International Online Conference
One-Parameter Semigroups of Operators

BOOK OF ABSTRACTS

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Preface

International online conference One-Parameter Semigroups of Operators (OPSO 2021), 5-9 April 2021, is organized by the International laboratory of dynamical systems and applications, and research group Evolution semigroups and applications, both located in Russia, Nizhny Novgorod city. The Laboratory was created in 2019 at the National research university Higher School of Economics (HSE).

Website of the Laboratory: https://nnov.hse.ru/en/bipm/dsa/

HSE is a young university (established in 1992) which rapidly become one of the leading Russian universities according to international ratings. In 2021 HSE is a large university focused on only on economics. There are departments of Economics (including Finance, Statistics etc), Law, Mathematics, Computer Science, Media and Design, Physics, Chemistry, Biotechnology, Geography and Geoinformation Technologies, Foreign Languages and some other.

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The OPSO 2021 online conference has connected 106 speakers and 26 participants without a talk from all over the world. The conference covered the following topics: 1. One-parameter groups and semigroups of linear operators; 2. Nonlinear flows and semiflows; 3. Interplay between linear infinite-dimensional systems and nonlinear finite-dimensional systems; 4. Quantum stochastic evolutions and dynamical semigroups; 5. Further applications of semigroups in mathematical physics; 6. Related topics.

Website of the OPSO 2021 conference: https://nnov.hse.ru/bipm/dsa/opso2021

Contents

1. One-parameter groups and semigroups of linear operators

On perturbations of one-parameter semigroups determined by covariant operator valued measures on the half-axis

G. G. Amosov[1], E. L. Baitenov[2]

Keywords: perturbations of semigroups; covariant operator valued measures; the semigroup of shifts on the half-axis.

MSC2010 codes: 47D06, 46L51

A one-parameter family of contractions $T_t : X \ ! \ X; \ t \quad 0;$ acting on a Banach space X is said to be a semigroup if $T_{t+s} = T_t T_s; \ t; s \quad 0;$ and $T_0 = I$ (the identical transformation). If orbits of the semigroup $T = (T_t)$ are continuous in some topology then there exists a linear operator L with the domain $D(L)$ dense in X in the same topology such that $T_t = exp(tL); \ t \quad 0$. The operator L is said to be a generator of the semigroup T [1]. For the case $X = B(H)$ (the algebra of all bounded operators in a Hilbert space H) a perturbation of T of the semigroup T can be defined as a solution to the integral equation [2]

$$T_t \quad \int_0^t M(ds) T_{t \quad s} = T_t; \ t \quad 0;$$

where M is a measure on the half-axis taking values in the set of all completely positive maps on the algebra $B(H)$. To define a semigroup the measure M should be covariant with respect to the action of T in the sense

$$T_r \quad M([t; s)) = M([t + r; s + r)); \ r \quad 0; \ s \quad t \quad 0:$$

Let us go back to an arbitrary Banach space X. If we consider a perturbation of the generator L by a bounded operator in X, then the operator $L = L +$ having the same domain $D(L) = D(L)$ is a generator of the semigroup that is a solution to the integral equation determined by the covariant measure

$$M([t; s)) = \int_t^s T_r \quad dr; \ s; t \quad 0:$$

More complicated cases that lead to a change of the domain of a generator are defined by non-trivial cohomologies of T. We consider two examples in which a crucial role is played by the semigroup $S = (S_t)$ consisting of right shifts in the Hilbert space $H = L^2(\mathrm{R}_+)$. In one example, perturbations of the semigroup S are introduced in [3-4]. The second example determines the construction of perturbation for the semigroup $T = (T_t)$ acting in $X = B(F(H))$ by the formula [5]

$$T_t(x) = \hat{S}_t x \hat{S}_t; \ x \ 2 \ B(F(H));$$

where $\hat{S}_t$ acts in the antisymmetric Fock space $F(H) = f\mathrm{C} \ g \quad H \quad H^{\ 2}_{\ a} \quad \quad H^{\ n}_{\ a} \quad :::$ over one-particle Hilbert space $H = L^2(\mathrm{R}_+)$ by the formula

$$\hat{S}_t \quad = \quad ; \ \hat{S}_t(f_1 \quad f_2 \quad ::: \quad f_n) = S_t f_1 \quad S_t f_2 \quad ::: \quad S_t f_n; \ t \quad 0; \ f_j \ 2 \ H:$$

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Continuous Kernels for Positive Semigroups: applications to boundary problems and semilinear equations

W. Arendt[3]

We consider (mainly self-adjoint) semigroups on L^2 as generated by the Laplacian with diverse boundary conditions. In the talk we will present a criterion for such a semigroup to have a continuous kernel. One important consequence is that the semigroup operates on a space of continuous functions and, if the semigroup is positive, it will be automatically positivity improving. This has interesting consequences. For example, the principal eigenvector is strictly positive (i.e. it is larger than a positive constant). This in turn allows one to solve semilinear equations and to define a dynamical system. Interesting concrete cases are given by the Laplacian with Robin boundary conditions, but also by the Dirichlet-to-Neumann Operator.

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Local eventual positivity of operator semigroups

Sahiba Arora[4]

Keywords: One-parameter semigroup of linear operators; eventually positive semigroup; locally eventually positive semigroup; positive spectral projection; eventually positive resolvent; locally eventually positive resolvent.

MSC2010 codes: 47D06, 47B65, 47A10, 34B09, 34G10

Introduction. The theory of positive C_0-semigroups on L^p-spaces, or more generally, on Banach lattices, is well-known and has a plethora of applications (see [4]). Recently, the study of semigroups that only become (and stay) positive after a sufficiently large time was initiated in [1] and [2], where the authors also presented a variety of applications.

Another related notion that appeared several years earlier is of *local eventual positivity*. This loosely means that the solution of the corresponding Cauchy problem becomes eventually positive in a part of the domain. For example, it has been shown that (see [3]) for a homogeneous biharmonic heat equation in $\mathbb{R}^d$, if a positive initial datum has compact support, then the solution of the corresponding Cauchy problem becomes positive on compact sets for large times and that such eventual positivity does not occur on the whole domain.

In this talk, we discuss the local eventual positivity of strongly continuous semigroups on L^p- spaces. In particular, if $(e^{tA})_{t \geq 0}$ is a C_0-semigroup on some L^p-space F, and S and T are bounded operators on F, then given a function $f \in F$ which is positive almost everywhere, we provide sufficient conditions for existence of $t_0 \geq 0$, so that the functions $Se^{tA}Tf$ are also positive almost everywhere for $t \geq t_0$. Moreover, we also look at some conditions that guarantee that the time t_0 is independent of the initial datum f. We will see that these results can be applied to concrete differential equations, for instance, the Dirichlet bi-Laplacian. Additionally, we will consider some spectral and convergence implications of local eventual positivity.

While we restrict ourselves to L^p-spaces during the talk, the results mentioned can be generalized to general Banach lattices.

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Telegrapher's systems on networks

J. Banasiak[5], A. Błoch[6]

Keywords: hyperbolic systems, networks, semigroups of operators, port-Hamiltonians, Saint-Venant system, Kirchho□'s conditions

MSC2010 codes: 47D08, 35C15, 35J10

In this talk we consider a system of linear hyperbolic differential equations on a network coupled through general transmission conditions of Kirchhoff's type at the nodes. We discuss the reduction of such a problem to a system of 1-dimensional hyperbolic problems, also called port-Hamiltonian, for the associated Riemann invariants and provide a semigroup theoretic proof of its well-posedness in any L_p.

In the second part of the talk we consider a reverse question, that is, we derive conditions under which such a port-Hamiltonian with general linear Kirchhoff's boundary conditions can be written as a system of 2 2 hyperbolic equations on a metric graph . This is achieved by interpreting the matrix of the boundary conditions as a potential map of vertex connections of and then showing that, under the derived assumptions, that matrix can be used to determine the adjacency matrix of .

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Rates of decay of energy via operator semigroups

C. Batty [7]

Abstract. One approach to obtaining rates of decay of energy of damped waves and other similar models is by considering an operator which generates an operator semigroup and finding or estimating the norm of the resolvent on the imaginary axis. This can be converted into a rate of decay of the semigroup by quantified versions of tauberian theorems or by methods of Fourier analysis. I will briefly discuss the outcomes of these methods.

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An explicit formula for the telegraph equation semigroup on a network

A. Błoch[8]

In [1] the authors study a system of linear hyperbolic differential equations on a network coupled through transmission conditions in the vertices of the underlying graph. Such systems arise in e.g. hydrodynamics (Saint-Venant equations), electrical engineering (telegraph equations) or random walk theory. The authors construct general boundary conditions of Kirchhoff's type leading to an initial-boundary value problem and study its well-posedness using semigroup theory. It turns out that in some special cases it is possible to obtain an explicit formula for the semigroup governing the solution to the aforementioned IBVP which allows to study its long-term behaviour.

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Semigroup-theoretic approach to thin layers: the role of transmission conditions

A. Bobrowski[9]

Keywords: convergence of Feller semigroups of operators; singular perturbations; transmission conditions; thin layers

MSC2010 codes: 35B25, 35K57, 35K58, 47D06, 47D07

Motivated by models of signaling pathways in B lymphocytes, which have extremely large nuclei, we study the question of how reaction-diffusion equations in thin domains may be approximated by diffusion equations in regions of smaller dimensions. In particular, we study how transmission conditions featuring in the approximating equations become integral parts of the limit master equation. We device a scheme which, by appropriate rescaling of coefficients and finding a common reference space for all Feller semigroups involved, allows deriving the form of the limit equation formally. The results obtained, are expressed as convergence theorems for the Feller semigroups.

In the first part of the talk we take a look at the question of approximating solutions to equations governing diffusion in thin $3D$ layers separated by a semi-permeable membrane (see Figure 1). We show that as thickness of the layers converges to 0, the solutions, which by nature are functions of 3 variables, gradually lose dependence on the vertical variable and thus may be regarded as functions of 2 variables. The limit equation describes diffusion on the lower and upper sides of a two-dimensional surface (the membrane) with jumps from one side to the other. The latter possibility is expressed as an additional term in the generator of the limit semigroup, and this term is build from permeability coefficients of the membrane featuring in the transmission conditions of the approximating equations (i.e., in the description of the domains of the generators of the approximating semigroups).

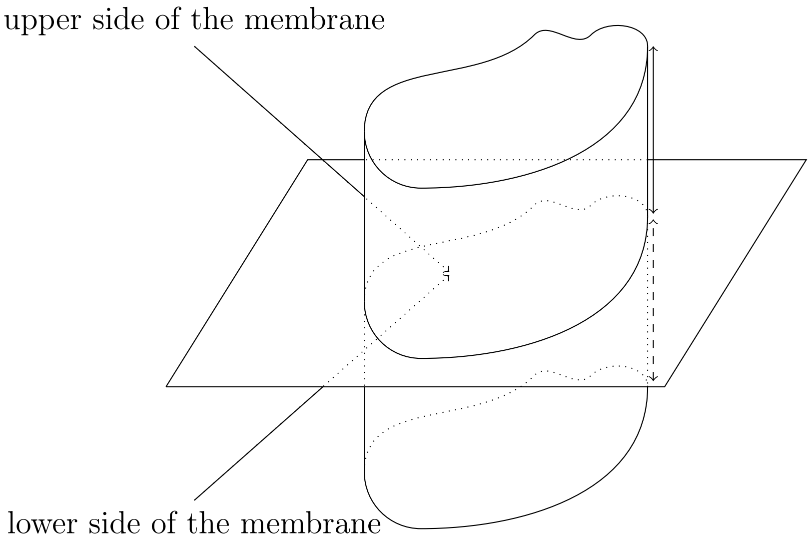

Figure 1: Two thin layers separated by a semi-permeable flat membrane

In the second part we work in two dimensions and deal with different geometry: we choose the example of a circular membrane as a case study. Again, the goal of this study is to establish the fact that in the thin layer approximation transmission conditions become integral parts of the limit equation in three natural scenarios.

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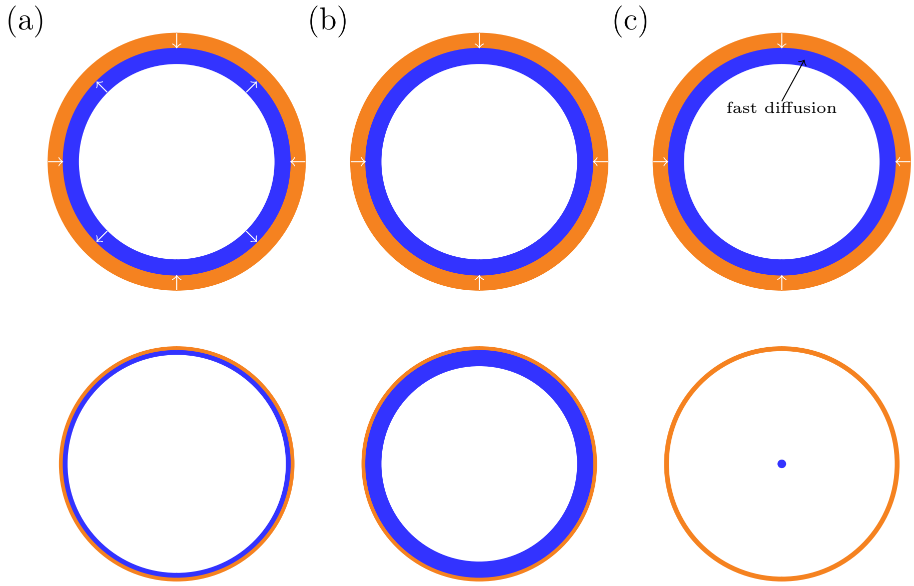

Figure 2: Different limit state-spaces resulting from different approximations: In the case (a) both layers are thin, and the limit state-space is composed of the upper and the lower sides of the unit circle. In the case (b) only the upper layer is thin, and thus the limit state-space is composed of the union of the unit circle and the lower layer/annulus. In the case (c), even though the lower layer is thick, diffusion there is fast and thus the limit state-space is the union of the unit circle and the point formed by compounding all elements of the lower annulus.

In the first scenario we assume that both the upper and the lower layers are thin, and in the limit obtain two coupled PDEs on a surface of dimension one (see Figure 2 (a)). Next, we turn to a 'thick' two-dimensional region bordering a 'thin' two-dimensional region (see Figure 2 (b)). In this case, in the limit we face a PDE on a two-dimensional region coupled with a PDE on a one-dimensional surface. Interestingly, here also one of permeability coefficients becomes an integral part of the main equation and describes jumps from the one-dimensional surface to the two-dimensional region.

In the third scenario, a diffusion in a thin layer is accompanied by a very fast diffusion in the bordering 'thick' two-dimensional region (see Figure 2 (c)). Then the limit equation involves a PDE on a surface coupled with an ODE: the fast diffusion averages out the concentration in the two-dimensional region, and thus, at any particular time, the concentration may be described by a single real number. Again, permeability coefficients become jump intensities between points of the surface and the isolated point formed by lumping all the points of the 'thick' two-dimensional region.

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On Hardy identities and inequalites

K. Bogdan[10]

We discuss the route from symmetric Markovian semigroups to Hardy inequalities, to non-explosion and contractivity results for Feynman-Kac semigroups in L^p. The emphasis will be on the fractional Laplacian on R^d, in which case the constants, estimates of the Feynman-Kac semigroups and the tresholds for contractivity and explosion are sharp. We will mention both published and new results, selected from:

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https://arxiv.org/abs/2103.06550.

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Asymptotic behaviour of a class of random evolution problems with application to combinatorial and metric graphs

S. Bonaccorsi[11]

We consider a family of graphs $\{G_k;\ k = 1, \dots, N\}$, each associated to the (discrete or continuous) *Laplacian* operator L_k acting on the function defined on the vertices (edges) of the graph.

Given a stochastic mechanism of switching the graphs during time, we get that the evolution is lead by an operator L_{X_k} (selected from the set $\{L_1, \dots, L_N\}$ according to some Markov chain X_k) during the (random) time interval $[T_k, T_{k+1})$

$$\begin{cases} \partial_t u(t,x) = L_{X_k} u(t,x), & t \in [T_k, T_{k+1}), \\ u(0,x) = f(x). \end{cases} \tag{1}$$

We can associate to (1) the (random) evolution operator

$$S(t) = e^{(t-T_n)L_{X_n}} \prod_{k=0}^{n-1} e^{(T_{k+1}-T_k)L_{X_k}}, \qquad t \in [T_n, T_{n+1}).$$

Our main problem can be stated as follows:

(P) under which condition the random evolution operator $S(t)$ converges? towards which limit?

The paper is based on a joint work with Francesca Cottini (Università di Milano Bicocca) and Delio Mugnolo (FernUniversität in Hagen).

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Positive Miyadera–Voigt perturbations of bi-continuous semigroups

Christian Budde[12]

We discuss positive Miyadera–Voigt type perturbations for bi-continuous semigroups on AL-spaces with an additional locally convex topology generated by additive seminorms. The main example of such spaces is the space of bounded Borel measures (on a Polish space).

Markov processes associated to stochastic differential equations or jointly continuous flows on metric spaces give rise to semigroups which are in general not strongly continuous with respect to the Banach space norm but they do enjoy strong continuity with respect to a weaker additional locally convex topology on the Banach space. An auspicious approach to such operator semigroups has been introduced by F.Kühnemund [9, 8] by means of bi-continuous semigroups. The theory of such semigoups has recently attract attention [7, 3, 5, 1] and especially perturbation theory of bi-continuous semigroups [4, 6].

Various models of physical processes ask for positive solutions in order to have a reasonable interpretation, e.g., consider solutions containing the absolute temperature or a density. The maximum principle for elliptic and parabolic partial differential equations guarantees positive solutions under positive initial data. This demonstrates the importance of positivity in the theory of operator semigroups on Banach lattices, hence in the theory of linear evolution equations.

This talk is based on [2] and generalizes the perturbation result of Voigt [10] to the class of bi-continuous semigroups.

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Asymptotic behaviour of biharmonic heat equations on unbounded domains

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Keywords: evolution equation, asymptotic behaviour, local eventual positivity, biharmonic operator

MSC2010 codes: 35G10, 35K30, 35B09, 35B40

Abstract. It is well-known that the heat equation $u_t \quad u = 0$ on R^n enjoys a *positivity preserving property*: if $u_0 \quad 0$ is a non-trivial function, then the solution $u = u(t; x)$ to the heat equation with initial datum u_0 satisfies $u(t; x) > 0$ for all $t > 0$. This phenomenon is closely connected with maximum principles for the Laplacian and second-order elliptic operators in general. On the other hand, one cannot expect the positivity preserving property to hold for higher-order elliptic operators. Nevertheless, it seems that positivity in some sense is "almost" preserved. As a particular case, the biharmonic heat equation $u_t + (\quad)^2 u = 0$ on R^n displays *local eventual positivity*, as shown in [5] and elaborated in [4]. Roughly speaking, this means that given non-trivial initial datum $u_0 \quad 0$, for every compact set $K \quad \mathrm{R}^n$, there exists a time $T > 0$ such that the corresponding solution $u = u(t; x)$ is positive on K for all $t \quad T$. Intuitively, this phenomenon occurs as a result of the oscillatory behaviour of the fundamental solution, and so far, local eventual positivity for these equations has been studied via explicit analysis of such biharmonic heat kernels.

From an abstract perspective, solutions to evolution equations may be studied via an appropriate semigroup of linear operators. The study of positive operator semigroups is by now a classic topic. However, as remarked above, this framework does not apply directly to higher-order evolution equations. One may instead use the theory of *eventually positive semigroups*, which was first developed systematically in [2,3]. Very recently, a localised version of the theory was initiated in [1]. The results of these papers are effective in particular for studying positivity in elliptic or parabolic problems on bounded domains where the associated operator has a simple principal eigenvalue. For this reason, however, the methods cannot be adapted to evolution equations on unbounded domains.

The current work therefore lies at the crossroads of the explicit methods of [4,5] and the semigroup methods in [1,2,3]. We study the asymptotic behaviour of solutions to the biharmonic heat equation on R^n as well as on 'infinite cylinders' of the form $\mathrm{R} \quad$, where $\quad \mathrm{R}^n$ is a bounded domain with C^1 boundary. The main tools are the Fourier transform and a classical blow-up argument, which reveals the asymptotic profile of the solutions. As a consequence of our results, we demonstrate how the local eventual positivity of solutions may be obtained qualitatively (i.e. without use of explicit heat kernels), and for a larger class of initial data than was previously considered. The analysis on the infinite cylinder is modelled loosely on the R^n problem, but in addition uses various properties of a family of fourth-order eigenvalue problems, which may be of independent interest.

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Long-term Behaviour of Flows in Infinite Networks

A. Dobrick[16]

Keywords: transport equations; infinite metric graphs; long-term behaviour; operator semigroups.

MSC2010 codes: 35B40; 47D06; 35R02; 35F46

Abstract. We study transport processes on infinite networks which can be modeled by an operator semigroup on a Banach space. Classically, such semigroups are strongly continuous and therefore their asymptotic behaviour is quite well understood. However, recently new examples of transport processes emerged in which the corresponding semigroup is not strongly continuous. Due to this lack of strong continuity, there are currently no results on the long-term behaviour of these semigroups. In this paper, we close this gap for a certain class of transport processes. In particular, it is proven that the solution semigroups behave asymptotically periodic with respect to the operator norm as a consequence of a more general result on the long-term behaviour by positive semigroups that contain a multiplication operator. Furthermore, we revisit known results on the asymptotic behaviour of transport processes on infinite networks and prove the asymptotically periodicity of the extensions of those semigroups to the space of bounded measures.

