- Concluding remarks

Classical (non-quantum) linear stochastic systems

$$dx(t) = Ax(t)dt + Bdw(t) + Eu(t)dt$$
$$dy(t) = Cx(t)dt + Ddw(t) + Fu(t)dt$$

- They are an important class of stochastic models in modern control theory, the basis for Kalman-Bucy filtering and stochastic optimal control theory, in particular linear quadratic Gaussian (LQG) control

Linear quantum stochastic systems

- There is a class of quantum stochastic models with equations like classical linear stochastic systems (in the Heisenberg picture)
- However, the variables appearing in the equations are operators (that do not necessarily commute) rather than real or complex valued functions

Linear quantum stochastic systems

- They model various linear quantum devices found in, for instance, quantum optics, optomechanics and superconducting circuits
- This includes, for example, optical cavities, linear parametric amplifiers, and optomechanical systems such as gravitational wave interferometers

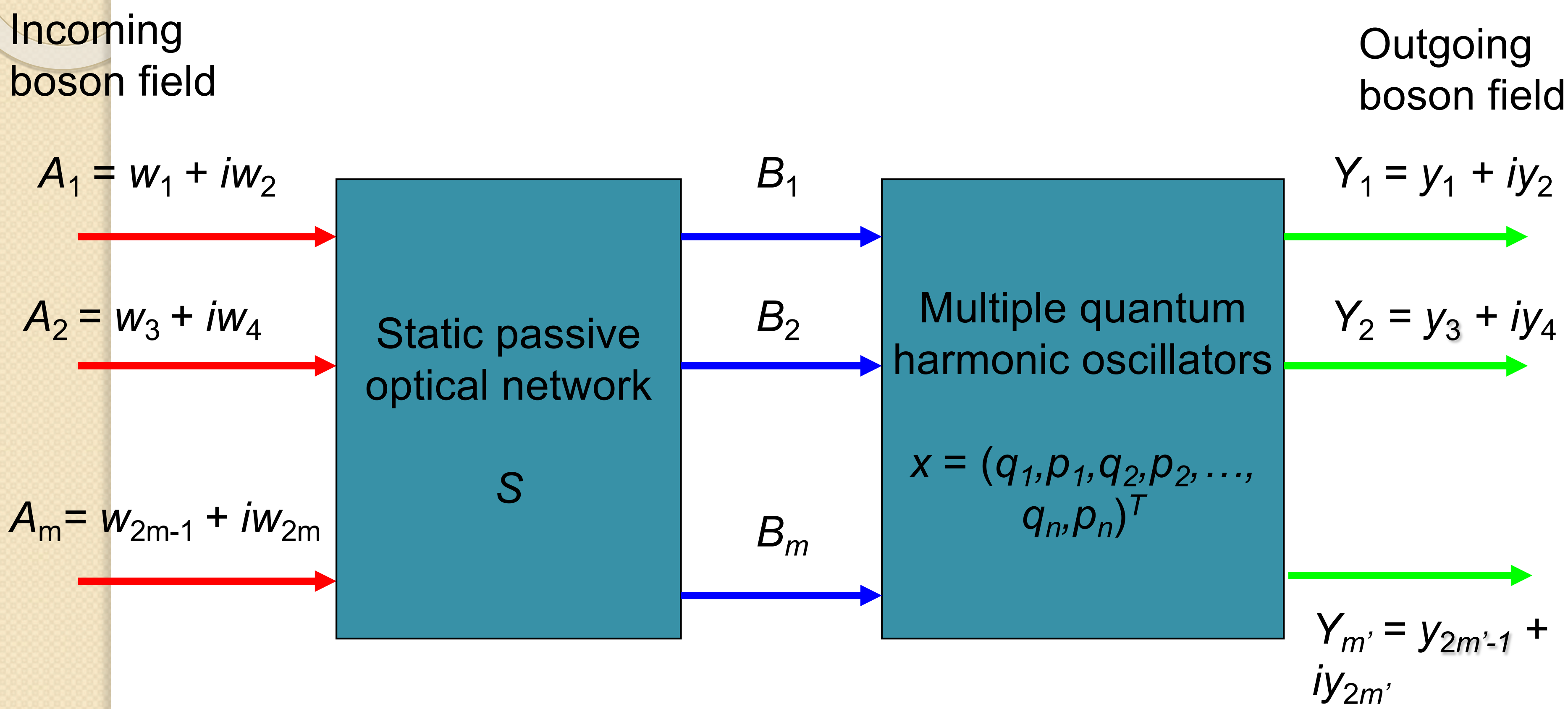
Boson field operators

- These are singular operators $b_j(t)$ and $b_k^*(t)$ satisfying the commutation relations $[b_j(t),b_k^*(s)]=\delta(t\text{-}s)\delta_{jk}$, $[b_j(t),b_k(s)] = [b_j^*(t),b_k^*(s)] = 0$
- It is convenient to work with integrated versions of these processes