Introduction. Consider a transport process on an infinite network, modeled by an infinite, directed graph $G = (V, E)$ which is assumed to be simple, locally finite and non-degenerate and consider G as a metric graph by identifying each edge with the unit interval $[0, 1]$ and parametrizing it contrarily to its direction.

The distribution of mass transported along one edge e_j, $j \in J$, at some time $t \geq 0$ is described by a function $u_j(t, x)$ for $x \in [0, 1]$ and the material is transported along e_j with a constant velocity $c_j > 0$ that suffices

$$0 < c_{\min} \leq c_j \leq c_{\max} < \infty.$$

Define $\mathbb{B}^C := C^{-1}\mathbb{B}C$, where $\mathbb{B}$ denotes the weighted (transposed) adjacency matrix and $C := \operatorname{diag}(c_j)$ denotes the diagonal velocity matrix. Moreover, suppose that the functions u_j satisfy the generalized Kirchhoff law

$$\sum_{j \in J} \varphi^-_{ij} c_j u_j(1, t) = \sum_{j \in J} \varphi^+_{ij} c_j u_j(0, t)$$

for all $i \in I$ and $t > 0$. Then the transport process can be modeled by the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} u_j(t, x) = c_j \frac{\partial}{\partial x} u_j(t, x), & x \in (0, 1),\ t \geq 0, \\ u_j(0, x) = f_j(x), & x \in (0, 1), \\ u_j(1, t) = \sum_{j \in J} \mathbb{B}^C_{jk} u_k(0, t), & t \geq 0, \end{cases}$$

where f_j, $j \in J$, are the initial distributions of mass along the edges of G.

The investigation of equation systems of the above form on metric graphs by employing the theory of strongly continuous semigroups has quite some history. The semigroup approach to this kind of transport problems was introduced first in [1]. This paper was followed by a series of papers [2,3,4,5] from several different authors using the semigroup approach to discuss transport processes on metric graphs. In all these papers, transport equations are considered on the state

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space $L^1([0,1];\ell^1)$, where the solution semigroups turn out to be strongly continuous, and a major point is the asymptotic behaviour of the solution semigroup. However, the investigation of the long-term behaviour relied on the theory of strongly continuous semigroups, the Jacobs-de Leeuw-Glicksberg decomposition and classical results from Perron-Frobenius theory.

Motivated by results from [6,7], the authors in [8] discuss transport processes on the state space $L^\infty([0,1];\ell^1)$, where the solution semigroup is only bi-continuous with respect to the weak*-topology on $L^\infty([0,1];\ell^1)$ (see [9] for a definition) but not strongly continuous. Although the same kind of equation is investigated (however in different state spaces), in contrast to the papers from the L^1-case, [8] does not discuss the asymptotic behaviour of the solutions in the bi-continuous case. In this paper, we close this gap by combining spectral theoretic observations, the concept of the semigroup at infinity and classical Perron-Frobenius theory. In particular, the following theorem holds without any regularity assumptions on the semigroup:

Theorem 1. Let $\Omega=(\Omega,\Sigma,\mu)$ be a measure space, E a Banach lattice and $(T(t))_{t\geq 0}$ a bounded, positive semigroup on $X:=L^p(\Omega;E)$, $1\leq p\leq\infty$. Suppose that there is $t_0\in[0,\infty)$ such that $T(t_0)=M_B$ for some irreducible operator B on E with $r(B)=1$. If $r(B)$ is a pole of the resolvent of B, then there is a strictly positive projection P commuting with the semigroup $(T(t))_{t\geq 0}$ with the following properties:

(i) $(T(t)|_{PX})_{t\geq 0}$ can be extended to a positive, periodic group on the Banach lattice PX.

(ii) $(T(t)|_{\ker P})_{t\geq 0}$ is uniformly exponentially stable, i.e., there are $M\geq 1$ and $\varepsilon>0$ such that $\|T(t)-T(t)P\|\leq Me^{-\varepsilon t}$ for all $t\geq 0$.

Using our new methods we also revisit a result on the asymptotic behaviour of solutions from [2] and improve slightly upon the statement as well as the proof. In particular, our argument shows that the asymptotical periodicity of the solution, obtained there, does not depend on the strong continuity of the semigroup.

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Operator semigroups on vector lattices

M. Kramar Fijavž[17]

Our goal is to develop operator semigroup theory on vector lattices, suitable for the study of evolution equations, that properly generalises the classical theory of strongly continuous operator semigroups on Banach spaces. To this aim we introduce the notion of relatively uniformly continuous operator semigroups on vector lattices. We characterise such semigroups on L^p spaces and compare them with C_0-semigroups on Banach lattices. We define appropriate generators and present a Hille-Yosida-type theorem on vector lattices.

This is a joint work with Marko Kandić and Michael Kaplin.

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Upper and lower estimates for the speed of convergence of Chernoff approximations of operator semigroups

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Keywords: Chernoff product formula, approxition of C_0-semigroup, speed of convergence, estimates, examples

MSC2010 codes: 47D03, 47D06, 35A35, 41A25

This talk is devoted to the speed (rate) of convergence of Chernoff approximations [3,4] to strongly continuous one-parameter semigroups [1, 2]. We provide simple natural examples for which this convergence: is arbitrary high; is arbitrary slow; holds in the strong operator topology but does not hold in the norm operator topology. We also prove general theorem that gives estimate from above for the speed of decay of the norm of the residual appearing in Chernoff approximations. We provide also supplementary theorems which makes it easier to check the conditions of the main theorem.

This talk is based on the first draft-preprint version [5] of the text e.g. it lacks up-to-date literature overview. We kindly ask authors of relevant research papers to forgive us. We will fill this gap before submitting to a journal.

Usually expressing the C_0-semigroup $\left(e^{tL}\right)_{t\geq 0}$ it terms of variable coefficients of operator L is a hard problem. However, if the so-called Chernoff function [3] is constructed, i.e. an operator-valued function G which satisfies the conditions of the Chernoff theorem (in particular, satisfied $G(t) = I + tL + o(t)$ in the strong operator topology as $t \to +0$), then the semigroup is given by the equality $e^{tL} = \lim_{n\to\infty} G(t/n)^n$. An advantage of this approach arises from the fact that usually it is possible to define G by an explicit and not very long formula which contains coefficients of operator L. Expressions $G(t/n)^n$ are called Chernoff approximations to e^{tL}.

We have constructed [5] the following examples.

Proposition 1. There exists a Banach space F, C_0-semigroup $(e^{tL})_{t\geq 0}$ in F with generator $(L, D(L))$, and Chernoff function G for operator $(L, D(L))$ such that Chernoff approximations converge on each vector but do not converge in operator norm. More precisely:

1. $\lim_{n\to\infty} \|G(t/n)^n f - e^{tL} f\| = 0$ for all $f \in F$,
2. $\|e^{tL}\| = \|G(t)\| = 1$,
3. for each $t > 0$ and each $n \in \mathbb{N}$ there exists $f_n \in F$ such that $\|f_n\| = 1$ and $\|G(t/n)^n f_n - e^{tL} f_n\| \geq \|f_n\|$ so $\|G(t/n)^n - e^{tL}\| \geq 1 \neq 0$ as $n \to \infty$.

Proposition 2. There exists C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space F, Chernoff function G and vector $f \in F$ such that $f \notin D(L)$ but the speed of convergence is arbitrary high. More precisely: for arbitrary chosen non-increasing continuous function $v\colon [0, +\infty) \to [0, +\infty)$ vanishing at infinity at arbitrary high rate (e.g. $v(x) = (1+x)^{-k}$, $v(x) = e^{-x}$, $v(x) = e^{-e^x}$) and all $T > 0$ we have $\sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL} f\| = Tv(n/T)$ for all $n = 1, 2, 3, \ldots$ such that $Tv(n/T) \leq 1$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = \|f\| = 1$.

Proposition 3. There exists C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space F, Chernoff function G and vector $f \in F$ such that $f \in \cap_{j=1}^{\infty} D(L^j)$ but the speed of convergence is arbitrary low, i.e. for arbitrary chosen non-increasing continuous function $u\colon [0, +\infty) \to [0, +\infty)$ vanishing at infinity at arbitrary low rate (e.g. $v(x) = (1+x)^{-1/k}$, $v(x) = 1/\log(x+e)$, $v(x) = 1/\log(\log(x+e^e))$) and all $T > 0$ we have $\sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL} f\| = Tv(n/T)$ for all $n = 1, 2, 3, \ldots$ such that $Tv(n/T) \leq 1/2$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = \|f\| = 1$.

Our main result is stated as follows.

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Theorem 1. Suppose that

1. Number $T>0$ is given, and C_0-semigroup $(e^{tA})_{t\geq 0}$ with generator $(A, D(A))$ in Banach space F satisfies for some $M_1\geq 1$ and $w\geq 0$ the condition $\|e^{tA}\|\leq M_1e^{wt}$ for all $t\in[0,T]$.

2. There is a mapping $S\colon (0,T]\to\mathscr{L}(F)$, i.e. $S(t)\colon F\to F$ is a linear bounded operator for each $t\in(0,T]$. There exists some constant $M_2\geq 1$ that $\|S(t)^k\|\leq M_2e^{kwt}$ for all $t\in(0,T]$ and all $k=1,2,3,\dots$

3. Numbers $m\in\{0,1,2,\dots\}$ and $p\in\{1,2,3,\dots\}$ are fixed. There is a $(e^{tA})_{t\geq 0}$-invariant subspace $D\subset D(A^{m+p})\subset F$, i.e. $(e^{tA})(D)\subset D$ for any $t\geq 0$ (for example D may be equal to $D(A^{m+p})$).

4. There exist such functions $K_j\colon(0,T]\to[0,+\infty)$, $j=0,1,\dots,m+p$ that for all $t\in(0,T]$ and all $f\in D$ we have

$$\left\|S(t)f-\sum_{k=0}^{m}\frac{t^kA^kf}{k!}\right\|\leq t^{m+1}\sum_{j=0}^{m+p}K_j(t)\|A^jf\|.$$

Then:

1. For all $t>0$, all integer $n\geq t/T$ and all $f\in D$ we have

$$\|S(t/n)^nf-e^{tA}f\|\leq\frac{M_1M_2t^{m+1}e^{wt}}{n^m}\sum_{j=0}^{m+p}C_j(t/n)\|A^jf\|,$$

where $C_{m+1}(t)=K_{m+1}(t)e^{-wt}+M_1/(m+1)!$ and $C_j(t)=K_j(t)e^{-wt}$ for $j\neq m+1$.

2. If D is dense in F and for all $j=0,1,\dots,m+p$ we have $K_j(t)=o(t^{-m})$ when $t\to+0$, then for all $g\in F$ and $T>0$ the following equality is true:

$$\lim_{n\to\infty}\sup_{t\in(0,T]}\left\|S(t/n)^ng-e^{tA}g\right\|=0.$$

Example 1. Let us consider particular modeling example. Suppose $\|e^{tA}\|\leq e^t$, $\|S(t)\|\leq e^t$, $\|S(t)f-f-tAf-\frac{1}{2}t^2A^2f\|\leq t^2\sqrt{t}\|A^3f\|$ for all $f\in D(A^3)$ and $t\in(0,1]$. Then $D=D(A^3)$, $m=2$, $M_1=M_2=w=1$, $K_0(t)=K_1(t)=0$, $K_2(t)=1/\sqrt{t}$ for any $t\in(0,1]$. So estimate in the item of theorem 1 states that for any fixed $t>0$ the following estimate is true for all $f\in D(A^3)$ and integer $n\geq t$, having the following asymptotic behaviour as $n\to\infty$:

$$\|S(t/n)^nf-e^{tA}f\|\leq\frac{t^3e^t}{n^2}\left(\frac{1}{\sqrt{t/n}}+\frac{e^{-t/n}}{3!}\right)\|A^3f\|=$$
$$=e^t\left(\frac{t^2\sqrt{t}}{n\sqrt{n}}+\frac{e^{-t/n}t^3}{6n^2}\right)\|A^3f\|=\frac{t^2\sqrt{t}e^t}{n\sqrt{n}}\|A^3f\|+O\left(\frac{1}{n^2}\right).$$

Lemma 1. Suppose that $n\in\{0,1,2,\dots\}$, suppose that functions $a,b,c\colon\mathbb{R}\to\mathbb{R}$ are differentiable $2\lfloor(n-1)/2\rfloor$ times and the inequality $\inf_{x\in\mathbb{R}}|a(x)|>0$ holds. Suppose, in addition, that operator L maps each twice differentiable function $u\colon\mathbb{R}\to\mathbb{R}$ to the function $Lu=au''+bu'+cu$. Then there are nonnegative constants $C_0,C_1,\dots,C_{\lfloor(n+1)/2\rfloor}$, such that for any $2\lfloor(n+1)/2\rfloor$ times differentiable function $v\colon\mathbb{R}\to\mathbb{R}$, the following inequality is true:

$$\|v^{(n)}\|\leq\sum_{k=0}^{\lfloor(n+1)/2\rfloor}C_k\|L^kv\|.$$

Remark 1. Let us use the symbol $UC_b(\mathbb{R})$ for the space of all bounded, uniformly continuous functions $f\colon\mathbb{R}\to\mathbb{R}$ with the norm $\|f\|=\sup_{x\in\mathbb{R}}|f(x)|$. Let us use symbol $HC_b(\mathbb{R})$ for the space of all Hölder continuous functions $u\colon \mathbb{R} \to \mathbb{R}$, and for each $n \in \{1, 2, 3, \ldots\}$ let us denote by $HC_b^n(\mathbb{R})$ the space of all such functions $u \in HC_b(\mathbb{R})$, that $u', \ldots, u^{(n)} \in HC_b(\mathbb{R})$. It is clear that $HC_b^n(\mathbb{R}) \subset UC_b(\mathbb{R})$ and $HC_b^n(\mathbb{R})$ is dense in $UC_b(\mathbb{R})$ for all $n \in \{1, 2, 3, \ldots\}$. Similarly, the space $UC_b^n(\mathbb{R})$ is defined. The following result is an example of usage of theorem 1 and lemma 1.

Theorem 2. Suppose that

1. Numbers $m, q \in \{1, 2, 3, \ldots\}$ are fixed, and $\hat{q} = 2\lfloor (q+1)/2 \rfloor$. Functions a, b, c from the class $HC_b^{2m+\hat{q}-2}(\mathbb{R})$ are given, and $\inf_{x\in\mathbb{R}} a(x) > 0$. Operator L on $UC_b(\mathbb{R})$ with domain $D(L) = HC_b^2(\mathbb{R})$ is defined by the formula

$$Lu = au'' + bu' + cu.$$

2. Numbers $T > 0$, $M \geq 1$ and $\omega \geq 0$ are given. For any $t \in (0, T]$ a bounded linear operator $S(t)$ on $UC_b(\mathbb{R})$ is defined such that $\|S(t)^k\| \leq Me^{k\omega t}$ for all $k = 1, 2, 3, \ldots$.

3. There exist nonnegative constants $K_0, K_1, \ldots, K_{2m+q}$ such that for all $t \in (0, T]$ and all $f \in UC_b^{2m+q}(\mathbb{R})$ we have

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k L^k f}{k!} \right\| \leq t^{m+1} \sum_{j=0}^{2m+q} K_j \|f^{(j)}\|.$$

Then:

1. The closure $\overline{L}$ of operator L is a generator of C_0-semigroup $(e^{t\overline{L}})_{t\geq 0}$ in Banach space $UC_b(\mathbb{R})$, and the condition $\|e^{t\overline{L}}\| \leq e^{\omega t}$ for all $t \geq 0$ is satisfied, where $\omega = \max(0, \sup_{x\in\mathbb{R}} c(x))$.

2. For all $t > 0$, all integer $n \geq t/T$ and all $f \in UC_b^{2m+\hat{q}}(\mathbb{R})$ we have

$$\|S(t/n)^n f - e^{t\overline{L}} f\| \leq \frac{M t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{2m+\hat{q}} C_j \|f^{(j)}\|,$$

where $w = max(\omega, \omega)$, $\hat{q} = 2\lfloor (q+1)/2 \rfloor$ and $C_0, C_1, \ldots, C_{2m+\hat{q}}$ are nonnegative constants that are independent of t and n.

3. For all $g \in UC_b(\mathbb{R})$ and all $T > 0$ the following equality is true:

$$\lim_{n\to\infty} \sup_{t\in(0,T]} \left\| S(t/n)^n g - e^{t\overline{L}} g \right\| = 0.$$

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Positivity properties of operator semigroups

J. Glück[20]

Keywords: positive semigroup; spectrum; convergence to equilibrium; eventual positivity.

MSC2010 codes: 47D06, 47D07, 47B65, 46B42

We give a brief but pointed tour through the subject of positive one-parameter semigroups. As a motivation, we start with a few examples that demonstrate the ubiquity of positive semigroups in various fields in the mathematical sciences. With these examples in mind, we then proceed to several theoretical highlights of the theory, ranging from classical spectral results in infinite-dimensional *Perron–Frobenius* and *Krein–Rutman theory* to recent insights into the long-term behaviour of positive semigroups without time regularity.

Our journey ends with an outlook on the emerging topic of *eventually positive semigroups*, i.e., semigroups that can have a sign change for small times but then become and stay positive for larger times. A systematic study of this concept has begun only recently, and is accompanied by the observation that the eventual positivity phenomenon occurs in various concrete evolutions equations.

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Bi-Kolmogorov type operators and weighted Rellich's inequalities

F. Gregorio [21]

Keywords: Higher order elliptic equations, Maximal regularity, Invariant measures, Weighted Rellich's inequalities

MSC2010 codes: 47D03, 35K35, 35A23, 35K65

We consider the symmetric Kolmogorov operator $L = \quad + \frac{r}{\quad} \quad r$ on $L^2(\mathrm{R}^N; d\ )$, where   is the density of a probability measure on R^N. Under general conditions on   we prove first weighted Rellich's inequalities with optimal constants and deduce that the operators L and $\ L^2$ with domain $H^2(\mathrm{R}^N; d\ )$ and $H^4(\mathrm{R}^N; d\ )$ respectively, generate analytic semigroups of contractions on $L^2(\mathrm{R}^N; d\ )$. We observe that $d\ $ is the unique invariant measure for the semigroup generated by $\ L^2$ and as a consequence we describe the asymptotic behaviour of such semigroup and obtain some local positivity properties. As an application we study the bi-Ornstein-Uhlenbeck operator and its semigroup on $L^2(\mathrm{R}^N; d\ )$.

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Fractional Kolmogorov operator and desingularizing weights

D. Kinzebulatov[22]

We establish sharp two-sided bounds on the heat kernel of the fractional Laplacian perturbed by Hardy-type drift by transferring it to an appropriate weighted space with singular weight. The talk is based on joint papers with Yu.A. Semenov and K. Szczypkowski.

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Propagators for the Equation of Internal Waves

K. Yu. Kotlovanov.[23]

Keywords: sobolev type equations; propagators; internal wave equation.

MSC2010 codes: 35Q35, 47D03

Introduction.

Equation of internal waves in a homogeneous incompressible non-rotating fluid is described by the Poincare equation

$$u_{tt} + N^2(u_{xx} + u_{yy}) = 0, \tag{1}$$

where N^2 – buoyancy frequency.

Let D a bounded domain from $\mathbb{R}^3$ with smooth boundary ∂D with the initial–boundary conditions

$$u(x,y,z,t) = 0, \quad (x,y,z,t) \in \partial D \times \mathbb{R}, \tag{2}$$

$$u(x,y,z,0) = u_0, \quad u_t(x,y,z,0) = u_1. \tag{3}$$

The solution to the problem (1) - (3) will be sought in the framework of the theory of Sobolev-type equations.

In this thesis, used methods based on the theory of polynomially bounded operator pencils [1].

(A,p)-bounded Operators.

Let $\mathfrak{U}, \mathfrak{F}$ be Banach spaces and operators $A, B_0, B_1, \dots, B_{n-1} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$. By $\vec{B}$ denote a pencil formed by operators $B_0, B_1, \dots, B_{n-1}$. Set $\rho^A(\vec{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1}B_{n-1} - \dots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$, and $\sigma^A(\vec{B}) = \overline{\mathbb{C}} \setminus \rho^A(\vec{B})$ are called A-resolvent set and A-spectrum of the pencil $\vec{B}$ respectively. Operator-function of a complex variable $R^A_\mu(\vec{B}) = (\mu^n A - \mu^{n-1}b_{n-1} - \dots - \mu B_1 - B_0)^{-1}$ with the domain $\rho^A(\vec{B})$ is called A-resolvent of the pencil $\vec{B}$.

Definition 1. The operator pencil $\vec{B}$ is called polynomially bounded with respect to an operator A (or polynomially A-bounded) if $\exists a \in \mathbb{R}_+ \ \forall \mu \in \mathbb{C}\ (|\mu| > a) \Rightarrow (R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{F};\mathfrak{U}))$.

Remark 1. Let operator exist $A^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$ then the pencil $\vec{B}$ polynomially A-bounded.

Condition (4) is necessary for the existence of projectors. [2]

$$\int_\gamma \mu^k R^A_\mu(\vec{B})\,d\mu \equiv \mathbb{O}, \quad k = 0,1,\dots,n-2, \tag{4}$$

where the circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Lemma 1. Let the operator pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled. Then the operators

$$P = \frac{1}{2\pi i}\int_\gamma R^A_\mu(\vec{B})\mu^{n-1}A\,d\mu, \quad Q = \frac{1}{2\pi i}\int_\gamma \mu^{n-1}AR^A_\mu(\vec{B})\,d\mu$$

are projectors in spaces $\mathfrak{U}$ and $\mathfrak{F}$ respectively.

Denote $\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{U}^1 = \operatorname{im} Q$, $\mathfrak{F}^1 = \operatorname{im} Q$. According to lemma 1 $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. By $A^k(B^k_l)$ denote restriction of operators$A(B_l)$ on $\mathfrak{U}^k(\mathfrak{F}^k)$, $k = 0,1$; $l = 0,1,\dots,n-1$.

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Theorem 1. [2] Let the operator pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled. Then

(*i*) $A^k \in L(U^k; F^k)$, $k = 0,1$;
(*ii*) $B_l^k \in L(U^k; F^k)$, $k = 0,1$, $l = 0,1,\dots,n-1$;
(*iii*) operator $(A^1)^{-1} \in L(U^1; F^1)$ exists;
(*iv*) operator $(B_0^0)^{-1} \in L(U^0; F^0)$ exists.

Using theorem 1 construct operators: $H_0 = (B_0^0)^{-1}A^0 \in L(U^0)$, $H_1 = (B_0^0)^{-1}B_1^0 \in L(U^0), \dots, H_{n-1} = (B_0^0)^{-1}B_{n-1}^0 \in L(U^0)$ and $S_0 = (A^1)^{-1}B_0^1 \in L(U^1), S_1 = (A^1)^{-1}B_1^1 \in L(U^1), \dots, S_{n-1} = (A^1)^{-1}B_{n-1}^1 \in L(U^1)$

Further a removable singularity of an A-resolvent of the pencil $\vec{B}$ will be called a pole of order 0 for brevity. If the operator pencil $\vec{B}$ is polynomially A-bounded and the point ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of an A-resolvent of the pencil $\vec{B}$ then the operator pencil $\vec{B}$ is called polynomially (A,p)-bounded.

Abstract Problem.

Consider the Cauchy problem

$$u(0) = u_0, \quad u_t(0) = u_1 \tag{5}$$

for the second-order Sobolev-type equation

$$Au_{tt} = B_1 u_t + B_0 u. \tag{6}$$

Operator-functions

$$U_0^t = \frac{1}{2\pi i}\int_\gamma R_\mu^A(\vec{B})(\mu A - B_1)e^{\mu t}d\mu, \quad U_1^t = \frac{1}{2\pi i}\int_\gamma R_\mu^A(\vec{B})Ae^{\mu t}d\mu$$

– are propagators, where the circuit $\gamma \subset \mathbb{C}$ bounds the domain containing the A-spectrum of the operator pencil $\vec{B}$.

The solution of the problem (1), (3) in terms of the theory of degenerate groups was obtained in [2], under the condition that the operator pencil $\vec{B}$ is polynomially bounded with respect to the operator A.

Theorem 2. [3] Let operator pencil $\vec{B}$ be polynomially A-bounded and condition (4) fulfilled, let ∞ be a pole of order $p \in \{0\} \cup \mathbb{N}$ of the A-resolvent of $\vec{B}$. There exists a unique solution $u \in C^\infty(\mathbb{R}; U)$ of problem (5), (6) of the form $u(t) = U_1^t u_1 + U_0^t u_0$, where $u_k \in$ im $U_1^0 =$ im U_0^0, $k = 0,1$, im U_1^0 ΓË im U_0^0 subspace in U.

Internal Wave Equation.

Consider the cases when the domain D is a parallelepiped and a cylinder. Let the domain D be the parallelepiped $[0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$.

Problem(1)–(3) can be reduced to Cauchy problem (5) for equation (6). Introduce the spaces $U = W_2^{l+2}(D)$, $F = W_2^l(D)$ and define the operators

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad B_1 = O, \quad B_0 = -N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

For any $l \in \{0\} \cup \mathbb{N}$ operators $A, B_1, B_0 \in L(U; F)$. Define $\lambda_{k,m,n}^2 = -\left(\frac{\pi k}{a}\right)^2 - \left(\frac{\pi m}{b}\right)^2 - \left(\frac{\pi n}{c}\right)^2$ the eigenvalues of the Dirichlet problem for the Laplace operator. Denote by $\varphi_{k,m,n} = \sin\left(\frac{\pi kx}{a}\right)\sin\left(\frac{\pi my}{b}\right)\sin\left(\frac{\pi nz}{c}\right)$ the orthogonal eigenfunctions that correspond to $\{\lambda_{k,m,n}^2\}$ in the sense of the scalar product in $L^2(D)$.

Since $\{\varphi_{k,m,n}\} \subset C^\infty(D)$, then

$$\mu^2 A - \mu B_1 - B_0 = \sum_{k,m,n=1}^{\infty} [\mu^2 \lambda_{k,m,n}^2 + N^2 \lambda_{k,m}^2] < \varphi_{k,m,n}, \cdot > \varphi_{k,m,n},$$

where $B_0\varphi_{k,m,n} = -\nu^2_{k,m}\varphi_{k,m,n}$, and $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2(D)$. Construct the equation to determine the relative spectrum:

$$\lambda^2_{k,m,n}\mu^2 + N^2\nu^2_{k,m} = 0, \qquad \mu^{1,2}_{k,m,n} = \pm\frac{N\nu_{k,m}}{\sqrt{\lambda^2_{k,m,n}}}i$$

The relative spectrum $\sigma^A(B) = \{\mu^{1,2}_{k,m,n}\}$ is bounded, because $|\mu^{1,2}_{k,m,n}| \le N$. Since the operator A is continuously invertible in the given spaces, then the condition (4) is satisfied. As a result, the conditions of Lemma 1 hold.

Construct propagators:

$$U_0^t u_0 = \sum_{k,m,n=1}^{\infty}\cos\left(\frac{N\nu_{k,m}}{\sqrt{\lambda^2_{k,m,n}}}t\right)\langle\varphi_{k,m,n},u_0\rangle\varphi_{k,m,n},$$
$$U_1^t u_1 = \sum_{k,m,n=1}^{\infty}\frac{N\nu_{k,m}}{\sqrt{\lambda^2_{k,m,n}}}\sin\left(\frac{N\nu_{k,m}}{\sqrt{\lambda^2_{k,m,n}}}t\right)\langle\varphi_{k,m,n},u_1\rangle\varphi_{k,m,n}.$$

Solution to problem (1)–(3) has the form

$$u(x,y,z,t) = U_0^t u_0 + U_1^t u_1.$$

Now consider the case when the domain D is a cylinder. Similarly, as in the case of the parallelepiped, the problem (1)–(3) can be reduced to the Cauchy problem (5) for the equation (6).

Using the operators A, B_0, B_1 of the form:

$$A = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}, \quad B_1 = \mathbb{O}, \quad B_0 = -N^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right).$$

Construct the equation to determine the relative spectrum:

$$\mu^2\left(\left(\mu_k^{(n)}\right)^2 + \left(\frac{m\pi}{l}\right)^2\right) + N^2\left(\mu_k^{(n)}\right)^2 = 0, \qquad \mu^{1,2}_{k,m,n} = \pm\frac{N\mu_k^{(n)}}{\sqrt{(\mu_k^{(n)})^2 + \left(\frac{m\pi}{l}\right)^2}}i.$$

where $(\mu_k^{(n)})^2 + (\frac{n\pi}{l})^2$ – the eigenvalues of the Dirichlet problem for the Laplace operator, $B_0\varphi_{k,m,n} = (\mu_k^{(n)})^2\varphi_{k,m,n}$.

Construct propagators:

$$U_0^t u_0 = \sum_{k,m,n=1}^{\infty}\cos\left(\frac{N\mu_k^{(n)}}{\sqrt{(\mu_k^{(n)})^2+\left(\frac{m\pi}{l}\right)^2}}t\right)\langle\varphi_{k,m,n},u_0\rangle\varphi_{k,m,n},$$
$$U_1^t u_1 = \sum_{k,m,n=1}^{\infty}\frac{\sqrt{(\mu_k^{(n)})^2+\left(\frac{m\pi}{l}\right)^2}}{N\mu_k^{(n)}}\sin\left(\frac{N\mu_k^{(n)}}{\sqrt{(\mu_k^{(n)})^2+\left(\frac{m\pi}{l}\right)^2}}t\right)\langle\varphi_{k,m,n},u_1\rangle\varphi_{k,m,n}.$$

Solution to problem (1)–(3) has the form

$$u(x,y,z,t) = U_0^t u_0 + U_1^t u_1.$$

As a result of the work, we obtained solutions of the initial-boundary value problem (2)–(3) in closed form on the considered domains for the equation of internal waves (1).

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Subordination for Semigroups in locally convex Spaces

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Keywords: Subordination; C_0-semigroups; locally convex spaces; bi-continuous semigroups; transition semigroups.

MSC2010 codes: 47D06, 46A03, 46A70

Introduction: Subordination in the sene of Bochner. Let X be a Banach space and $(T(t))_{t\geq 0}$ a bounded C_0-semigroup on X with generator $-A$. Given a so-called Bernstein function f, there exists a unique vaguely continuous convolution semigroup of sub-probability measures $(\mu_t)_{t\geq 0}$ on $[0,\infty)$ associated with f, and we can define a new semigroup $(S(t))_{t\geq 0}$ by

$$S(t)x := \int_{[0,\infty)} T(s)x \,\mathrm{d}\mu_t \quad (x \in X, t \geq 0).$$

It turns out that $(S(t))_{t\geq 0}$ is strongly continuous and its generator is given by $-f(A)$. This sketched construction is called *subordination in the sense of Bochner*, see e.g. [1, 2].

Subordination in locally convex spaces.

In many applications, in particual those coming from stochastic processes, there do appear semigroups which are not strongly continuous w.r.t. the norm topology of a Banach space, but admit continuous orbits only for a coarser locally convex topology.

In this talk we will show that subordination can also be performed for locally (sequentially) equicontinuous, equibounded C_0-semigroups on (sequentially) complete Hausdorff locally convex spaces. This generalisation gives rise to applications on bi-continuous semigroups [3] as well as transition semigroups for C_b-Feller processes. In the context of stochastic processes, subordination corresponds to a random time change. Thus, we obtain an analytic description of the semigroup induced by the time-changed process.

Remark. Subordination can be viewed as a functional calculus technique, which can thus also be used in locally convex spaces.

The results are contained in [4].

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On admissible singular drifts of symmetric -stable process

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We consider the problem of existence of a (unique) weak solution to the SDE describing symmetric -stable process with a locally unbounded drift $b : \mathbb{R}^d \to \mathbb{R}^d$, $d \quad 3$, $1 < \quad < 2$. In this talk, b belongs to the class of weakly form-bounded vector fields, the class providing the L^2 theory of the non-local operator behind the SDE, i.e. $(\quad)^{\frac{}{2}} + b \cdot \nabla$. It contains as proper sub-classes other classes of singular vector fields studied in the literature in connection with this operator, such as the Kato class, the weak $L^{d/(\quad 1)}$ class and the Campanato-Morrey class (in general, such b makes invalid the standard heat kernel estimates in terms of the heat kernel of the fractional Laplacian). We show that the operator $(\quad)^{\frac{}{2}} + b \cdot \nabla$ with weakly form-bounded b admits a realization as (minus) Feller generator, and that the probability measures determined by the Feller semigroup (uniquely in appropriate sense) admit description as weak solutions to the corresponding SDE. The proof is based on detailed regularity theory of $(\quad)^{\frac{}{2}} + b \cdot \nabla$ in L^p, $p > d \quad + 1$.

The talk is based on joint work with Damir Kinzebulatov (Université Laval).

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Degenerate Nonlinear Semigroups for General Mathematical Filtration Boussinesq Model

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Keywords: semilinear Sobolev type equations; Cauchy problem; generalized Boussinesq filtration equations.

MSC2010 codes: 35Q35

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary of class C^∞. In the cylinder $\Omega \times \mathbf{R}_+$ consider the Cauchy initial condition

$$u(0) = u_0 \tag{1}$$

and Dirichlet boundary condition

$$u(s,t) = 0, \ (s,t) \in \partial\Omega \times \mathbf{R}_+ \tag{2}$$

for the generalized filtration Boussinesq equation [1]

$$(\lambda - \Delta)u_t = \Delta(|u|^{p-2}u), \ p \geq 2. \tag{3}$$

Equation (3) is the most interesting particular case of the equation obtained by E.S. Dzektser [1]. Here the desired function $u = u(s,t)$ corresponds to the potential of speed of movement of the free surface of the filtered liquid; the parameter $\lambda \in \mathbf{R}$ characterizes the medium, and this parameter λ can take negative values. In [2], a study of the phase space is given. The paper [3] was the first to obtain the conditions for the existence of a local solution to the Showalter–Sidorov–Dirichlet problem. Also, the paper [3] shows that the phase space of the generalized filtration Boussinesq equation is a smooth Banach manifold.

Let $\mathbf{H} = W_2^{-1}(\Omega)$, $H = L_2(\Omega)$, $B = L_p(\Omega)$ (all functional spaces are defined on domain Ω). Note that there exists the dense and continuous embedding $\overset{\circ}{W}{}_2^1(\Omega) \hookrightarrow L_q(\Omega)$ for $p \geq \frac{2n}{n+2}$, therefore $L_p(\Omega) \hookrightarrow W_2^{-1}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In $\mathbf{H}$, define the scalar product by the formula

$$\langle x, y \rangle = \int_\Omega x\tilde{y}\,ds \quad \forall x, y \in \mathbf{H},$$

where $\tilde{y}$ is the generalized solution to the homogeneous Dirichlet problem for Laplace operator $(-\Delta)$ in the domain Ω. Let $B^* = (L_p(\Omega))^*$ and $H^* = (L_2(\Omega))^*$, where $(L_p(\Omega))^*$ is conjugate space with respect to duality

$$B \hookrightarrow H \hookrightarrow \mathbf{H} \hookrightarrow H^* \hookrightarrow B^*.$$

Define the operators L and M as follows:

$$\langle Lu, v \rangle = \int_\Omega (\lambda u\tilde{v} + uv)\,ds, \quad u, v \in H;$$

$$\langle M(u), v \rangle = -\int_\Omega |u|^{p-2}uv\,ds, \quad u, v \in B.$$

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Let $\{\varphi_k\}$ be the sequence of eigenfunctions of the homogeneous Dirichlet problem for Laplace operator $(\quad)$ in the domain $\;$, and $\{\ _k\}$ be the corresponding sequence of eigenvalues numbered in non-increasing order taking into account the multiplicity.

Lemma 1. [4] (i) For all $\quad _1$ the operator $L \in L(H; H)$ is self-adjoint, Fredholm, and non-negatively defined, and the orthonormal family $\{\varphi_k\}$ of its functions is total in the space H.

(ii) Operator $M \in C^1(B; B)$ is dissipative and p-coercive.

If $\quad _1$

$$\ker L = \begin{cases} \{0\}; \ if \quad > \quad _1; \\ \operatorname{span}\{\varphi_1\}; \ if \quad = \quad _1. \end{cases}$$

Therefore

$$\operatorname{im} L = \begin{cases} H; \ if \quad > \quad _1; \\ \{u \in H : \langle u, \varphi_1 \rangle = 0\}; \ if \quad = \quad _1; \end{cases}$$

$$\operatorname{coim} L = \begin{cases} H; \ if \quad > \quad _1; \\ \{u \in H : \langle u, \varphi_1 \rangle = 0\}; \ if \quad = \quad _1. \end{cases}$$

Hence, the projectors

$$P = Q = \begin{cases} \mathbf{I}; \quad > \quad _1; \\ \mathbf{I} \quad \langle \ ; \varphi_1 \rangle \ ; \quad = \quad _1. \end{cases}$$

Construct the set

$$\mathbf{M} = \begin{cases} B; \ if \quad > \quad _1; \\ \{u \in B : \int |u|^{p \ 2} u \varphi_1 \, ds = 0\}; \ if \quad = \quad _1. \end{cases}$$

Theorem 1. [4] Suppose that $p \quad \frac{2n}{n+2}; \quad _1$. Then

(i) the set $\mathbf{M}$ is a simple Banach C^1-manifold modelled by the space $\operatorname{coim} L \cap B$;

(ii) $\forall u_0 \in \mathbf{M}$ there exists the unique solution $u \in C^k((0; +\infty); \mathbf{M})$ to problem (1) – (3).

Define the shift operator $U^t(u_0) \quad u(t)$, where $u(t)$ is a solution to problem (1) – (3). Then $\{U^t : t \in \mathbf{R}_+\}$ forms a nonlinear semigroup of operators with domain $D(U) = \mathbf{M}$.

Theorem 2. [4] Suppose that $p \quad \frac{2n}{n+2}; \quad _1$. Then there exists a resolving semigroup of contractive operators $\{U^t : t \in \mathbf{R}_+\}$ of equation (1) defined on the manifold $\mathbf{M}$.

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Positive Holomorphic Semigroups of Operators. Sobolev Type Equations

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Keywords: Sobolev type equations; positive degenerate holomorphic semigroups of operators; positive solution.

MSC2010 codes: 43A35

Let $\mathfrak{U};\ \mathfrak{F}$ be real Banach spaces, the operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (i.e. L is linear and continuous operator) and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$ (i.e. M is linear, closed, and densely defined operator). The foundation of our research is the theory of degenerate semigroups of operators and the phase space method described in [1]. Let us give the necessary information on the theory of degenerate semigroups of operators in Banach spaces. Consider the *L-resolvent set* $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and the *L-spectrum* $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ *of the operator* M, as well as the operator function $R^L_\mu(M) = (\mu L - M)^{-1}L$, which is called the *right L-resolvent of the operator M*.

Let us consider the nonhomogeneous linear Sobolev type equation

$$L\dot{u} = Mu + f, \quad \ker L \neq \{0\}. \tag{1}$$

It turns out that if the operator M is (L,p)-sectorial for some $p \in \{0\} \cup \mathbb{N}$ [1, Ch. 2], then the resolving semigroup of homogeneous equation (1) ($f \equiv 0$) is a degenerate holomorphic semigroups of operators $U^\bullet = \{U^t : t \in \mathbb{R}\}$ ($F^\bullet = \{F^t : t \in \mathbb{R}\}$). If, in addition, the operator M is strongly (L,p)-sectorial on the right (on the left) for some $p \in \{0\} \cup \mathbb{N}$ [1, Ch. 2] consider the Cauchy problem

$$\lim_{t \to 0+} (u(t) - u_0) = 0 \tag{2}$$

or the Showalter–Sidorov problem

$$\lim_{t \to 0+} P(u(t) - u_0) = 0 \tag{3}$$

for equation (1). Here the operator $P = s\text{-}\lim_{t \to 0} U^t$ is the unit of the semigroup $U^\bullet$ and $Q = s\text{-}\lim_{t \to 0+} F^t$ is the unit of the semigroup $F^\bullet$, which are projectors by construction [1]. There exists the splitting of the space $\mathfrak{U}$ and $\mathfrak{F}$, i.e.

$$\mathfrak{U}^0 \oplus \mathfrak{U}^1 = \mathfrak{U} \text{ and } \mathfrak{F}^0 \oplus \mathfrak{F}^1 = \mathfrak{F}.$$

Definition 1. The real Banach space $\mathfrak{B} = (\mathfrak{B}; \|\cdot\|_{\mathfrak{B}})$ is called *the ordered Banach space* $\mathfrak{B} = (\mathfrak{B}; \|\cdot\|_{\mathfrak{B}}; \leq_{\mathfrak{B}})$ if there exists the order relation $\leq_{\mathfrak{B}}$, which satisfies the axioms of reflexivity, transitivity, antisymmetry, and is consistent with the vector structure of the space $\mathfrak{B}$, i.e.

(i) $(x \leq_{\mathfrak{B}} y) \Rightarrow (\alpha x \leq_{\mathfrak{B}} \alpha y)$ for any $x, y \in \mathfrak{B}$ and for all $\alpha \in \mathbb{R}_+$,

(ii) $(x \leq_{\mathfrak{B}} y) \Rightarrow (x + z \leq_{\mathfrak{B}} y + z)$ for any $x, y \in \mathfrak{B}$ and for all $z \in \mathfrak{B}$.

If, in addition, the order relation $\leq_{\mathfrak{B}}$ is consistent with the metric structure of the space $\mathfrak{B}$, i.e. for all $x \in \mathfrak{B}$ there exist $x_+, x_- \in \mathfrak{B}$ such that

(iii) $(x_+ \geq_{\mathfrak{B}} 0) \wedge (x_- \geq_{\mathfrak{B}} 0) \wedge (x = x_+ - x_-)$,

(iv) $(|x| \leq_{\mathfrak{B}} |y|) \Rightarrow (\|x\|_{\mathfrak{B}} \leq \|y\|_{\mathfrak{B}})$ for any $y \in \mathfrak{B}$,

then the ordered Banach space $\mathfrak{B} = (\mathfrak{B}; \|\cdot\|_{\mathfrak{B}}; \leq_{\mathfrak{B}})$ is called the Banach lattice. (Here $|x| = x_+ + x_-$).