$$A_j(t) = \int_0^t b_j(\tau)d\tau$$

$$A_j^*(t) = \int_0^t b_j^*(\tau)d\tau$$

Linear quantum stochastic systems



$H = 1/2 x^T R x$

Quadratic Hamiltonian,
R a real symmetric matrix

$L = Kx$

Linear coupling operator,
K a complex matrix

$S^\dagger S = S S^\dagger = I$

Scattering matrix S

Linear quantum stochastic dynamics

- In linear quantum stochastic systems, the evolution of the vector x in the Heisenberg picture takes the form of the linear quantum stochastic differential equation (QSDE):

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t) \\ dy(t) &= Cx(t)dt + Ddw(t) \end{aligned}$$

$$\begin{aligned} w(t) &= (w_1(t), w_2(t), \ldots, w_{2m-1}(t), w_{2m}(t))^T \\ y(t) &= (y_1(t), y_2(t), \ldots, y_{2m'-1}(t), y_{2m'}(t))^T \end{aligned}$$

Linear quantum stochastic systems

- The vectors $x(t)$ and w(t) satisfy the commutation relations

$$
\begin{aligned}
[x(t), x(t)^T] &= x(t)x(t)^T - (x(t)x(t)^T)^T \\
&= 2i\mathbb{J}_n \\
[dw(t), dw(t)^T] &= dw(t)dw(t)^T - (dw(t)dw(t)^T)^T \\
&= 2i\mathbb{J}_m \\
\underbrace{dw(t)dw(t) + (dw(t)dw(t)^T)^T}_{\text{Gaussian state!}} &= 2F dt
\end{aligned}
$$

$$\mathbb{J}_k = I_k \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$F + i\mathbb{J}_m \geq 0.$$

Linear quantum stochastic dynamics

- Though $q_1(t)$, $p_1(t)$, $q_2(t)$, $p_2(t)$, …, $q_n(t)$, $p_n(t)$ evolve linearly, the Heisenberg evolution of other system operators are not linear
- The matrices A, B, C, D cannot be arbitrary for linear quantum systems due to the constraints of quantum mechanics
- There is no such restriction on these matrices in the classical setting

Physical realizability and structural constraints

- A notion of physical realizability has been introduced for linear QSDEs
- A system is physically realizable if and only if

$$
\begin{aligned}
A\mathbb{J}_n + \mathbb{J}_n A^T + B\mathbb{J}_m B^T &= 0 \\
\mathbb{J}_m C^T + B\mathbb{J}_m B^T &= 0 \\
D\mathbb{J}_m D^T &= \mathbb{J}_m.
\end{aligned}
$$

M. R. James, H. I. Nurdin, and I. R. Petersen, *IEEE Trans. Automat. Control*, vol. 53, no. 8, pp. 1787–1803, 2008

H. I. Nurdin and N. Yamamoto, Linear Dynamical Quantum Systems: Analysis, Synthesis, and Control, Cham: Switzerland: Springer, 2017

Preservation of quantum states

- If the oscillator and the boson fields are initially in a Gaussian state, due to the linear dynamics the oscillator states remain in a Gaussian state at all times
- Thus linear quantum systems are important for the realization of quantum information processing with Gaussian states

Quantum filtering

- *Commuting* components of the output $y(t)$ can be measured (via homodyne detection) and used to construct a mean square optimal estimate of $x(t)$. This is *quantum filtering*.
- For quantum harmonic oscillators and input fields in a Gaussian state this leads to a quantum Kalman-Bucy filter for linear quantum stochastic systems, pioneered by Slava Belavkin.

Application – Quantum LQG control

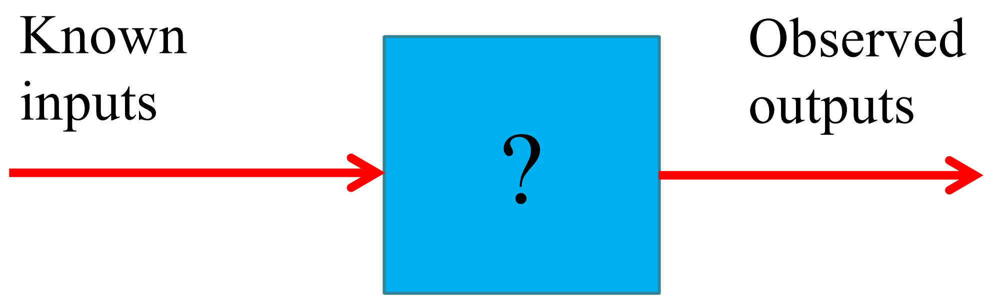
- A quantum version of the linear quadratic Gaussian (LQG) stochastic optimal control theory was also developed by Belavkin.
- Other control problems for linear quantum systems have since been investigated
- A network synthesis theory for linear quantum systems shows how to go from an (A,B,C,D) to a physical realization using quantum optical components

Transfer function

- As with classical linear systems, we can associate a transfer function to a linear quantum system

$$G(s) = C(sI - A)^{-1}B + D$$

System identification



- System identification is inferring a model for a black-box system by observing the system's output response to known inputs
- Typically, this uses a single stochastic input-output record obtained from the system

L. Ljung, System Identification: Theory for the User, 2nd ed. Prentice-Hall, 1999

System identification

- If the unknown system is assumed to be a linear quantum system, Guta & Yamamoto and Levitt & Guta have shown that using coherent state inputs and observing the output response, the (A,B,C,D) matrices can be identified up to a similarity transformation $(TAT^{-1}, TB, CT^{-1}, D)$

M. Guta and N. Yamamoto, IEEE Trans. Automat. Contr., vol. 61, no. 4, pp. 921–936, 2016
M. Levitt and M. Guta, Phys. Rev. A, vol. 95, p. 033825, 2017

System identification

- The power spectral density (PSD) of the output $y(t)$ of a linear quantum system is given by

$$\Phi_y(i\omega) = G(-s)FG(s)^T\Big|_{s=i\omega}$$

System identification

- If the input has a zero-mean Gaussian state, with a covariance matrix which is not invariant under symplectic transformations, Levitt, Guta & Nurdin showed that from knowledge of the output PSD, the matrices (A,B,C,D) of a globally minimal linear quantum system model can be determined up to a similarity transform

System identification

- Globally minimal = there's no model with a smaller number of internal oscillators with the same PSD
- However, an algorithm to empirically estimate the PSD and identify a model from a single input-output observation data has not been developed (open problem!)

M. Levitt, M. Guta, and H. I. Nurdin Automatica, vol. 90, pp. 255–262, 2018

System identification

- Nurdin, Amini and Chen proposed a two-step identification algorithm for an unknown linear quantum system driven by coherent input fields.
- The data was continuous homodyne measurements of commuting quadratures of the output $y(t)$. That is, $y^q(t) = (y_1, y_3, \ldots, y_{2m'-1})$ or $y^p(t) = (y_2, y_4, \ldots, y_{2m'})$.

System identification

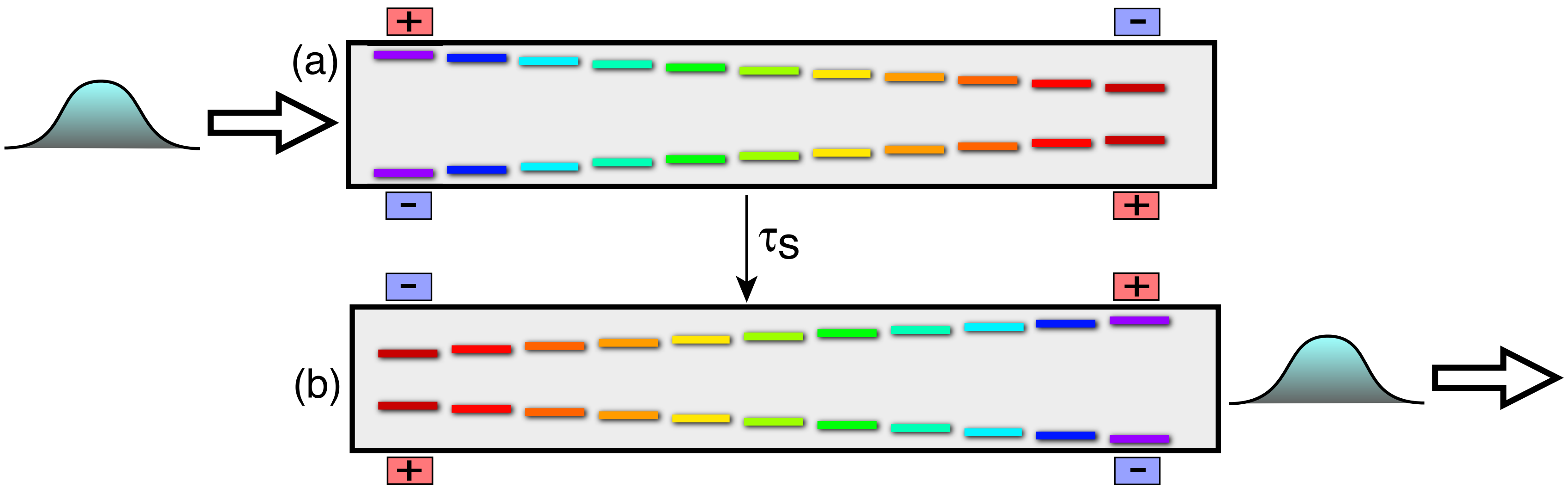
- Two steps:
  - (i) Estimate a classical linear stochastic model using a classical identification algorithm. The model need not meet physical realizability conditions
  - (ii) solve an optimization problem to find a physically realizable linear quantum model close to the identified classical model from Step *i*
- This is under the assumption that D is known, this needs to be relaxed