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Definition 2. Let $\mathfrak{B}$ be a Banach space. A convex set $\mathfrak{C} \subset \mathfrak{B}$ such that $\alpha\mathfrak{C} + \beta\mathfrak{C} \subset \mathfrak{C}$ for all $\alpha, \beta \in \mathbb{R}_+$ is called a *cone*. A cone $\mathfrak{C}$ is called *proper*, if $\mathfrak{C} \cap (-\mathfrak{C}) = \{0\}$, and *generative*, if $\mathfrak{B} = \mathfrak{C} - \mathfrak{C}$.

Let $\mathfrak{B}$ be a Banach lattice, then (it is easy to see) $\mathfrak{B}_+ = \{x \in \mathfrak{B} : x \geq 0\}$ is a proper generative cone. On the other hand, let $\mathfrak{B}$ be a Banach space, and $\mathfrak{C} \subset \mathfrak{B}$ be a proper generative cone. Let us introduce the order relation $\geq_{\mathfrak{B}}$ by the formula $(x \geq_{\mathfrak{B}} y) \Leftrightarrow (x - y \in \mathfrak{C})$. Further, we well distinguish the ordered Banach space $(\mathfrak{B}; \|\cdot\|_{\mathfrak{B}}; \geq_{\mathfrak{B}})$ and the Banach lattice $(\mathfrak{B}; \|\cdot\|_{\mathfrak{B}}; \mathfrak{B}_+)$.

Definition 3. (i) Let $(\mathfrak{B} = (\mathfrak{B}; \|\cdot\|_{\mathfrak{B}}; \mathfrak{B}_+)$ be a Banach lattice. The operator $A \in \mathcal{L}(\mathfrak{B})$ is called *positive*, if $A\mathfrak{B}_+ \subset \mathfrak{B}_+$.

(ii) A semigroup of operators $V^\bullet = \{V^t : t \in \mathbb{R}_+\}$ is called *positive*, if $V^t\mathfrak{V}_+ \subset \mathfrak{V}_+$ for all $t \in \mathbb{R}_+$.

Theorem 1. [2] Let the operator M be strongly (L, p)-sectorial for some $p \in \{0\} \cup \mathbb{N}$, and the Banach space $\mathfrak{U}$ be a Banach lattice, $\mathfrak{U} = (\mathfrak{U}; \|\cdot\|_{\mathfrak{U}}; \mathfrak{U}_+)$. Then the following statements are equivalent.

(i) The operator $[R^L_\mu(M)]^{p+1}$ is positive for all sufficiently large $\mu \in \mathbb{R}_+$.

(ii) The degenerate holomorphic semigroup $U^\bullet$ is positive.

Since the semigroups $U^\bullet$ and $F^\bullet$ are holomorphic, we set $\mathfrak{U}^0 = \ker U^\bullet$, $\mathfrak{F}^0 = \ker F^\bullet$. Let L_0 be the restriction of the operator L to $\mathfrak{U}^0$, and M_0 be the restriction of the operator M to $\mathfrak{U}^0 \cap \operatorname{dom} M$.

Theorem 2. [1] Let the operator M be (L, p)-sectorial for some $p \in \{0\} \cup \mathbb{N}$. Then

(i) $L_0 \in \mathcal{L}(\mathfrak{U}^0; \mathfrak{F}^0)$ and $M_0 : \mathfrak{U}^0 \cap \operatorname{dom} M \to \mathfrak{F}^0$;

(ii) there exists the inverse operator $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$;

(iii) the operator $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ is nilpotent of degree less than or equal to p.

Since the semigroups $U^\bullet$ and $F^\bullet$ are holomorphic, we set $\mathfrak{U}^1 = \operatorname{im} U^\bullet$, $\mathfrak{F}^1 = \operatorname{im} F^\bullet$. Let L_1 be the restriction of the operator L to $\mathfrak{U}^1$. If the operator M be strongly (L, p)-sectorial for some $p \in \{0\} \cup \mathbb{N}$. Then there exists the inverse

operator $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$.

Theorem 3. [1] Let the operator M be strongly (L, p)-sectorial for some $p \in \{0\} \cup \mathbb{N}$. Then for any vector $u_0 \in \mathfrak{U}$ and any vector function $f : (0, \infty) \to \mathfrak{F}$ such that $f^0 = (\mathbb{I} - Q)f \in C^{p+1}((0, \infty); \mathfrak{F}^0)$ and $f^1 = Qf \in C([0, \infty]; \mathfrak{F}^1)$ there exists a unique solution $u = u(t)$ to problem (1), (3), and the solution has the form

$$u(t) = -\sum_{q=0}^{p} H^q M_0^{-1}\frac{d^q f^0}{dt^q}(t) + U^t u_0 + \int_0^t U^{t-s}L_1^{-1}f^1(s)ds, \quad t \in (0, \infty). \tag{4}$$

In addition, if the initial vector u_0 satisfies the relation

$$(\mathbb{I} - Q)u_0 = -\lim_{t \to 0+}\sum_{q=0}^{p} H^q M_0^{-1}\frac{d^q f^0}{dt^q}(t), \tag{5}$$

then there exists a unique solution $u = u(t)$ to problem (1), (2), and the solution has the form (4).

Definition 4. Let the operator M be strongly (L, p)-sectorial for some $p \in \{0\} \cup \mathbb{N}$. The Banach lattices $\mathfrak{U} = (\mathfrak{U}; \|\cdot\|_{\mathfrak{U}}; \mathfrak{U}_+)$ and $\mathfrak{F} = (\mathfrak{F}; \|\cdot\|_{\mathfrak{F}}; \mathfrak{F}_+)$ are called *concordant with respect to the pair* (L, M) (briefly, *(L, M)-concordant*) if

(i) $\mathfrak{U}^k_+ = \mathfrak{U}^k \cap \mathfrak{U}_+$ and $\mathfrak{F}^k_+ = \mathfrak{F}^k \cap \mathfrak{F}_+$ are the proper generative cones, $k = 0, 1$,

(ii) the operators $L \in \mathcal{L}(\mathfrak{U}^k_+; \mathfrak{F}^k_+)$ and $M \in \mathcal{C}l(\mathfrak{U}^k_+; \mathfrak{F}^k_+)$, moreover, there exist the operators $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1_+; \mathfrak{U}^1_+)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0_+; \mathfrak{U}^0_+)$.

Corollary 1. [2] Let the operator M be strongly $(L;p)$-sectorial for some $p \in \{0\} \cup \mathbb{N}$, the Banach lattices $\mathfrak{U} = (\mathfrak{U}; \|\cdot\|_{\mathfrak{U}}; \mathfrak{U}_+)$ and $\mathfrak{F} = (\mathfrak{F}; \|\cdot\|_{\mathfrak{F}}; \mathfrak{F}_+)$ be $(L;M)$-concordant, and the degenerate holomorphic semigroup of operators $U^\bullet$ be positive. Then for any vector function $f : (0;\infty) \to \mathfrak{F}$ such that $\frac{d^q f^0}{dt^q} \in C((0;\infty);\mathfrak{F}^0_+)$, $q = \overline{1;p}$, and $f^1 = C([0;\infty];\mathfrak{F}^1_+)$, and for any vector $u_0 \in \mathfrak{U}_+$ there exists a unique positive solution to problem (1), (3), and the solution has the form (4). If, in addition, the initial vector $u_0 \in \mathfrak{U}_+$ satisfies condition (5), then there exists a unique positive solution to problem (1), (2), and the solution has the form (4).

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Operator semigroups associated with stochastic processes within the framework of semigroup and Gelfand–Shilov classifications

I. V. Melnikova,[32] U. A. Alekseeva,[33] V. A. Bovkun.[34]

Keywords: semigroup of operators, pseudo-di□erential operator, Levy process, transition probability, Levy–Khintchine formula.

MSC2010 codes: 60G51, 60J35, 46F10, 47G30

Introduction. A wide class of processes arising in various fields of natural science, economics and social phenomena can be mathematically described by stochastic differential equations (SDE). Recently, great interest in the problems of financial mathematics has led to significant advances in this area.

The most studied is the class of diffusion SDEs with Wiener processes being the randomness sources. The solutions of such equations, due to the continuity properties of Wiener processes have continuous trajectories. Therefore, modeling based on diffusion-type equations is most suitable for describing processes that do not have jumps. Simulation based on Levy and more general Levy type processes allows one to study along with continuous, jump processes.

At the same time, both in applications and in fundamental science, often what is needed is not the random process itself, defined by SDE or a set of properties, but its probabilistic characteristic. The study of the relationship between SDEs and deterministic equations for probabilistic characteristics is one of the main directions of stochastic analysis.

Main results. The talk is devoted to solution properties of equations for probabilistic characteristics, defined by stochastic Levy processes. It is shown that, in contrast to the PDEs for probabilistic characteristics determined by Wiener processes, the equations determined by Levy processes and more general Markov processes are pseudo-differential.

The semigroup technique underlies the study of Cauchy problems for the obtained pseudo-differential equations. The central place is occupied by semigroups with kernels formed by transition probabilities of Markov processes and their important subclasses — Feller and Levy processes. The kernels are considered in spaces of tempered distributions.

The connection between the semigroup classification based on the spectral properties of generators, which are generally pseudo-differential operators, and the GelfandB—Shilov classification for differential systems based on generalized Fourier transform techniques, has been shown. The embedding scheme has been constructed.

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L^p estimates for the Caffarelli-Silvestre extension operators

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Keywords: Elliptic operators, discontinuous coefficients, kernel estimates, maximal regularity.

MSC2010 codes: 47D07, 35J70

Introduction. In this talk I present some new results, obtained in [4], on solvability and regularity of elliptic and parabolic problems associated to the degenerate operators

$$L = \Delta_x + D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2} \quad \text{and} \quad D_t - L$$

in the half-space $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N, y > 0\}$ or $(0,\infty) \times \mathbb{R}^{N+1}_+$ under Dirichlet and Neumann boundary conditions at $y = 0$.

Here b, c are constant real coefficients and we use

$$L_y = D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2}.$$

Note that singularities in the lower order terms appear when either b or c is different from 0.

When $b = 0$, then L_y is a Bessel operator (I shall denote it by B_y) and both $L = \Delta_x + B_y$ and $D_t - L$ play a major role in the investigation of the fractional powers $(-\Delta_x)^s$ and $(D_t - \Delta_x)^s$, $s = (1-c)/2$, through the "extension procedure" of Caffarelli and Silvestre, see [2].

The main results concern with the classical parabolic L^p estimates

$$\|D_t v\|_p + \|Lv\|_p \le C\|(D_t - L)v\|_p$$

(L^p norms on $(0,\infty) \times \mathbb{R}^{N+1}_+$) and their elliptic counterpart

$$\|\Delta_x u\|_p + \|L_y u\|_p \le C\|Lu\|_p$$

(L^p norms on $\mathbb{R}^{N+1}_+$) which we prove.

We make an extensive use of semigroup theory and vector-valued harmonic analysis relying upon explicit bounds on the heat kernels and weighted estimates, see [1] and [3].

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Random evolution equations on graphs and beyond

D. Mugnolo[38]

We begin our talk by studying diffusion-type equations supported on combinatorial and metric graphs that are randomly varying in time. We hence follow the evolution of a system along the path of a random walk whose states are diffusion equations driven by different graph Laplacians. After settling the issue of well-posedness, we focus on the asymptotic behavior of solutions and show convergence of the propagator towards a deterministic steady state. If time allows, I will also briefly turn to a different viewpoint and follow the evolution of a system not any more along a tree-like time structure corresponding to all possible paths of the Markov chain, but rather along a time structure given by a general network. In this rather general setting we can prove well-posedness and certain qualitative properties of the solution. This talk is based on joint articles with Stefano Bonaccorsi (Trento), Francesca Cottini (Milano-Bicocca) and Amru Hussein (Kaiserslautern).

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Approximation of fractional equations in Banach spaces

S. Piskarev [39]

In this talk we have a deal with the well-posedness and approximation for nonhomogeneous fractional differential equations in Banach spaces E:

$$(\mathbf{D}_t u)(t) = Au(t) + f(t), \quad t \in [0,T], \quad u(0) = x,$$

where $\mathbf{D}_t$ is the Caputo-Dzhrbashyan derivative $0 < \alpha < 1$, the operator A generates analytic C_0-semigroup, the function $f(\cdot) : [0,T] \to E$ is smooth enough.

The same way as in [1-2] we get the necessary and sufficient condition for the coercive well-posedness of nonhomogeneous fractional Cauchy problems in the spaces $C_0([0,T];E)$, $L^p[0,T];E$. Then using implicit difference scheme and explicit difference scheme, we deal with the full discretization of the solutions of nonhomogeneous and semilinear fractional differential equations in time variables and as in [3] we get the stability of the schemes and the order of convergence.

We discuss also the full discretization of autonomous semilinear problem.

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The Bi-Laplacian with Wentzell boundary conditions on Lipschitz domains

D. Ploss[40]

Keywords: Fourth-order differential operator, Wentzell boundary condition, Lipschitz boundary, analytic semigroup, eventual positivity

MSC2010 codes: 35K35 (primary); 47A07, 47D06, 35B65 (secondary)

Problem Setting. This talk is based on a joint work (cf. [1]) with Robert Denk and Markus Kunze and investigates the Bi-Laplacian with Wentzell boundary conditions in a bounded domain $\subset \mathbb{R}^d$ with Lipschitz boundary , using form methods. The aim of our paper is to study the evolution equation for a fourth-order operator on a Lipschitz domain with Wentzell boundary conditions:

$$\partial_t u + \ (\ \ )u = 0 \text{ in } (0, \infty) \times \ ; \tag{1}$$

$$(\ \ )u + \ \partial\ (\ \ )u \ \ u = 0 \text{ on } (0, \infty) \times \ ; \tag{2}$$

$$\partial\ u = 0 \text{ on } (0, \infty) \times \ ; \tag{3}$$

$$u|_{t=0} = u_0 \text{ in } \ . \tag{4}$$

Method and results. To tackle this problem, we formally decouple the domain into interior and boundary, using the L^2-space $H = L^2(\ ) \ L^2(\ )$ and considering the following form

$$a(u, v) := \int \ \ u_1 \ \overline{\ v_1}\, dx + \int \ u_2 \overline{v_2} \ \ ^{1}\, dx$$

for

$$u, v \in D(a) := \{u \in H \mid u_1 \in D(\ _N), u_2 = \operatorname{tr} u_1\}.$$

Here, $D(\ _N)$ denotes the domain of the Neumann Laplacian, which is known to be given by

$$\{u \in H^{\frac{3}{2}}(\ ) \mid \partial\ u = 0\} = \{u \in H^{\frac{3}{2}}(\ ) \mid \ u \in L^2(\ ), \partial\ u = 0\}$$

on Lipschitz domains. The associated operator A to the form a is self-adjoint and semibounded. It has compact resolvent and can be fully charaterized as

$$A = \begin{pmatrix} (\ \ ) & 0 \\ \partial\ (\ \ ) & \end{pmatrix},$$

defined on the domain

$$D(A) = \left\{u \in H : u_1 \in H^{3/2}(\ ); \ \ u_1 \in H^{3/2}(\ ), \partial\ u_1 = 0, u_2 = \operatorname{tr} u_1\right\}.$$

Moreover, A generates a strongly continuous, real, and analytic semigroup $(T(t))_{t \ 0}$ of self-adjoint operators on H, which solves (1) – (4).

We also prove that for $u \in D(A^{1})$, we have that the component u_1 has a Hölder continuous representative and u_2 is the classical trace of u_1. Among other this allows for classical interpretation of the Wentzell boundary condition (2).

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Analytic semigroups in the quaternionic framework

V. Recupero[41]

Abstract. In the last two decades a good deal of attention has been dedicated to the development of a functional calculus for operators acting on a quaternionic Banach space. In this talk I will present the fundamental special case of the exponential function leading to the new notions of spherical sectorial operators and of slice regular semigroups, which represent the quaternionic counterpart of the analytic semigroup of the classical complex framework. This is a joint work with Riccardo Ghiloni.

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How to obtain $\exp(-itH)$ for arbitrary self-adjoint H if for each $t > 0$ you know $\exp(-tH)$ or $\exp(tH)$ or even less

I. D. Remizov[42]

Keywords: Schrödinger equation, approximation of C_0-group, Chernoff product formula, Chernoff tangency

MSC2010 codes: 81Q05, 47D08, 35C15, 35J10, 35K05

Answer to the question from the title. Consider Hilbert space F over the field $\mathbb{C}$ and (unbounded in interesting cases) densely defined self-adjoint operator H in F. The main result of the talk is my formula

$$R(t) = \exp(ia(S(t) - I)) \tag{1}$$

where a is a non-zero real number and $(S(t))_{t \geq 0}$ is a family of self-adjoint bounded linear operators that is Chernoff-tangent to H. Omitting some minor details one can say that Chernoff-tangency means that

$$S(t)f = f + tHf + o(t) \text{ as } t \to +0 \tag{2}$$

for all f from $D \subset F$ where D is a core of H. Then thanks to the Chernoff theorem [1] we have the following approximations that hold in the strong operator topology

$$\exp(aitH) = \lim_{n \to \infty} R(t/n)^n = \lim_{n \to \infty} \exp(ian(S(t/n) - I)) \tag{3}$$

where in the exponent there is a bounded linear operator $ian(S(t/n) - I)$ so exponent is well-defined by a power series $\exp(ian(S(t/n) - I)) = \sum_{k=0}^{\infty} \frac{(ian)^k}{k!}(S(t/n) - I)^k$. This approach was introduced in [2] with exact definitions, full proofs and comments.

For example, if $\exp(tH)$ exists and you have any explicit formula for it then you can set $a = -1$ and $S(t) = \exp(tH)$ will be Chernoff-tangent to H, so in the strong operator topology we obtain the limit expression

$$\exp(-itH) = \lim_{n \to \infty} \exp\left(-in\left(e^{\frac{t}{n}H} - I\right)\right). \tag{4}$$

Similarly, if $\exp(-tH)$ exists and you have any explicit formula for it then you can set $a = 1$ and $S(t) = \exp(-tH)$ will be Chernoff-tangent to $-H$, so in the strong operator topology we obtain the limit expression

$$\exp(-itH) = \lim_{n \to \infty} \exp\left(in\left(e^{-\frac{t}{n}H} - I\right)\right). \tag{5}$$

If e.g. $H = \Delta$ is the Laplacian then formula (4) means that we solve the heat equation $u_t' = \Delta u$ obtaining solution $u(t) = e^{t\Delta}u_0$, then mix this solution with the imaginary unit i via formula (1) and obtain the solution of the Schrödinger equation $i\psi_t' = \Delta\psi$ given by $\psi(t) = e^{-it\Delta}u_0$ where $e^{-it\Delta}$ is given by (4). This example is elementary but the formula (1) is universal and it works for all self-adjoint operators H, which can contain derivatives of arbitrary high order and variable coefficients.

What to do if $\exp(tH)$ and $\exp(-tH)$ are both not known. This situation is more difficult, but still formula (1) is helpful. The way is to find such $S(t)$ that $S(t)^* = S(t)$ and $S(t)f = f + tHf + o(t)$ or $S(t)f = f - tHf + o(t)$ as $t \to +0$ and apply formula (3). Construction of such $S(t)$ is much more simpler than constructing $\exp(tH)$ or $\exp(-tH)$ because S does not need to hold the composition property: it is ok when $S(t_1 + t_2) \neq S(t_1)S(t_2)$. Moreover, if $S_1(t)f = f + tH_1f + o(t)$ and $S_2(t)f = f + tH_2f + o(t)$ then $S_1(t) + S_2(t) - I$ is self-adjoint and

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Chernoff-tangent to $H_1 + H_2$. This gives a lot of freedom and allows to find representations for sulutions of rather complicated Schrödinger equations. As far as I know now (4 April 2021) this approach was used at least by V.Zh.Sakbaev, M.S.Businov, D.V.Grishin&A.V.Smirnov and can be found in the overview [1] by Ya.A.Butko. You are welcome to use it also. The only problem is that representations for $\exp(-itH)$ provided by (3) are rather lengthy.

Example 1. Solution to one-dimensional Schrödinger equation with derivatives of arbitrary high order and variable coefficients.

Theorem 1. (see in [3] with full proof) Fix arbitrary $K \in \mathbb{N}$. Suppose that for $k = 0, 1, \dots, K$ functions $a_k\colon \mathbb{R} \to \mathbb{R}$ are given. Suppose that for each $k = 1, \dots, K$ function a_k belongs to space $C_b^{2k}(\mathbb{R})$ of all bounded functions $\mathbb{R} \to \mathbb{R}$ with bounded derivatives up to $(2k)$-th order. Suppose that function $a_0\colon \mathbb{R} \to \mathbb{R}$ is measurable and belongs to space $L_2^{loc}(\mathbb{R})$, i.e. $\int_{-R}^{R} |a_0(x)|^2 dx < \infty$ for each real number $R > 0$. Define

$$(H\varphi)(x) = a_0(x)\varphi(x) + \sum_{k=1}^{K} \frac{d^k}{dx^k}\left(a_k(x)\frac{d^k}{dx^k}\varphi(x)\right)$$

for each φ from the space $C_0^\infty(\mathbb{R})$ of all functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ wich are bounded together with their derivatives of all orders and have compact support (are zero outside of some closed interval). We also use the following condition for coefficients a_k, $k = 0, 1, \dots, K$: operator H defined on $C_0^\infty(\mathbb{R})$ is essentially self-adjoint in $L_2(\mathbb{R})$, i.e. the operator $(H, C_0^\infty(\mathbb{R}))$ is closable and its closure — let us denote it as $(H, D(H))$ — is a self-adjoint operator.

Suppose that function $w\colon \mathbb{R} \to \mathbb{R}$ is continuous, bounded, differentiable at zero and $w(0) = 0$, $w'(0) = 1$ (examples include: $w(x) = \arctan(x)$, $w(x) = \sin(x)$, $w(x) = \tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$, etc). For each $t \geq 0$, $k = 1, 2, \dots, K$, each $x \in \mathbb{R}$, and each $f \in L_2(\mathbb{R})$ define:

$$(B_{a_k} f)(x) = a_k(x) f(x),$$

$$(A(t)f)(x) = f(x+t), \quad (A(-t)f)(x) = f(x-t),$$

$$F_k(t) = \left(A(t^{1/2k}) - I\right)^k B_{a_k}\left(I - A(-t^{1/2k})\right)^k, \quad F_0(t)f(x) = w(ta_0(x))f(x),$$

$$F(t) = \sum_{k=0}^{K} F_k(t), \quad S(t) = I + F(t) = I + \sum_{k=0}^{K} F_k(t),$$

where I is the identity operator ($If = f$), and expression such as Z^k means the composition $ZZ \dots Z$ of k copies of linear bounded operator Z.

THEN for each initial condition $\psi_0 \in L_2(\mathbb{R})$ the Cauchy problem

$$\begin{cases} \psi'_t(t) = -iH\psi(t), \\ \psi(0) = \psi_0, \end{cases}$$

and has a unique (in sense of $L_2(\mathbb{R})$) solution $\psi(t)$ that depends on ψ_0 continuously with respect to norm in $L_2(\mathbb{R})$, and for all $t \geq 0$ and almoust all $x \in \mathbb{R}$ can be expressed in the form

$$\psi(t, x) = \left(e^{-itH}\psi_0\right)(x) = \left(\lim_{n\to\infty} \lim_{j\to+\infty} \sum_{q=0}^{j} \frac{(-in)^q}{q!} \left(\sum_{k=0}^{K} F_k(t/n)\right)^q \psi_0\right)(x).$$

Here linear bounded operators $F_0(t), \dots, F_K(t)$ are defined above in conditions of the theorem for all $t \geq 0$ (hence $F_0(t/n), \dots, F_K(t/n)$ are defined for all $t \geq 0$ and all $n \in \mathbb{N}$), and the power q in $\left(\sum_{k=0}^{K} F_k(t/n)\right)^q$ stands for a composition of q copies of linear bounded operator $\sum_{k=0}^{K} F_k(t/n)$.

Example 2. Solution to multi-dimensional Schrödinger equation with Laplacian and measurable locally square integrable potential.

Theorem 2. (see in [3] with full proof) Suppose that function $V\colon \mathbb{R}^d \to \mathbb{R}$ belongs to the space $L_2^{loc}(\mathbb{R}^d)$, i.e. V is measurable and $\int_{\|x\|\le R} V(x)^2 dx < \infty$ for each $R > 0$, where $\|x\| = (x_1^2 + \dots + x_d^2)^{1/2}$. Suppose that $a \in \mathbb{R}, a \neq 0$. Suppose that function $w\colon \mathbb{R} \to \mathbb{R}$ is bounded, continuous, differentiable at zero and $w(0) = 0$, $w'(0) = 1$; for example, one can take $w(x) = \sin(x)$, $w(x) = \arctan(x)$, $w(x) = \tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$ etc. Suppose that for each $j = 1, \dots, d$ constant vector $e_j \in \mathbb{R}^d$ has 1 at position j and has 0 at other $d-1$ positions. For each function $f \in L_2(\mathbb{R}^d)$, each smooth function $\varphi\colon \mathbb{R}^d \to \mathbb{C}$ and each $x \in \mathbb{R}$, $t \ge 0$ define

$$(W(t)f)(x) = \frac{1}{2d}\sum_{j=1}^{d}\left[f\left(x + \sqrt{d}\sqrt{t}e_j\right) + f\left(x - \sqrt{d}\sqrt{t}e_j\right) - 2f(x)\right] + w(-tV(x))f(x); \quad (6)$$

$$(H\varphi)(x) = \frac{1}{2}\Delta\varphi(x) - V(x)\varphi(x).$$

Suppose also that at least one of these two conditions is satisfied:

A) if we use the symbol $C_0^\infty(\mathbb{R}^d)$ for the set of all infinitely smooth functions $\mathbb{R}^d \to \mathbb{R}$ with compact support then the closure of the operator $(H, C_0^\infty(\mathbb{R}^d))$ is a self-adjoint operator in $L_2(\mathbb{R}^d)$;

B) $V(x) \ge 0$ for all $x \in \mathbb{R}^d$.

Consider Cauchy problem for Schrödinger equation

$$\begin{cases} \psi_t'(t,x) = iaH\psi(t,x); & t \in \mathbb{R}^1, x \in \mathbb{R}^d;\\ \psi(0,x) = \psi_0(x); & x \in \mathbb{R}^d. \end{cases} \quad (7)$$

where the Hamiltonian is equal to $-aH$.

THEN for each $t \ge 0$ and $\psi_0 \in L_2(\mathbb{R}^d)$ Cauchy problem (7) have the unique (in $L_2(\mathbb{R}^d)$) solution $\psi(t,x) = \left(e^{iatH}\psi_0\right)(x)$, that continuously, with respect to norm in $L_2(\mathbb{R}^d)$, depends (for fixed t) on ψ_0. For almost all $x \in \mathbb{R}^d$ and all $t \ge 0$ this solution satisfies the formula

$$\psi(t,x) = \left(\lim_{n\to+\infty}\lim_{j\to+\infty}\sum_{k=0}^{j}\frac{(ian)^k}{k!}W(t/n)^k\psi_0\right)(x);$$

where $W(t/n)$ is obtained by substitution of t with t/n in (6), and $W(t/n)^k$ is a composition of k copies of linear bounded operator $W(t/n)$.

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Asymptotic decomposition of substochastic semigroups and applications

R. Rudnicki[43]

A *substochastic semigroup* is a C_0-semigroup of linear operators $\{P(t)\}_{t\geq 0}$ defined on some L^1 space such that if $f \geq 0$ then $P(t)f \geq 0$ and $\|P(t)f\| \leq \|f\|$. If additionally $\|P(t)f\| = \|f\|$ for $f \geq 0$, then $\{P(t)\}_{t\geq 0}$ is called *stochastic semigroup* or *Markov semigroup*. Substochastic semigroups have been intensively studied because they play a special role in applications. They are used to investigate the long-time behaviour of the distributions of Markov processes like diffusion processes and piecewise deterministic processes. We present some results concerning the long-time behaviour of substochastic semigroups [1,2,3]. We apply these results to study asymptotic stability of semigroups generated by piecewise deterministic Markov processes. We illustrate mathematical results by applications to gene expression models [4] and kinetic equations [5].

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Around Baillon's theorem on maximal regularity

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Keywords: maximal regularity, strongly continuous semigroup, admissible operator, uniformly continuous semigroup

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Maximal regularity of an abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{dt}}x(t) = Ax(t) + f(t), \quad t > 0, \quad x(0) = 0,$$

which refers to the property that the regularity of the inhomogeneity f is preserved by $\frac{\mathrm{d}}{\mathrm{dt}}x$ and Ax, where A generates a strongly continuous semigroup on a Banach space X, is omnipresent in the study of (parabolic) evolution equations. In contrast to regularity with respect to L^p, for $p \in (1, \infty)$, Baillon's theorem [1] states that it is rare when considered with respect to supremum norms; it only occurs when the corresponding semigroup is uniformly continuous or the geometry of the state space is sufficiently coarse, that is, X contains c_0. In this talk we show that the latter alternative can be excluded for a refined notion, which is based on an analogy to the concept of admissibility, a notion appearing in infinite-dimensional systems theory. We moreover comment on the "dual" situation of maximal regularity with respect to L^1.

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Degenerate analytic resolving groups of operators for solutions of the Barenblatt–Zheltov–Kochina equation in "noise" spaces on a Riemannian manifold

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Keywords: Sobolev type equations; Showalter – Sidorov condition; Laplace – Beltrami operator; Nelson – Gliklikh derivative; stochastic process; differential forms.

MSC2010 codes: 35S11, 58A10, 60H15

Introduction. It is known (see [1]) that the linear Barenblatt – Zheltova – Kochina equation

$$(\quad)\dot{u} = \quad u + f; \tag{1}$$

describing the dynamics of the pressure of the fluid filtered in a fractured porous medium. The coefficient corresponds to the ratio of cracks and pores in rock, and the coefficient corresponds for the visco-elastic properties of the liquid. The beginning of the investigation of equation (1) should be related to [2], where this equation was considered for the first time as a linear inhomogeneous Sobolev type equation

$$L\dot{u} = Mu + f. \tag{2}$$

Here the operators L and M are the operators and , given in some functional spaces $L, M : \mathfrak{U} \to \mathfrak{F}$. Equation (2) is equipped with the initial Showalter – Sidorov condition [3]

$$P(u(0) \quad u_0) = 0; \tag{3}$$

where the projector P is constructed with the use of operators L and M. Note that in the case of existence operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ (those linear and bounded operators), condition (3) becomes the Cauchy condition

$$u(0) = u_0. \tag{4}$$

We also note [4], where the Barenblatt–Zheltov–Kochina equation (1) is reduced to the form (2) defined in the spaces of differential forms on smooth Riemannian manifolds without boundary.

We will be interested in the stochastic interpretation of the deterministic of equation (2), namely:

$$L \overset{o}{\ } = M \quad + N \quad . \tag{5}$$

Here the operators L and M are the same as in (2), the operator $N \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$; $= (t)$ is required, and $= (t)$ is a given stochastic process with values in the Hilbert space $\mathfrak{U}$. Through $\overset{o}{\ }$ we denote the Nelson – Gliklikh derivative of the stochastic process $= (t)$ (for details see [5]).

Preliminary information. Let $(\ ; \mathcal{A}; \mathsf{P})$ be a complete probability space, $\mathbb{R}$ be a set of real numbers endowed with a Borel -algebra. Measurable mapping $: \ \to \mathbb{R}$ is called *a random variable*. A set of random variables with a zero mathematical expectation ($\mathbf{E}$) and finite variance ($\mathbf{D}$) forms a Hilbert space with scalar product $(_1, _2) = \mathbf{E} \ _1 \ _2$. The resulting Hilbert space is denoted by symbol $\mathbf{L_2}$.

Let $\in \mathbf{L_2}$ then is called *the conditional mathematical expectation* of a random variable and denoted by the symbol $\mathbf{E}(\ |\mathcal{A}_0)$. Recall also that the minimal -subalgebra $\mathcal{A}_0 \quad \mathcal{A}$,

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with respect to which the random variable is measurable, is called the *-algebra generated by* . Let $I \subset \mathbb{R}$ be a certain interval. Consider two mappings: $f : I \to \mathbf{L_2}$, which puts to each $t \in I$ a random variable $\in \mathbf{L_2}$, and $g : L_2 \to \mathbb{R}$, which puts to each pair $(; \omega)$ the point $(\omega) \in \mathbb{R}$. The map $: I \to \mathbb{R}$, having the form $= (f(t); \omega)$, is called *a stochastic process*. The stochastic process is called *continuous*, if a.s. (almost surely) all its trajectories are continuous (for almost all $\omega \in$ the trajectories $(; \omega)$ are continuous). By the symbol $\mathbf{CL_2}$ we denote the set of the continuous stochastic processes. Let's call *Gaussian* continuous stochastic process the process, if its (independent) random variables are Gaussian.

We use derivative was founded by E. Nelson, and the theory of such derivative was developed by Yu.E. Gliklikh (see [5]), then further, for brevity, the derivative of a stochastic process will be called the *Nelson–Gliklikh derivative* and denote by $\overset{o}{}$.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, the operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. Following [2] we introduce the L-resolvent set $^L(M) = \{ \in \mathbb{C} : (L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and the L-spectrum $^L(M) = \mathbb{C} \setminus {}^L(M)$ of the operator M. If the L-spectrum $^L(M)$ of the operator M is bounded, then the operator M is said to be $(L;)$-bounded. If the operator M is $(L;)$-bounded, then there exist projectors

$$P = \frac{1}{2 i} \int R^L(M) d \in \mathcal{L}(\mathfrak{U}); \quad Q = \frac{1}{2 i} \int L^L(M) d \in \mathcal{L}(\mathfrak{F}). \tag{6}$$

where expression $R^L(M) = (L - M)^{-1} L$ is the *right*, and $L^L(M) = L(L - M)^{-1}$ is the *left L-resolvent* of the operator M, and the closed contour $\subset \mathbb{C}$ bounds a domain containing $^L(M).$ We set $\mathfrak{U}^0(\mathfrak{U}^1) = \ker P(\operatorname{im} P)$, $\mathfrak{F}^0(\mathfrak{F}^1) = \ker Q(\operatorname{im} Q)$ and denote by $L_k(M_k)$ the restriction of the operator $L(M)$ on $\mathfrak{U}^k$, $k = 0, 1$.

Theorem 1. [2] Let the operator M be $(L;)$-bounded, then
(i) the operators $L_k(M_k) \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;
(ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$.

We construct the operators $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1} M_1 \in \mathcal{L}(\mathfrak{U}^1).$

Corollary 1. [2] Suppose that the operator M is $(L;)$-bounded, then for all $\in {}^L(M)$

$$(L - M)^{-1} = -\sum_{k=0}^{\infty} {}^k H^k M_0^{-1}(\mathbb{I} - Q) + \sum_{k=1}^{\infty} {}^k S^{k-1} L_1^{-1} Q. \tag{7}$$

The operator M is called *$(L; p)$-bounded*, $p \in \{0\} \cup \mathbb{N}$, if ∞ is a removable singular point (that is, $H \equiv \mathbb{O}, p = 0$) or a pole of order $p \in \mathbb{N}$ (that is $H^p \not\equiv \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of the L-*resolvent* $(L - M)^{-1}$ of the operator $M.$

We consider a n-dimensional smooth compact oriented connected Riemannian manifold without boundary and the space of differential q-forms on we denote by $E^q = E^q( ), 0 \le q \le n$. In particular $E^0(\mathbb{R}^n)$ is the space of functions of n variables. Note that there exists a linear Hodge operator $: E^q \to E^{n-q}$, which associates the q-form with $(n - q)$-form. In the double application of the Hodge operator, the equality $= (-1)^{q(n-q)}$ holds. In addition, there is an operator for taking the external differential $d : E^q \to E^{q+1}$. We define the operator $: E^q \to E^{q-1}$, setting $= (-1)^{n(q+1)+1} d .$ The Laplace-Beltrami operator $: E^q \to E^q$ is defined by the equality $= d + d ;$ and it is a linear operator on space $E^q, 0 \le q \le n$. We introduce the space of harmonic q-form $H^q = \{\omega \in E^q : \omega = 0\}.$

Theorem 2. [4] (Hodge's splitting theorem) For any integer $q, 0 \le q \le n$, space H^q is finite-dimensional and there is the following decomposition of the space of smooth q-forms on into an orthogonal direct sum $E^q = (E^q) \oplus H^q = d (E^q) \oplus d(E^q) \oplus H^q.$

By the formula $(,)_0 = \int_n \wedge ,\ , \in E^q$ where is the Hodge operator, we define a scalar product in the space E^q, $q = 0, 1, \dots, n$, and denote the corresponding norm by $\| \ \|_0$.

We continue this scalar product to a direct sum $\sum_{q=0}^{n} E^q$, requiring that different spaces E^q were orthogonal. Completion of space E^q in the norm $|| \cdot ||_0$ we denote by H_0^q. We denote by P_q the projector on H^q :

We similarly define H_1^q and H_2^q respectively. Actually upper index means how many times differentiable in the generalized sense of the q-form in the corresponding spaces.

The spaces $H_l^q, l = 1, 2$ are Hilbert spaces, and we have continuous and dense embedding $H_2^q \quad H_1^q \quad H_0^q$.

Corollary 2. For any $q = 0, 1, \dots, n$ there are splitting spaces

$$H_l^q = H_l^{q1} \quad H^q ;$$

where $H_l^{q1} = (I \quad P \quad)[H_l^q]; \ l = 0, 1, 2.$

Main result. Let consider $U = \mathbf{UL_2H_2^q}$ $; F = \mathbf{FL_2H_0^q}$ be a real separable Hilbert space by differential q-form with stochastic coefficient defined on a smooth compact oriented Riemannian manyfold without boundary two and zero times derivative respectively. Our operator $M =$ be $(L, 0)$-bounded$(L = \quad)$.

Consider a linear stochastic equation of Sobolev type

$$L \overset{o}{\ } = M \quad + N \quad . \tag{8}$$

We supplement equation (8) with the initial Showalter-Sidorov condition

$$\left[R^L(M)\right]^{p+1} (\quad (0) \quad {}_0) = 0, \tag{9}$$

where

$$_0 = \sum_{k=1}^{\infty} \ _k \ _k \varphi_k, \tag{10}$$

$\{\varphi_k\}$ is an orthonormal basis of the space U, and pairwise independent random Gaussian variables $_k \in \mathbf{L}_2$ are such that $D \ _k \quad C_0$, and $\{ \ _k\}$ is the spectrum of some nuclear operator $(\mathbf{K} = \{ \ _\mathbf{k}\}, \sum_{k=0}^{\infty} \ _k^2 < \infty)$.

Theorem 3. For any $\in \mathbb{R} \setminus \{0\}, \quad \in \mathbb{R} \setminus \{0\}$ and any operator $N \in L(F)$ and $_0 \in \mathbf{L}_2$, that does not depend on and satisfies (10) exists a unique classical solution $= (t)$ of problem (8), (9).

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The Kato square root problem for some classes of elliptic functional differential operators with smooth coefficients

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Keywords: the Kato square root problem, elliptic functional differential operators with degeneration

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In 1961, T. Kato has formulated the following problem: "Is it true that the domain of square root from regular accretive operator is equal to the domain of square root from adjoint operator?" Sufficient conditions for fulfilment of the Kato conjecture were studied by T. Kato, J. Lions and others. J. Lions has proved that strongly elliptic differential operators of 2m order with smooth coefficients and homogeneous Dirichlet conditions on a smooth boundary satisfy the Kato conjecture. For strongly elliptic differential operators of 2m order with measurable bounded coefficients corresponding result was obtained by P. Auscher, S. Hofman, A. McIntosh, and P. Tchamitchian. In this lecture we consider two types of nonlocal elliptic operators: strongly elliptic functional differential operators and elliptic differential difference operators with degeneration. We prove that these operators satisfy the Kato square root conjecture, see [1, 2].

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Investigation of positive solutions to the Sobolev-type equations in sequence spaces

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Keywords: positive solutions; positive degenerate semigroups of operators; linear Sobolev type equations; Sobolev spaces of sequences.

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Introduction. We investigate positive solutions to the Cauchy and Showalter–Sidorov problem based on the properties of Banach lattices in sequence spaces, the theory of degenerate holomorphic groups of operators [1] and the theory of the positive semigroups of operators [2]. When studying positive solutions, we rely on the theory of positive degenerate holomorphic groups of operators generated by linear and continuous operators L and M, besides the operator M is (L, p)-bounded. Using these groups, we obtain positive solutions to linear homogeneous and inhomogeneous Sobolev type equations. As an example, we consider positive degenerate holomorphic groups in Sobolev spaces of sequences. Our aim is to find the necessary and sufficient conditions for the positivity of such groups. In the future, we intend to adapt the theory of positive degenerate groups for non-classical models of equations of mathematical physics [3], including the Barenblatt–Zheltov–Kochina model with a multipoint initial-final condition.