H. I. Nurdin, N. H. Amini, and J. Chen, in *Proceedings of the 59th IEEE CDC*, 2020, pp. 3829–3835.

Infinite-dimensional linear quantum systems

- A class of optical quantum memories called photon-echo memories have been developed that can store not only the amplitudes of a quantum state but also the temporal profile of the optical pulse that encoded the quantum information
- An example is gradient echo memories (GEMs)

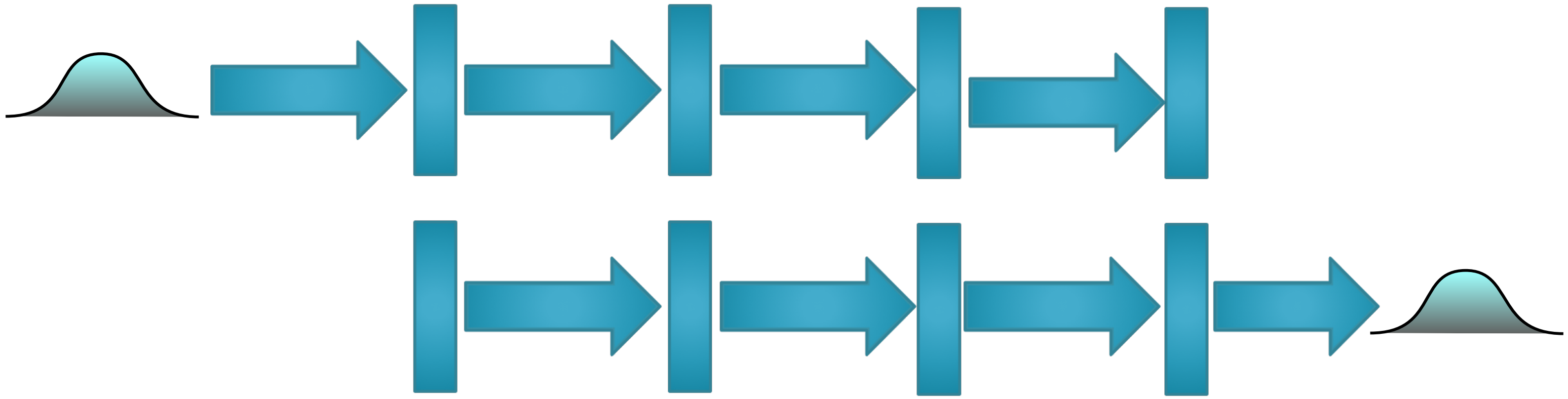
Infinite-dimensional linear quantum systems



A gradient echo memory. Figure from G. Hétet et al, arXiv:0801.3860

Infinite-dimensional linear quantum systems

- Under certain assumptions, a GEM can be modelled as infinitesimal slices along the length of the GEM, put together in a series/cascade connection



M. Hush, A. R. R. Carvalho, M. Hedges, and M. R. James, New J. Phys., vol. 15, pp. 085 020–1–085 020–33, 2013

Infinite-dimensional linear quantum systems

- To a slice at position x one can associate field operators $b(x)$ and $b^*(x)$ satisfying $[b(x),b^*(x')]=\delta(x-x')$, $[b(x),b(x')]=[b^*(x),b^*(x')] = 0$.
- The model is a genuine quantum model
- This leads to linear quantum dynamics for the field operators of the quantum memory but with an infinite dimensional vector $x(t)$!

Infinite-dimensional linear quantum systems

- The treatment of this type of infinite-dimensional linear quantum system is mainly at the heuristic level
- A general and systematic theory for infinite-dimensional linear quantum systems still needs to be developed

Concluding remarks

- This talk has given a brief introduction to linear quantum stochastic systems as quantum analogues of classical linear stochastic systems
- They represent a large class of linear quantum devices that are found in practice, including gravitational wave-interferometers and optical quantum memories
- Some new research directions and open problems were highlighted

Further references:

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- H. I. Nurdin, M. R. James, and I. R. Petersen, Automatica, vol. 45, pp. 1837–1846, 2009
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- H. I. Nurdin, IEEE Trans. Automat. Contr., vol. 55, no. 4, pp. 1008–1013, 2010

That’s all folks

THANK YOU FOR LISTENING!