Main result. Let us consider Banach lattice $\mathfrak{B} = (\mathfrak{B}; C)$. Here C is a proper generating cone, and note that C may not coincide with *the canonical cone* $\mathfrak{B}_+$. The operator $A \in \mathcal{L}(\mathfrak{B})$ is called *positive* if $Ax \geq 0$ for any $x \in C$. The group of operators $V^\bullet = \{V^t : t \in \mathbb{R}\}$, acting on the space $\mathfrak{B}$, is called *positive*, if $V^t x \geq 0$ for any $x \in C$ and $t \in \mathbb{R}$. If $V^\bullet$ is a degenerate group, then its unit V^0 is a projector that splits the space $\mathfrak{B}$ into a direct sum $\mathfrak{B} = \mathfrak{B}^0 \oplus \mathfrak{B}^1$, where $\mathfrak{B}^0 = \ker V^0$ and $\mathfrak{B}^1 = \operatorname{im} V^0$. Since $V^t = V^0 V^t V^0$, then $\mathfrak{B}^0 = \ker V^t$, and $\mathfrak{B}^1 = \operatorname{im} V^t$. Hence, we can define $\ker V^\bullet = \mathfrak{B}^0$ and $\operatorname{im} V^\bullet = \mathfrak{B}^1$. If the degenerate group $V^\bullet$ is, in addition, positive, then $\mathfrak{B}^1$ is Banach lattice with proper generating cone $C^1 = \{x \in C : V^0 x = x\} = \mathfrak{B}^1 \cap C$. If it turns out that the space $\mathfrak{B}^0$ is also Banach lattice with a generating cone C^0 and an order relation $\leq_0$, then the cone $C_\bullet = C^0 \oplus C^1$ can generate a new Banach lattice of the space $\mathfrak{B}$ with an order relation $\leq_\bullet$, that is

$$(x \leq_\bullet y) \leftrightarrow (x_0 \leq_0 y_0) \wedge (x_1 \leq_1 y_1).$$

The main goal of the discussion is to indicate the conditions for the positivity of degenerate holomorphic groups $U^\bullet = \{U^t : t \in \mathbb{R}\}$ and $F^\bullet = \{F^t : t \in \mathbb{R}\}$. Further we assume that the Banach space $\mathfrak{U}(\mathfrak{F})$ is a Banach lattice $\mathfrak{U} = (\mathfrak{U}; C_{\mathfrak{U}})$ $(\mathfrak{F} = (\mathfrak{F}; C_{\mathfrak{F}}))$, where $C_{\mathfrak{U}}$ $(C_{\mathfrak{F}})$ is the proper generating cone.

Moreover, as Banach spaces $\mathfrak{U}$ and $\mathfrak{F}$ we take the Sobolev spaces of sequences

$$l_q^m = \left\{u = \{u_k\} : \sum_{k=0}^{\infty} \lambda_k^{\frac{mq}{2}} |u_k|^q < \infty\right\}, \; m \in \mathbb{R}, \; q \in [1, +\infty).$$

Theorem 1. Let the operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the following statements are equivalent.

(i) $(\mu R^L_\mu(M))^{p+1}$ $((\mu L^L_\mu(M))^{p+1})$ is positive for all sufficiently large $\mu \in \mathbb{R}_+$.

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(ii) Degenerate holomorphic group U $(F$ $)$ is positive.

Let $\mathfrak{U} = (\mathfrak{U}; C_U)$ and $\mathfrak{F} = (\mathfrak{F}; C_F)$ be Banach lattices, where C_U and C_F are own generating cones. Let $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and $M \in Cl(\mathfrak{U}; \mathfrak{F})$ be operators. Consider the linear homogeneous Sobolev type equation

$$L\dot{u} = Mu. \quad (1)$$

A vector function $u \in C^1(\mathbb{R}; \mathfrak{U})$, satisfying this equation is called *a solution to equation* (1). A solution $u = u(t)$ is called *a solution to the Cauchy problem*, if it satisfies the condition

$$u(0) = u_0. \quad (2)$$

for some $u_0 \in \mathfrak{U}$. A solution $u = u(t)$ is called *a solution to the Showalter B– Sidorov problem* [4], if it satisfies the condition

$$P(u(0) \quad u_0) = 0. \quad (3)$$

The vector function $u(t) = U^t u_0$ is solution to equation (1) for any $u_0 \in \mathfrak{U}$, and it is also a solution to problem (3) also for any $u_0 \in \mathfrak{U}$. Now we arrive at the question about the existence and uniqueness of the solution to problem (1), (2) and the question about the uniqueness of the solution to problem (1), (3).

Here $\{U^t : t \in \mathbb{R}\}$ is a degenerate holomorphic group of operators of the form

$$U^t = \frac{1}{2\ i} \int R^L(M) e^{\ t} d\ , \quad t \in \mathbb{R},$$

$$F^t = \frac{1}{2\ i} \int L^L(M) e^{\ t} d\ , \quad t \in \mathbb{R},$$

defined on the spaces $\mathfrak{U}$ and $\mathfrak{F}$ respectively. Here the contour $\ \in \mathbb{C}$, $\ = \{\ \in \mathbb{C} : |\ | = r > a\}$ $R^L(M) = (\ L \quad M)^{\ 1} L$ is *the right L-resolvent of the operator M* and $L^L(M) = L(\ L \quad M)^{\ 1}$ is *the left one*.

To solve the question about the existence and the uniqueness of solutions, recall that a set $\mathfrak{B} \quad \mathfrak{U}$ is called the phase space of equation (1) if any of its solutions $u(t) \in \mathfrak{B}$ for each $t \in \mathbb{R}$; and for any $u_0 \in \mathfrak{B}$ there exsists a unique solution $u \in C^1(\mathbb{R}; \mathfrak{U})$ to problem (2) for equation (1). The following theorem is true.

Theorem 2. Let the operator M be $(L; p)$-bounded, $p \in \{0\} \cup \mathbb{N}$. Then
(i) the phase space of equation (1) is the subspace $\mathfrak{U}^1$;
(ii) for any $u_0 \in \mathfrak{U}$ there exists a unique solution $u = u(t)$ to problem (1), (3) of the form $u(t) = U^t u_0.$

By virtue of statement (i) of this theorem, any solution to equation (1) belongs to the space $\mathfrak{U}^1$ pointwise, that is $u(t) \in \mathfrak{U}^1$ for all $t \in \mathbb{R}$. This means that if we represent an arbitrary initial vector u_0 in the form $u_0 = u_0^0 + u_0^1$, where $u_0^k \in \mathfrak{U}^k, k = 0, 1$ (according to the splitting theorem [1]), then the solution $u(t) = U^t u_0$ to problem (1), (3) is also the unique solution to this problem under the initial condition $v_0 = v_0^0 + u_0^1$, where the vector $v_0^0 \in \mathfrak{U}^0$ is arbitrary.

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Exponential Stability for Port-Hamiltonian Systems

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Keywords: Exponential Stability, Port-Hamiltonian Systems.

We consider port-Hamiltonian Systems; i.e., Cauchy problems of the form

$$\partial_t u + P_1 \partial_x H u + P_0 H u = 0, \quad u(0) = u_0,$$

where $P_0 = -P_0^\top \in \mathbb{R}^{n\times n}$, $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$ and $H \in L_\infty([a,b];\mathbb{R}^{n\times n})$ such that there exists $c > 0$ with

$$cI_{n\times n} \leq H(x) = H(x)^\top \quad (x \in [a,b] \text{ a.e.}).$$

In order to solve the above problem, one is interested in m-accretive realisations of the spatial operator $P_1\partial_x H + P_0 H$ in the weighted L_2-space

$$L_{2,H}([a,b])^n = (L_2([a,b])^n, \langle \cdot, H\cdot\rangle_{L_2([a,b])^n})$$

by imposing suitable boundary conditions; that is, one considers restrictions $A \subseteq P_1\partial_x H + P_0 H$ with domain

$$\operatorname{dom}(A) = \left\{ v \in L_2([a,b])^n \,;\, Hv \in H^1([a,b])^n, W_B \begin{pmatrix} (Hv)(b) \\ (Hv)(a) \end{pmatrix} = 0 \right\}$$

where $W_B \in \mathbb{R}^{N\times 2N}$ is a suitable matrix. Such m-accretive realisations are well-studied and can be characterised by properties of the matrix W_B, see e.g. [1, Theorem 1].
Assuming now the m-accretivity of a restriction A, the question arises whether the associated C_0-semigroup $(e^{-tA})_{t\geq 0}$ is exponentially stable. The latest theorem in this direction seems to be the following:

Theorem ([3, Theorem 3.5]) If there exists $c \in \{a,b\}$ and $\kappa > 0$ such that

$$\operatorname{Re}\langle v, Av\rangle_{L_{2,H}([a,b])^n} \geq \kappa \|(Hv)(c)\|^2 \quad (v \in \operatorname{dom}(A))$$

and H is of bounded variation, then $-A$ generates an exponentially stable C_0-semigroup.

Similar results were known before under stronger regularity assumptions (e.g. Lipschitz-continuity) on the mapping H. The reason for this additional regularity lies in the method of proof for these result, which is based on a-priori estimates for solutions of Port-Hamiltonian systems in terms of their boundary values, see e.g. [2, Lemma 9.1.2].

We will show that we can remove this additional regularity assumption for H and prove an analogous result for $H \in L_\infty([a,b])^n$. In fact, we will provide a characterisation result for exponential stability in terms of the matrices P_0, P_1, W_B and the function H.

The talk is based on an ongoing project together with Rainer Picard (TU Dresden), Marcus Waurick (TU Freiberg) and Bruce Watson (Univ. of Witwatersrand).

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Extinction time of stochastic SIRS epidemic models: application of Chernoff approximation for bi-continuous semigroups

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Keywords: stochastic SIRS epidemic model; degenerate diffusion; phase transition; Chernoff product formula; bi-continuous semigroup.

MSC2010 codes: 47D06, 60J27

Introduction. The relationship between strongly continuous semigroups on Banach spaces and the solutions of initial value problems for partial differential equations is well established. However, very often we encounter PDEs whose corresponding semigroups are not strongly continuous. One way to treat such cases is by introducing a weaker topology to the underlying Banach space, for which to achieve that the corresponding semigroup is strongly continuous. Proposed by [4], the concept of bi-continuous semigroups describes such treatments in a general framework, with analogous generation theorems and approximation formulas.

In the area of epidemiology and population genetics, the limits of many multi-dimensional discrete stochastic models turn out to be degenerate. In this work, we focus on a particular degenerate diffusion, occurring as the limit of stochastic epidemic SIRS models, when the population size tends to infinity. In solving the hitting time problem for this diffusion, we encounter the problem of a lack of strong continuity as described above, and subsequently prove that the corresponding degenerate operator generates a bi-continuous semigroup, using a generalised Chernoff product formula [3]. Since the corresponding second-order operator is associated with the squared Bessel process and the corresponding semigroup can be computed explicitly, we obtain an explicit Chernoff equivalent to the semigroup.

Motivation. Stochastic compartmental models are widely used in modelling the spread of epidemic diseases, such as SIS, SIR and SIRS models. We consider the stochastic SIRS model, a two-dimensional continuous-time Markov chain which represents diseases with no incubation period and temporary immunity in a closed, homogeneously mixing population of size N. A stochastic SIRS model is said to be "at criticality" when its basic reproduction number R_0 is 1. It is known that SIS and SIR models exhibit 'critical behaviours' not only at criticality, but also when the parameter converges to the criticality as $N \to \infty$. Studying such "near-critical" behaviours is regarded as one of the significant challenges for stochastic epidemic modelling, since many diseases, especially those under an eradication campaign, are near-critical [1].

In this work, we focus on the critical behaviour of the extinction time, i.e., the time taken for a population to reach zero infections. The study of the extinction time of stochastic epidemic models has drawn wide interests from both epidemiological and mathematical point of view.

We identify the critical parameter domain, where a phase transition between rapid extinction and exponential-time prevalence of the epidemic can be observed. We then apply a suitable scaling in both time and space such that the scaled stochastic SIRS model converges to a particular limit diffusion. These findings are analogous to what has been proven for stochastic SIS/ SIR models [2].

Little is known about the distribution of the extinction time of critical stochastic epidemic models in general. We contribute to the existing knowledge by expressing the tail distribution of the SIRS extinction time using bi-continuous semigroups, which allows for future exploration of its qualitative properties. The relatively simple form of the iterative approximation scheme also provides the possibility of numerical analysis.

Model setting. For a closed population of size N, the stochastic SIRS model is defined as a two-dimensional continuous-time Markov chain $(I_t^N, R_t^N)_{t \geq 0}$ with parameter space

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$(\ _o(N),\ _o(N))$, where I_t^N represents the size of the infected population at time t, and R_t^N represents the size of the immune population at time t. The model is associated with the transition rates:

$$\begin{aligned} (i,r) &\to (i+1,r), && \text{at rate } \ _o(N - i - r)i/N, \\ (i,r) &\to (i,r-1), && \text{at rate } \ _o r, \\ (i,r) &\to (i-1,r+1), && \text{at rate } i. \end{aligned}$$

In other words, each susceptible individual is expected to contract the disease at rate $\ _o I_t^N/N$. Once infected, each individual is immediately infectious and will recover at rate $\ _o = 1$ independently of other individuals. Each recovered individual loses immunity at rate $\ _o$ and becomes susceptible independently. The basic reproduction number is $R_0 = \ _o/\ _o$. The extinction time is defined as $T_o^N := \inf\{t : I_t^N = 0\}$.

Method and main result. We first identify the critical scaling through a heuristic argument as

$$Y_t^N := \frac{I^N_{N^{1/3}t}}{N^{1/3}}, \quad Z_t^N := \frac{R^N_{N^{1/3}t}}{N^{2/3}},$$

and the parameters are assumed to satisfy $(1 - \ _o(N))N^{1/3} \to \hat{} \in \mathbb{R}$, $\ _o(N)N^{1/3} \to \ 0$. We denote the scaled extinction time as $T^N := \inf\{t : Y_t^N = 0\}$.

Secondly, we prove that $(Y_t^N, Z_t^N)_{t \geq 0}$ has a degenerate limit diffusion as $N \to \infty$:

$$\begin{aligned} dY &= -(\hat{} + Z)Y\,ds + \sqrt{2Y}\,dW, \\ dZ &= (Y - Z)ds, \end{aligned}$$

and the scaled extinction time T^N weakly converges to $T := \inf\{t : Y_t = 0\}$.

Lastly, we obtain the asymptotic distribution of T^N. Denote

$$L := u\frac{\partial^2}{\partial u^2} \text{ and } H := -(\hat{} + v)u\frac{\partial}{\partial u} + (u - v)\frac{\partial}{\partial v}.$$

For any $t_0 > 0$, the tail distribution of T, $\mathrm{P}[T > t_0 - t | (Y_0, Z_0) = (u, v)]$, can be expressed as the solution of a PDE

$$\frac{\partial U}{\partial t} = -LU - HU,$$

with the end condition $U(u, v, t_0) = \mathbf{1}_{\mathbb{R}^2_+}(u, v)$, and the boundary condition

$$\lim_{u \downarrow 0} U(u, v, t) = 0, \ t \in [0, t_0),$$

where $\mathbf{1}_{\mathbb{R}^2_+} A$ denotes the indicator function of set A.

We prove by the generalised Chernoff product formula that the bi-closure of $(L + H, C_c^\infty(\mathbb{R}^2_+))$ generates a bi-continuous semigroup on the Banach space of bounded continuous functions with continuous extensions to $[0, \infty)^2$, equipped with the uniform norm. The Chernoff equivalent is constructed by combining the known expression of the squared-Bessel semigroup generated by the closure of L and the shift operator generated by H.

Explicitly,

$$\mathrm{P}[T > t | (Y_0, Z_0) = (u, v)] = \lim_{n \to \infty} \left(V\left(\frac{t}{n}\right)\right)^n \mathbf{1}_{\mathbb{R}^2_+}(u, v),$$

where

$$V(t)f(u, v) := \int_0^\infty g(t; ue^{-(\hat{} + v)t}; m) f(m; ve^{-t} + ut)\,dm,$$

$$g(t; u; m) := \frac{1}{t} u^{1/2} m^{-1/2} e^{-(u+m)/t} I_1 \left(\frac{2m^{1/2} u^{1/2}}{t} \right);$$

and $I_1(\cdot)$ denotes the modified Bessel function of the first kind of index 1.

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2. Nonlinear flows and semiflows

Quadratic Lyapunov functionals and geometry of inertial manifolds

M. M. Anikushin[55]

Keywords: inertial manifolds; quadratic functionals; frequency theorem; parabolic equations; delay equations.

MSC2010 codes: 35B42, 37L25, 37L45, 37L15

Introduction. We propose a geometric theory [2] based on integral geometrical conditions (i. e. the geometrical conditions that do not appeal explicitly to infinitesimal generators), which allow to unify and generalize some of known results concerned with inertial manifolds and adjacent problems including the pioneering work of C. Foias, G.R. Sell and R. Temam [6] for semilinear parabolic equations; works of Yu.A. Ryabov, R.D. Driver and C. Chicone [5] for delay equations in $\mathbb{R}^n$ with small delays; R.A. Smith's frequency-domain developments on the Poincaré-Bendixson theory and inertial manifolds for reaction-diffusion equations and delay equations in $\mathbb{R}^n$ [10,11]. Our theory is based on the use of quadratic Lyapunov functionals, which can be constructed in applications with the aid of various versions of the Frequency Theorem [8], especially those recently obtained by the present author [3,4]. This also allows to compare the mentioned results with the Spatial Averaging Principle proposed by J. Mallet-Paret and G. R. Sell [9], where the essential role of quadratic functionals was established by A. Kostianko and S. Zelik (see the review of A. Kostianko et al. [7]). In our talk we will discuss some nuances concerned with the exponential tracking, differentiability and normal hyperbolicity of these manifolds.

Main result. Let $\mathcal{Q}$ be a complete metric space with a given flow $\vartheta^t \colon \mathcal{Q} \to \mathcal{Q}$, $t \in \mathbb{R}$. A *cocycle* in a real Banach space $\mathbb{E}$ is a family of maps $\psi^t(q, \cdot) \colon \mathbb{E} \to \mathbb{E}$, where $t \geq 0$ and $q \in \mathcal{Q}$, satisfying the following conditions:

1. $\psi^0(q, v) = v$ for every $v \in \mathbb{E}, q \in \mathcal{Q}$.

2. $\psi^{t+s}(q, v) = \psi^t(\vartheta^s(q), \psi^s(q, v))$ for all $v \in \mathbb{E}, q \in \mathcal{Q}$ and $t, s \geq 0$.

3. The map $\mathbb{R}_+ \times \mathcal{Q} \times \mathbb{E} \to \mathbb{E}$ defined as $(t, q, v) \mapsto \psi^t(q, v)$ is continuous.

For each cocycle there is an associated skew-product semiflow π on $\mathcal{Q} \times \mathbb{E}$ given by $\pi^t(q, v) = (\vartheta^t(q), \psi^t(q, v))$ for all $t \geq 0$, $q \in \mathcal{Q}$ and $v \in \mathbb{E}$.

By $\langle v, f \rangle := f(v)$ we denote the dual pairing between $v \in \mathbb{E}$ and $f \in \mathbb{E}^*$. We say that a bounded linear operator $P \in \mathcal{L}(\mathbb{E}; \mathbb{E}^*)$ is *symmetric* if $\langle v_1, P v_2 \rangle = \langle v_2, P v_1 \rangle$ for all $v_1, v_2 \in \mathbb{E}$. For a subspace $\mathbb{L} \subset \mathbb{E}$ we say that P is *positive* (resp. *negative*) on $\mathbb{L}$ if $\langle v, P v \rangle > 0$ (resp. $\langle v, P v \rangle < 0$) for all $v \in \mathbb{L}$.

Let $\mathbb{E}$ be continuously embedded into some Hilbert space $\mathbb{H}$. We do not distinguish between the elements of $\mathbb{E}$ and $\mathbb{H}$ under the embedding. We study the cocycle under the following conditions:

(H1) There is a symmetric bounded linear operator $P \in \mathcal{L}(\mathbb{E}; \mathbb{E}^*)$ and $\mathbb{E}$ splits into the direct sum $\mathbb{E} = \mathbb{E}^+ \oplus \mathbb{E}^-$ such that P is positive on $\mathbb{E}^+$ and negative on $\mathbb{E}^-$.

(H2) We have $\dim \mathbb{E}^- = j < \infty$.

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(H3) For $V(v) := \langle v, Pv \rangle$ and some numbers > 0, $_P \ 0$ and > 0 we have

$$e^{2 \ r}V(\ ^r(q;v_1) \ \ ^r(q;v_2)) \ \ e^{2 \ l}V(\ ^l(q;v_1) \ \ ^l(q;v_2)) \int_l^r e^{2 \ s}|\ ^s(q;v_1) \ \ ^s(q;v_2)|^2_{\mathbb{H}} ds; \quad (1)$$

for every $v_1, v_2 \in \mathbb{E}$, $q \in Q$ and $0 \ \ l \ \ r$ with $r \ \ l \ \ _P$.

It should be noted that under (H1) and (H2) the space $\mathbb{E}^+$ can be always chosen such that $V(v) = V(v^+) + V(v\ )$, where $v = v^+ + v$ is the unique decomposition with $v^+ \in \mathbb{E}^+$ and $v \ \in \mathbb{E}$. We consider the corresponding (to the decomposition) *V-orthogonal* projector $: \mathbb{E} \to \mathbb{E}$ defined by $v := v$.

An essential part of our construction of inertial manifolds is the following compactness property.

(UCOM) There exists $_{ucom} > 0$ such that the set $^{ucom}(C; B)$ is precompact (in $\mathbb{E}$) for every precompact $C \ \ Q$ and bounded $B \ \ \mathbb{E}$.

In applications this property follows from the following uniform Lipschitz property and smoothing properties of delay or parabolic equations in bounded domains.

(ULIP) There exists $_S \ \ 0$ such that for any $T > 0$ there exist constants $C'_T > 0$ and $C''_T > 0$ such that for all $q \in Q$ and $v_1, v_2 \in \mathbb{E}$ we have

$$\| \ ^t(q;v_1) \ \ ^t(q;v_2)\|_{\mathbb{E}} \ \ C'_T|v_1 \ \ v_2|_{\mathbb{H}} \text{ for all } t \in [t_S, t_S + T] \quad (2)$$

and also

$$\| \ ^t(q;v_1) \ \ ^t(q;v_2)\|_{\mathbb{E}} \ \ C''_T\|v_1 \ \ v_2\|_{\mathbb{E}} \text{ for all } t \in [0, T]. \quad (3)$$

The ideas behind our construction of inertial manifolds are inspired by R.A. Smith's paper on non-autonomous ODEs [10].

A continuous function $v(\cdot): \mathbb{R} \to \mathbb{E}$ is a *complete trajectory* of the cocycle if there is $q \in Q$ such that $v(t+s) = \ ^t(\vartheta^s(q); v(s))$ for all $t \ \ 0$ and $s \in \mathbb{R}$. In this case we also say that $v(\cdot)$ is *passing through $v(0)$ at q*. Under (H3) a complete trajectory $v(\cdot)$ is called *amenable* if

$$\int_{-\infty}^0 e^{2 \ s}|v(s)|^2_{\mathbb{H}} ds < +\infty. \quad (4)$$

For any $q \in Q$ let $\mathfrak{A}(q)$ be the set of all $v \in \mathbb{E}$ such that there exists an amenable trajectory passing through v at q.

We also make use of the following assumption.

(BA) For any $q \in Q$ there is a bounded in the past complete trajectory $w_q(\cdot)$ at q and there exists a constant $M_b > 0$ such that $\sup_{t \ 0}\|w_q(t)\|_{\mathbb{E}} \ \ M_b$ for all $q \in Q$.

For $q \in Q$ let $_q$ denote the restriction of the V-orthogonal projector $: \mathbb{E} \to \mathbb{E}$ to $\mathfrak{A}(q)$. We have the following theorem as a corollary of results from [2] (a part of these results for the case $\mathbb{E} = \mathbb{H}$ is given in [1]).

Theorem 1. Let (H1), (H2), (H3), (UCOM), (ULIP) and (BA) be satisfied. Then $_q: \mathfrak{A}(q) \to \mathbb{E}$ is a bi-Lipschitz homeomorphism for any $q \in Q$. Moreover,

1. (Continuous dependence of amenable fibres) $\mathfrak{A} = \bigcup_{q \in Q}\{q\} \ \ \mathfrak{A}(q)$ is a bundle over Q, which is homeomorphic to $Q \ \ \mathbb{E}$ with the bundle homeomorphism given by $\mathfrak{A} \ni (q, v) \mapsto (q, \ _q v) \in Q \ \ \mathbb{E}$.

2. (Invertibility) The skew-product semiflow restricted to $\mathfrak{A}$ is a flow.
3. (Exponential tracking) There is a constant $M_{tr} > 0$ such that for any $q \in \mathcal{Q}$ and $v_0 \in \mathbb{E}$ there is a unique $v_0 \in \mathfrak{A}(q)$ such that

$$\| {}^{t}(q; v_0) \quad {}^{t}(q; v_0)\|_{\mathbb{E}} \quad M_{tr} e^{\quad t} \operatorname{dist}(v_0, \mathfrak{A}(q)) \text{ for all } t \quad 0. \tag{5}$$

Moreover, we have

$$\int_0^{+\infty} e^{2 \quad s} | {}^{t}(q; v_0) \quad {}^{t}(q; v_0)|^2_{\mathbb{H}} ds < +\infty. \tag{6}$$

Discussions. In applications, constructions of the operator P, which satisfies **(H1)**, **(H2)** and **(H3)**, can be done via the Frequency Theorem [3,4]. This theorem reduces the problem to the verification of the so-called ***frequency inequality***, which usually has the form of an inequality between the norm of a modified resolvent on some vertical line and the Lipschitz constant of the nonlinearity. In concrete situations this inequality takes the form of some well-known conditions. Namely, relationship between the Spectral Gap Condition and R.A. Smith’s frequency inequality was noted by the author in [3]. For the relation concerned with Yu.A. Ryabov, R.D. Driver and C. Chicone results we refer to our work [4].

In [2] it is considered a more general case, where the operator P depends on $q \in \mathcal{Q}$. Such operators can be constructed via the Frequency theorem for non-stationary problems, which are still awaiting developments for infinite-dimensional problems.

From the point of view concerned with **(H3)**, the Spatial Averaging Principle leads to a more general assumption, where the exponent depends on t and v_1, v_2. In this case, general ideas behind the construction of inertial manifolds used in Theorem 1 remain the same, but some obstacles arise in the study of normal hyperbolicity and corresponding bundles (see [2]). Namely, the normal hyperbolic structure of inertial manifolds, which are constructed via the Spatial Averaging Principle, seems to be not uniform.

For every $q \in \mathcal{Q}$ and $v_0 \in \mathfrak{A}(q)$ we can define the ***stable fibre*** over v_0 at q as the set of all $v_0 \in \mathbb{E}$ for which (6) is satisfied. We denote this set by $\mathfrak{A}^{st}(q, v_0)$. It can be shown that $\mathrm{Id} \quad : \mathfrak{A}(q, v_0) \to \mathbb{E}^{+}$ is a homeomorphism onto the image. We do not know how to show that the image is entire $\mathbb{E}^{+}$ in the general case.

In our talk, we plan to discuss how to define some other fibres.

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Stochastic semigroups and stochastic di erential equations related to stochastic flows with singularities

A. A. Dorogovtsev[56]

Keywords: stochastic semigroup, Arratia flow, stochastic exponent, Hilbert-valued martingale.

MSC2010 codes: 60H25, 60H05, 60G48.

In the talk we consider stochastic semigroups of strong random operators in Hilbert space. The possibility to obtain such semigroup from the good linear noise is discussed. We will present negative example related to Arratia low and sufficient conditions under which the noise can be constructed from the semigroup and has Levy structure.

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Dilations of classical diffusion processes via quantum stochastic calculus

F. Fagnola,[57]

Keywords: quantum Markov semigroups; diffusion processes; quantum stochastic calculus.)

MSC2010 codes: 81S22, 81S25, 60H10 (1-5 codes recommended)

Introduction. Restrictions of quantum Markov processes (semigroups) on a non-commutative von Neumann algebra often determine classical Markov process (semigroups) by restriction to commutative subalgebras. A natural question in quantum probability is: what classical Markov processes (semigroups) can arise in this way?

In this talk we discuss the construction of quantum diffusion semigroups and their dilations as quantum diffusions in Fock space satisfying an Hudson-Parthasarathy quantum stochastic differential driven by the fundamental noises of quantum stochastic calculus in a Boson Fock space.

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Disordering in a discrete-time stochastic flows

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Keywords: random interaction systems, discrete-time stochastic flow; singular interaction.

MSC2010 codes: 60H25, 60K37, 60H40, 60K35, 60H35.

Introduction.

The discrete-time approximation of the Arratia flow [1] are considered. This approximations $\{x^n_k(u); k = 0, \ldots, n\}$ are given by a difference equation with random perturbation generated by a sequence of independent stationary Gaussian processes $\{\ ^n_k(u); u \in \mathbb{R}; k = 0, \ldots, n\}$ with covariance function $\ _n$:

$$x^n_{k+1}(u) = x^n_k(u) + \frac{1}{\sqrt{n}}\ ^n_{k+1}(x^n_k(u)); \ x^n_0(u) = u; \ u \in \mathbb{R}.$$

Define the random process $x_n(u; \ )$ on $[0; 1]$ as the polygonal line with edges $\left(\frac{k}{n}; x^n_k(u)\right); k = 0; \ldots; n$. It was proved in [3] that if the covariance $\ _n$ approximates in some sense the function $I_{\{0\}}$ then m point motion of x_n weakly converges to the m point motion of the Arratia flow.

Results.

We obtain an explicit form of the Ito-Wiener expansion for $f(x_n(u_1); \ldots; x_n(u_m))$ with respect to noise that produced by the processes $\{\ ^n_k(u); u \in \mathbb{R}; k = 0; \ldots; n\}_{n \ 1}$. This expansion can be regarded as a discrete-time analogue of the Krylov-Veretennikov representation formula [4].

In contrasts to the flow of Brownian particles on the line, in the discrete-time approximations the order between particles can change in time. We define a measure of disordering for 2-point motion as follows

$$\ _n = \int_0^1 I_{\{x_n(u_2;s) \ x_n(u_1;s) < 0\}} ds;$$

where $u_1 < u_2$. If the discrete-time flow approximates the Arratia flow then the following asymptotics holds [5]:

$$\overline{\lim_{n \to \infty}} \frac{2C_n}{n} \ln P\{\ _n > 0\} \quad 1$$

$$\underline{\lim_{n \to \infty}} \frac{2C_n}{n} \ln P\{\ _n > \varepsilon\} \quad K^2;$$

where $C_n = \sup_{\mathbb{R}} \frac{2 \ 2 \ _n(x)}{x^2}$ and $K > 0$.

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Numerical research of the Barenblatt-Zheltov-Kochina model on the interval with wentzell boundary conditions

N. S. Goncharov,[59]

Keywords: Wentzell boundary conditions, Barenblatt – Zheltov – Kochina model, Galerkin method, numerical investigation,Cauchy–Wentzell problem.

MSC2010 codes: 35G15

Introduction. In terms of numerical investigation, we study Barenblatt – Zheltov – Kochina model, which describes dynamics of pressure of a filtered fluid in a fractured-porous medium with general Wentzell boundary conditions. Based on the theoretical results associated with Galerkin method, we developed an algorithm and implementation for the numerical solution of the Cauchy–Wentzell problem on the segment $[0;1]$. In particular, we examine the asymptotic approximation of the spectrum of the one-dimensional Laplace operator and present result of computational experiment. In the paper, these problems are solved under the assumption that the initial space is a contraction of the space $L^2(0;1)$.

Let us consider the Cauchy–Wentzell problem

$$\begin{aligned} &u(x;0) = u_0(x); x \; 2 \; [0;1] \\ &u_{xx}(0;t) + \;\; {}_0 u_x(0;t) + \;\; {}_1 u(0;t) = 0; \\ &u_{xx}(1;t) + \;\; {}_0 u_x(1;t) + \;\; {}_1 u(1;t) = 0 \end{aligned} \tag{1}$$

for the BarenblattB–ZheltovB–Kochina equation on the interval $[0;1]$

$$u_t(x;t) \quad u_{txx}(x;t) = \quad u_{xx}(x;t) + f(x;t); (x;t) \; 2 \; [0;1] \quad \mathrm{R}_+; \tag{2}$$

which describes dynamics of pressure of a filtered fluid in a fractured-porous medium. Here and are the material parameters characterizing the environment; the parameter $2 \; \mathrm{R}_+$; the function $f = f(x;t)$ plays the role of external loading.

The purpose of this work is to show new approach for resolvability of problem (1)-(2) with Wentzell boundary conditions. Namely, according to the modified Galerkin method, describe the solution of the Cauchy–Wentzell problem.

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On topology of ambient manifolds admitting A-di eomorphisms

V. Grines[60].

The report is devoted to exposition of a result obtained in collaboration with E.V. Zhuzhoma and V.S. Medvedev.

Let M^n, $n \quad 3$, be a closed orientable n-manifold and $G(M^n)$ the set of A-diffeomorphisms $f : M^n \ ! \ M^n$ satisfying the following conditions:

- if a basic set belonging to nonwandering set $NW(f)$ of a diffeomorphism $f \ 2 \ G(M^n)$ is nontrivial (that is different from periodic orbit) then it is either an orientable codimension one expanding attractor or an orientable codimension one contracting repeller;
- invariant manifolds of isolated saddle periodic points intersects transversally;
- separatrices of isolated saddle periodic points with Morse index one can intersect only (n-1)-dimensional separatrices of other saddle isolated periodic orbits and separatrices of isolated saddle periodic points with Morse index (n-1) can intersect only one-dimensional separatrices of other saddle isolated periodic orbits.

Let us recall that Morse index of hyperbolic periodic point $p \ 2 \ NW(f)$ of a diffeomorphism $f : M^n \ ! \ M^n$ is called the dimension of unstable manifold $W^u(p)$ of the point p.

For $f \ 2 \ G(M^n)$ denote $\ _f \quad 0$ the number of all nodal periodic points (sinks and sources), $\ _f \quad 0$ the number of isolated saddle periodic points with Morse index 1 or $n \quad 1$ and $\ _f \quad 0$ the number of all periodic points whose Morse index dose not belong to the set $f0; 1; n \quad 1; ng$. Moreover denote by $k_f \quad 0$ the number of nontrivial basic sets and $\ _f$ the number all bunches belonging to union of all nontrivial basic sets of f.

Below, S^m is an m-sphere, T^n is an n-torus. Denote T^n_m a manifold that is either empty set if $m = 0$ or is connected sum of $m \quad 1$ copies of n-torus T^n if $m > 0$

$$\underbrace{T^n] \quad]T^n}_{m \quad 1}:$$

Denote S^n_m a manifold that is either the sphere S^n if $m = 0$ or is connected sum of $m \quad 1$ copies of $S^{n \quad 1} \quad S^1$ if $m > 0$:

$$\underbrace{(S^{n \quad 1} \quad S^1)\,] \quad]\,(S^{n \quad 1} \quad S^1)}_{m \quad 1}:$$

Denote N_m a manifold that is either empty set if $m = 0$ or is connected sum of simply-connected manifolds N^n_i if $m > 0$:

$$N^n_1] \quad]N^n_m;$$

and each manifold N_i admits polar Morse-Smale diffeomorphisms $f_i : N_i \ ! \ N_i$ such that $NW(f_i)$ does not contain saddle periodic points with Morse index 1 or $n \quad 1$.

Theorem. Let M^n be a closed orientable n-manifold, $n \quad 3$, supporting a diffeomorphism $f \ 2 \ G(M^n)$. Then there are interges $k_f \quad 0$, $g_f \quad 0$, $l_f \quad 0$ such that M^n is homeomorphic to connected sum:

$$T^n_{k_f}]\, S^n_{g_f}]\, N_{l_f};$$

where $g_f \quad\ _f + \ _f \quad 1$, $l_f \quad\ _f$.

Remark If $k_f = 0$ then $g_f = \frac{\ _f \quad _f + 2}{2}$.

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Mathematical model of traffic flow at a regulated intersection

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Keywords: Multipoint initial-final condition, geometric graph, traffic flows, Oskolkov equation.

MSC2010 codes: 05C10, 35K70.

Introduction. To begin with, consider a intersection - a place of intersection, abutment or branching of roads at the same level, bounded by imaginary lines connecting, respectively, opposite, most distant from the center of the intersection, the beginning of the curving of the carriageway. Exits from adjacent territories are not considered as intersections

Problem setting. Imagine an intersection with a changing mode of its passage (traffic light is on) in the form of an eight-edge geometric graph $\mathbf{G_1}$ (pic. 1). In this case, the conditions continuity and flow balance [1] will look like

$$u_1^1(l_1;t) = u_2^1(0;t) = u_3^1(l_3;t) = u_4^1(0;t) = \\ = u_5^1(l_5;t) = u_6^1(0;t) = u_7^1(l_7;t) = u_8^1(0;t);$$

$$d_1u_{1x}^1(l_1;t) - d_2u_{2x}^1(0;t) + d_3u_{3x}^1(l_3;t) - d_4u_{4x}^1(0;t) + \\ + d_5u_{5x}^1(l_5;t) - d_6u_{6x}^1(0;t) + d_7u_{7x}^1(l_7;t) - d_8u_{8x}^1(0;t) = 0;$$

$$u_{1x}^1(0;t) = u_{2x}^1(l_2;t) = u_{3x}^1(0;t) = u_{4x}^1(l_4;t) = \\ = u_{5x}^1(0;t) = u_{6x}^1(l_6;t) = u_{7x}^1(0;t) = u_{8x}^1(l_8;t) = 0.$$

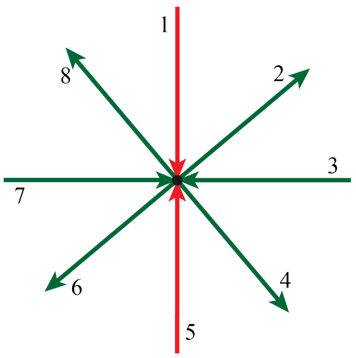

Figure 3: The intersection before the change of the traffic signal, during the time period $[\tau_{j-1}; \tau_j]$

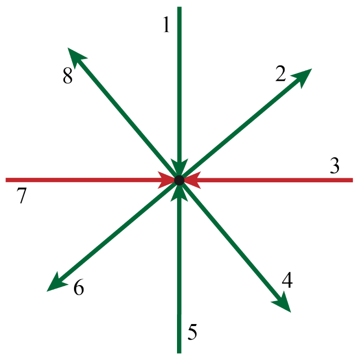

Figure 4: The intersection after changing the traffic signal, in the time period $[\tau_j; \tau_{j+1}]$

Let be k rib length l_k measured in linear metric units (kilometers or miles), however, in the mathematical model of traffic flow, the value is dimensionless. The number of lanes on the carriageway in one direction d_k will be called the capacity, similarly, in the context of the mathematical model, the value is dimensionless. Suppose that all adjacent roads at the intersection under consideration are equivalent, so we will assume that the capacity of each direction will be the same, i.e. $d_1 = d_2 = \dots = d_8 = d$.

The traffic flow will be determined using the Oskolkov equations given on the graph $\mathbf{G_1}$

$$\lambda u_{kt}^1 - u_{ktxx}^1 = \nu u_{kxx}^1 + f_k^1, \; k = \overline{1,8}. \tag{1}$$

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Here $u_k^1 = u_k^1(x, t)$, $x \in [0, l_k]$, $t \in \overline{\mathbb{R}}_+$ ($\{0\} \cup \mathbb{R}_+$), $k = \overline{1, 8}$, characterizes the average speed of the traffic flow on the set of edges E_k characterizes the average speed of the traffic flow on the set of edges of the graph $\mathbf{G_1}$. The average force that makes the wheels of vehicles spin, we will consider $f_k = f_k(x, t)$, $(x, t) \in [0, l_{ik}] \times \overline{\mathbb{R}}_+$.

Coefficient is equal to one divided by the retardation coefficient, which can take negative values, so we consider $\in \mathbb{R}$. The viscosity of the traffic flow, namely, its ability extinguish sudden changes in speed, sets the coefficient , by virtue of the physical meaning $\in \mathbb{R}_+$.

Consider the intersection at the initial moment of turning on the traffic light, when the intersection mode changes from unregulated (yellow blinking traffic light mode) to regulated and designate this moment $t = {}_0$. Suppose that at this moment on the first and fifth edges the traffic signal is red, i.e. for definiteness, we take the flow velocity on these edges to be equal to zero $u_1^1(x, {}_0) = u_5^1(x, {}_0) = 0$, on the remaining ribs $u_k^1(x, {}_0) = u_{0k}^1(x)$, $k = 2, 3, 4, 6, 7, 8$. In general, we write these conditions

$$P(u^1(x, {}_0) - u_0^1(x)) = 0. \tag{2}$$

When the time is reached $t = {}_1$ the traffic signal will change, the traffic flows at the intersection will be different, so we will consider a new graph $\mathbf{G_2}$ (pic. 2).

On this graph, the equations take the form

$$u_{kt}^2 - u_{ktxx}^2 = u_{kxx}^2 + f_k^2, \quad k = \overline{1, 8}; \tag{3}$$

and the conditions continuityPë and flow balance

$$u_1^2(l_1, t) = u_2^2(0, t) = u_3^2(l_3, t) = u_4^2(0, t) =$$
$$= u_5^2(l_5, t) = u_6^2(0, t) = u_7^2(l_7, t) = u_8^2(0, t);$$

$$du_{1x}^2(l_1, t) - du_{2x}^2(0, t) + du_{3x}^2(l_3, t) - du_{4x}^2(0, t) +$$
$$+ du_{5x}^2(l_5, t) - du_{6x}^2(0, t) + du_{7x}^2(l_7, t) - du_{8x}^2(0, t) = 0;$$

$$u_{1x}^2(0, t) = u_{2x}^2(l_2, t) = u_{3x}^2(0, t) = u_{4x}^2(l_4, t) =$$
$$= u_{5x}^2(0, t) = u_{6x}^2(l_6, t) = u_{7x}^2(0, t) = u_{8x}^2(l_8, t) = 0.$$

When changing the traffic signal at the time $t = {}_1$ the average speed of the third and seventh ribs will tend to zero, i.e.

$$\lim_{t \to {}_2} u_3^2(x, t) = 0, \quad \lim_{t \to {}_2} u_7^2(x, t) = 0.$$

In this case, on the remaining edges, the speed will be the one that is reached on the corresponding edge by the time ${}_1$, $u_k^2(x, {}_1) = u_k^1(x, {}_1) = u_{1k}^2(x)$, $k = 1, 2, 4, 5, 6, 8$. In general terms, these conditions take the form

$$P(u^2(x, {}_1) - u_1^2(x)) = 0. \tag{4}$$

Continuing the procedure for switching traffic lights at times $t = {}_j$, $j = \overline{0, n}$, and for even n treat an intersection as a graph $\mathbf{G_1}$, and for odd n – as a count $\mathbf{G_2}$. In general, the multipoint initial-final condition takes the form

$$P(u^m(x, {}_j) - u_j^m(x)) = 0, \quad j = \overline{0, n}, \quad m = 1, 2; \tag{5}$$

where ${}_j$ – the moment of switching the traffic light [2].

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Bi-Lipschitz Mane projections and finite-dimensional reduction for complex Ginzburg-Landau equation

A. Kostianko[64]

Abstract. In this talk I will discuss the problem of the finite-dimensional reduction for 3D complex Ginzburg-Landau equation (GLE) with periodic boundary conditions. Using spatial averaging principle, which was introduced by J. Mallet-Paret and G. Sell to handle the 3D reaction-diffusion equations, together with temporal averaging we will be able to show that the attractor of our problem possesses Mane projections with Lipschitz continuous inverse. This work can be considered as the first step to proof the existence of an inertial manifold for GLE.

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Topological conjugacy Morse-Smale flows with finite number of moduli on surfaces

V. E. Kruglov[65]

Keywords: Morse-Smale flow; moduli of topological conjugacy; equipped graph

MSC2010 codes: 37D15

Introduction. Two flows $f^t, f'^t\colon M \to M$ on a manifold M are called *topologically equivalent* if there exists a homeomorphism $h\colon M \to M$ sending trajectories of f^t into trajectories of f'^t preserving orientations of the trajectories. Two flows are called *topologically conjugate* if h sends trajectories into trajectories preserving not only directions but in addition the time of moving. To find an invariant showing the class of topological equivalence or topological conjugacy of flows in some class means to get a *topological classification* for one. Note, that for some classes their classifications in sense of equivalence and conjugacy coincide; for other classes these classifications completely differ.

The *Morse-Smale flows* were introduced on the plane for the first time in the classical paper of A.A. Andronov and L.S. Pontryagin in [1]. The non-wandering set of such flows consists of a finite number of hyperbolic fixed points and finite number of hyperbolic limit cycles, besides, saddle separatrices cross-sect only transversally, which means on surfaces that there is no a trajectory connecting saddle points. The most important for us combinatorial invariants for Morse-Smale flows are the *Leontovich-Maier's scheme* [2], [3] for flows on the plane, the *Peixoto's directed graph* [4] for Morse-Smale flows on any closed surface and the *Oshemkov-Sharko's molecule* [5] for Morse-Smale flows on any closed surface.

J. Palis in [6] proved that the class of topological equivalence of a flow can contain any volume of topological conjugacy classes, describing by parameters called *moduli*. For example, a modulus appears when a flow has a separatrix common for two saddle points.

Obviously, any limit cycle generates a modulus equals to the period of one. Additionally, in [7] it was proved that the presence of a cell bounded by limit cycles gives infinite number of moduli connected with the uniqueness of invariant foliation in the basin of the limit cycle.

The results. The first result solves the problem of a flow class with a finite number of modulus.

Theorem 1. A Morse-Smale surface flow has finite number of moduli iff it has no a trajectory going from one limit cycle to another.

Second, we use the complete topological classification with respect to equivalence for Morse-Smale surface flows [5], [8] by means of an *equipped graph* $_t$ describing dynamics of ^{t}.

To distinguish topological conjugacy classes we add to the equipped graph an information on the periods of the limit cycles. It gives a new equipped graph $_t$. In this way we get the following result.

Theorem 2. Morse-Smale surface flows $^t; {}^{0t}$ without trajectories going from one limit cycle to another are topologically conjugate iff the equipped graphs $_t$ and $_{0t}$ are isomorphic.

Acknowledgments. The results were obtained in collaboration with O.V. Pochinka. The reported study was funded by RFBR, project number 20-31-90067.

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Resonant localized patterns in the Swift-Hohenberg equation

L. M. Lerman,[66] N. E. Kulagin.[67]

Keywords: Swift-Hohenberg equation; localized pattern, symmetry, elliptic PDE.

MSC2010 codes: 35J35, 35J60, 35J91

The generalized Swift-Hohenberg equation (briefly, SHE)

$$u_t = \quad u \quad (1 + \quad)^2 u + \quad u^2 \quad u^3$$

is the well known pattern-forming model equation and studying its solutions with different spatial structure is a very interesting problem both theoretically and in view of its various applications. We are interested here in its solutions with the localization property

$$\lim_{r\to\infty} u(t,\mathbf{r}) = 0, \quad \mathbf{r} = (x,y), \quad r^2 = x^2 + y^2.$$

Such solutions are important in many applications and were found in a various experiments, both natural and numerical ones.

In the Sobolev space of $H^2(\mathbf{R^2})$ this equation defines a gradient-like differential equation with the functional

$$F = \int_{\mathbf{R^2}} \left\{ \frac{1}{2}[(1 + \quad)u]^2 \quad \frac{}{2}u^2 \quad \frac{}{3}u^3 + \frac{1}{4}u^4 \right\} dxdy,$$

therefore its stationary solutions are of the primary importance.

The existence of localized stationary solutions being rotationally invariant or radial, i.e. when u depends only on r, $u(r)$, is rather well studied ([1], [2], [3], and others). But numerical and natural experiments demonstrate also an existence of non-radial localized stationary patterns to this equation. Thus, it is an interesting problem to understand a genesis of such solutions and their possible shape.

Our strategy of searching non-radial stationary localized solution is to select some branch of radial solutions, for instance, fix   and vary , and find those points on the branch, where the linearization of the equation on that radial solution experiences a bifurcation of the appearance of non-radial solution with the lesser symmetry group. The latter means a possible appearance of a solution being invariant w.r.t. a discrete symmetry group $\mathbf{Z}_n$ instead of symmetry group $\mathbf{S^1}.$ Appearance of such solutions reminds the resonance phenomena when an elliptic equilibrium passes through the resonance of frequencies, from here is the title.

When studying the phenomenon we found the localized resonant solutions with invariant w.r.t the group $\mathbf{Z}_n$, $n = 2,3,4,5,6.$ After a related bifurcation we continued the newborn solution numerically. To substantiate the results, we address to the polar coordinates and use the Galerkin approximations expanding solutions in the Fourier series in angular variable $\varphi.$ This gives a system of differential equations in radial variable that is investigated.

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The massive modular Hamiltonian

R. Longo,[68]

Abstract. A solution of the Klein-Gordon equation can be viewed as a signal carried by a classical wave packet, or as the wave function of a quantum particle, or as a coherent state in Quantum Field Theory. Our recent work concerns the definition, computation and interpretation of the local entropy of this object and its relation to quantum energy inequalities. The Operator Algebraic approach, in particular the Tomita-Takesaki modular theory, provides a natural framework and powerful methods for our analysis. In this talk, I will discuss part of the general ground for our analysis and some key results, in particular the solution of an old problem in QFT: the description of the modular Hamiltonian associated with a space ball B in the free scalar massive QFT; this sets up the formula for the entropy density of a real wave packet.

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Two-dimensional attractors of A-flows and fibered links on 3-manifolds

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Keywords: A-flow, attractor, fibered link

MSC2010 codes: 37D05

Introduction. Dynamical systems satisfying an Axiom A (in short, A-systems) were introduced by S.Smale. By definition, a non-wandering set of A-system is the topological closure of periodic orbits endowed with a hyperbolic structure. Due to Smale's Spectral Decomposition Theorem, the non-wandering set of any A-system is a disjoint union of closed, invariant, and topologically transitive sets called *basic sets*. E.Zeeman proved that any n-manifold, $n \geq 3$, supporting nonsingular flows supports an A-flow with a one-dimensional nontrivial basic set. It is natural to consider the existence of two-dimensional (automatically non-trivial) basic sets on n-manifolds beginning with closed 3-manifolds M^3. We prove that any closed orientable 3-manifolds supports A-flows with two-dimensional attractors. Our main attention concerns to embedding of non-mixing attractors and its basins (stable manifolds) in M^3.

Main results. Let f^t be an A-flow on a closed orientable 3-manifold M^3 and Λ_a a two-dimensional non-mixing attractor of f^t. The stable manifold (in short, a basin) $W^s(\Lambda_a)$ of Λ_a is an open subset of M^3 consisting of the trajectories whose ω-limit sets belong to Λ_a. First, we construct a special compactification of $W^s(\Lambda_a)$ called a casing by a collection of circles that form a fiber link in the casing.

Theorem 1. Let f^t be an A-flow on an orientable closed 3-manifold M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor Λ_a. Then there is a compactification $M(\Lambda_a) = W^s(\Lambda_a) \cup_{i=1}^{k} l_i$ of the basin $W^s(\Lambda_a)$ by the family of circles $l_1, \ldots,$ l_k such that

- $M(\Lambda_a)$ is a closed orientable 3-manifold;
- the flow $f^t|_{W^s(\Lambda_a)}$ is extended continuously to the nonsingular flow $\tilde{f}^t$ on $M(\Lambda_a)$ with the non-wandering set $NW(\tilde{f}^t) = \Lambda_a \cup_{i=1}^{k} l_i$ where $l_1, \ldots, l_k$ are repelling isolated periodic trajectories of $\tilde{f}^t$;
- the family $L = \{l_1; \ldots; l_k\} \subset M(\Lambda_a)$ is a fibered link in $M(\Lambda_a)$.

The second result of the paper, in a sense, is reverse to the first one.

Theorem 2. Let $\{l_1; \ldots; l_k\} \subset M^3$ be a fibered link in a closed orientable 3-manifold M^3. Then there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor and the repelling isolated periodic trajectories $l_1, \ldots, l_k$.

Corollary. Given any closed orientable 3-manifold M^3, there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a two-dimensional attractor.

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LARGE-TIME ASYMPTOTICS OF FUNDAMENTAL SOLUTIONS FOR DIFFUSION EQUATIONS IN PERIODIC MEDIA

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Keywords: large time asymptotics; homogenization; operator-type estimates of homogenization error.

MSC2010 codes: 35K15

The diffusion equation is considered in an infinite 1-periodic medium of R^d. We find large-time approximations for its fundamental solution. The approximation precision has pointwise and integral estimates of orders $O(t^{\ (d+j+1)=2})$ and $O(t^{\ (j+1)=2})$, j = 0, 1, B. . . , respectively. The approximations are constructed on the base of the known fundamental solution of the homogenized equation with constant coefficients, its derivatives, and solutions of a family of auxiliary problems on the periodicity cell which is the unit cube in R^d. The family of problems on the cell is generated recurrently. These results are used to construct approximations of the operator exponential of the diffusion equation with precision estimates in operator norms in L^p-spaces, $1 \quad p \quad 1$. For the analogous equation in an ″-periodic medium, where ″ is a small parameter, we obtain approximations of the operator exponential in L^p-operator norms for a fixed time with precision of order $O(''^n)$, n = 1, 2, B. . .

To obtain these results, we use spectral approach based on the Bloch–Gelfand transformation.

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On the embedding of Morse-Smale di eomorphisms in a topological flow

O. Pochinka[72].

The resuls were obtained in collaboration with V. Z. Grines and E. Ya. Gurevich.

This review presents the results of recent years on solving the problem of G. Palis [1] on finding the necessary and sufficient conditions for the embedding of the Morse-Smale cascade into a topological flow. To date, the problem has been solved by Palis for Morse-Smale diffeomophisms given on manifolds of dimension two. The result for the circle is a trivial exercise. In dimension three and higher, new effects arise related to the possibility of wild embedding of closures of invariant varieties of saddle periodic points, which leads to additional obstructions for the embedding of Morse-Smale diffeomophisms into a topological flow. The progress made in solving the Palis problem in dimension three is connected with a recent complete topological classification of Morse-Smale diffeomorphisms on three-dimensional manifolds and the introduction of new invariants describing the embedding of separatrices of saddle periodic points in the ambient manifold [2]. The transition to a higher dimension requires the use of the latest results of the topology of manifolds.

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Coalescing stochastic flows on metric graphs

G. V. Riabov[73]

Keywords: stochastic flow; coalescence; metric graph.

MSC2010 codes: 37H05, 60J60.

Introduction. Let $\varphi = \{\varphi_{s,t} : -\infty < s \le t < \infty\}$ be a stochastic flow on a locally compact separable metric space M, that is a family of measurable random mappings of M such that $\varphi_{s,t}(\varphi_{r,s}(x)) = \varphi_{r,t}(x)$ a.s., $\varphi_{s,s}(x) = x$ a.s., for any sequence $t_1 < t_2 < \ldots < t_n$ mappings $\varphi_{t_1,t_2}, \ldots, \varphi_{t_{n-1},t_n}$ are independent, for any $s < t$ mappings $\varphi_{s,t}$ and $\varphi_{0,t-s}$ are equally distributed and for all $f \in C_0(M), s \le t$ and $x \in M,$

$$\lim_{(u,v)\to(s,t)} \sup_{y\in M} E(f(\varphi_{u,v}(y)) - f(\varphi_{s,t}(y)))^2 = 0,$$

$$\lim_{y\to x} E(f(\varphi_{s,t}(y)) - f(\varphi_{s,t}(x)))^2 = 0, \ \lim_{x\to\infty} E(f(\varphi_{s,t}(x)))^2 = 0.$$

It is known (see [1]) that the relation

$$P^{(n)}(x,B) = \mathrm{P}((\varphi_{0,t}(x_1),\ldots,\varphi_{0,t}(x_n)) \in B), n \ge 1, x \in M^n, B \in \mathcal{B}(M^n),$$

establishes a one-to-one correspondence between stochastic flows on M and consistence sequences $\{P^{(n)} : n \ge 1\}$ of coalescing Feller transition probabilities. The sequence $\{P^{(n)} : n \ge 1\}$ defines distributions of finite-point motions of the flow.

Problem setting. We will be interested in the existence of a strong stochastic flow that corresponds to a consistent sequence $\{P^{(n)} : n \ge 1\}$ of coalescing Feller transition probabilities. By a strong flow we understand a stochastic flow φ such that
for all $\omega \in \Omega, x \in M, r \le s \le t,$

$$\varphi_{s,t}(\omega, \varphi_{r,s}(\omega,x)) = \varphi_{r,t}(\omega,x), \ \varphi_{s,s}(\omega,x) = x.$$

Existence of a strong stochastic flow is well-known when its finite-point motions are families of solutions of an SDE with smooth enough coefficients (see [2]). In this case the flow is a flow of homeomorphisms of a corresponding manifold. On the contrary we will deal with flows in which coalescence occurs. In the case $M = \mathbb{R}$ existence of strong coalescing stochastic flows was proved in [3] for a large family of sequences $\{P^{(n)} : n \ge 1\}$ (see [4] and [5] for applications to the study of coalescing stochastic flows).

Main Result. Let M be a metric graph. By $\mathrm{P}^{(n)}_x$ we will denote the distribution of the Feller process $X^{(n)}$ in M^n that has transition probabilities $P^{(n)}$ and starts from $x \in M^n.$

Theorem. Assume that transition probabilities $P^{(n)}$ satisfy following properties:

- for any $x \in M$ and $\varepsilon > 0$ $P^{(1)}_t(x, B(x,\varepsilon)^c) = o(t), \ t \to 0+;$
- for any compact $K \subset M$ and any $t > 0$

$$\lim_{c\to\infty} \sup_{n\ge 1, x\in K^n} \mathrm{P}^{(n)}_x(\# X^{(n)}(t) \ge c) = 0.$$

Then there exists a strong stochastic flow that corresponds to the sequence $\{P^{(n)} : n \ge 1\}.$

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On non-singular flows on n-manifolds with two limit cycles

D. D. Shubin,[74] O. V. Pochinka.[75]

Keywords: non-singular flows; Morse-Smale flows.

MSC2010 codes: 37D15

Introduction. The results were obtained in collaboration with O. Pochinka. This talk is devoted to the so-called *NMS-flows* (non-singular Morse-Smale flows) which are Morse-Smale flows without fixed points. Such flows have close connection with topology of ambient manifold. Exhaustive classification of this systems with exactly two limit cycles on closed n-manifolds was obtained and will be presented.

General theory (see e.g. [1]) implies that ambient manifold M^n is the union of stable manifolds and simultaneously the union of unstable manifolds. Thus, on of this trajectories is attracting and another is repelling.

Due to Poincaré-Hopf theorem, Euler characteristic of the ambient manifold is 0. It leaves only torus and Klein bottle for two-dimensional manifold. Classification of such flows is a part of the problem solved in [2], [3], [4]. Namely, there are two equivalent classes of considered flows on the torus and three on the Klein bottle.

For three-dimensional manifolds the fact that Euler characteristic is equal to zero does not contract the class of manifolds since all three-dimensional manifolds have Euler characteristic equal to zero. Necessary and su□cient conditions follow from [5] where author considers wider class of dynamical systems. However, the results are not contain realisation and it is impossible to judge whether one or other flow is admissible.

In case of two non-twisted orbits the topology of ambient manifolds is known from [6]: they are so called *lens spaces*, which are obtained by attaching of two solid tori along their boundary. We establish that every lens space, except sphere, admits exactly two equivalent classes of considered flows. On the sphere S^3 there is unique class due to [6]. If orbits are twisted then there is only one ambient manifold admitting such flow, it is $S^2\tilde{\ }S^1$ and it admits two equivalent classes of considered flows.

For $n > 3$ the ambient manifold M^n; $n > 3$ is two generalised solid tori $D^{n\ 1}\ \ S^1$ glued along boundaries. The results [7] and [8] imply that only two manifolds which admit NMS-flow with two limit cycles, they are $S^{n\ 1}\ \ S^1$, $S^{n\ 1}\tilde{\ }S^1$ and each of them admits two equivalent classes of considered flows.

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3. Interplay between linear infinite-dimensional systems and nonlinear finite-dimensional systems

Geometry and Good Dictionaries for Koopman Analysis of Dynamical Systems

E. M. Bollt[76]

Keywords: Koopman operator; spectral analysis; reduced order model; dynamical system; ROM; good dictionary; DMD; EDMD

MSC2010 codes: 37M25, 37C10, 34M45, 47E05, 42-04

In the spirit of optimal approximation and reduced order modelling the goal of DMD methods and variants is to describe the dynamical evolution as a linear evolution in an appropriately transformed lower rank space, as best as possible. That Koopman eigenfunctions follow a linear PDE that is solvable by the method of characteristics yields several interesting relationships between geometric and algebraic properties. We focus on contrasting cardinality, algebraic multiplicity and other geometric aspects with the introduction of an equivalence class, "primary eigenfunctions," for those eigenfunctions with identical sets of level sets. We present a construction that leads to functions on the data surface that yield optimal Koopman eigenfunction DMD, (oKEEDMD).

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Betti numbers and the spectrum of dynamics on metrizable compact spaces

N. Edeko[77]

Keywords: maximal equicontinuous factor, locally connected, monotone, torus, Betti number.
MSC2010 codes: 54H20, 37B05

In topological dynamics, the study of properties of a dynamical system in terms of the underlying space is a central problem: In which way does the geometry of a space constrain the possible dynamics on it? And which information can one recover from a space by knowing that a dynamical system on it has certain properties? I will give an example of how these questions can be approached in terms of operator theory. This will be done by studying a certain quotient of a dynamical system, the *maximal equicontinuous factor* which is closely linked to a system's spectral properties. The starting point of the inquiry is the following result of Hauser and Jäger.

Theorem (Theorem 3.1, [1]). Suppose that is a homeomorphism of the two-torus $\mathbb{T}^2$. If the maximal equicontinuous factor of $(\mathbb{T}^2; \ )$ is minimal[78], then it must be one of the following three:

(i) an irrational translation on the two-torus,

(ii) an irrational rotation on the circle,

(iii) the identity on a singleton.

Thus, the geometric properties of the two-torus imply that the maximal equicontinuous factor of a homeomorphism on it must have a relatively simple structure if it is minimal: It is a rotation on a compact abelian Lie group of dimension less than two. As it turns out, this is representative of the following general phenomenon.

Theorem. Let be a homeomorphism of a compact metric space K such that K is locally path-connected and the first Betti number $b_1(K)$ is finite. If all -invariant functions $f \in C(K)$ are constant, then every equicontinuous factor of $(K; G)$ is isomorphic to a minimal flow on some compact abelian Lie group of dimension less than $\frac{b_1(K)}{b_0(K)}$.

We will see how operator theory can help prove this result and what it in turn means for dynamical systems in terms of their spectral theory.

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[78] i.e., there are no nontrivial, closed, invariant subsets

Transfer operators and dynamical systems

G. Froyland[79]

Keywords: Exponential Stability, Port-Hamiltonian Systems.

I will introduce transfer operators, which are time-parameterised composition operators that have played a central role in dynamical systems for the last 50 years. I will describe the heavily-used spectral approach for analysing, and more recently optimising, the statistics and mixing properties of dynamical systems, and illustrate these on geophysical flows if time permits.

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Koopmanism for dynamical systems on completely regular spaces

H. Kreidler[80]

Keywords: Koopman semigroup; jointly continuous flow; completely regular space.

MSC2010 codes: 37B02, 46A70, 47D06

The Koopman linearization of semiflows has proven to be an effective instrument to study dynamical systems with the tools of linear functional analysis and operator theory. Classically, this global linearization is considered for continuous semiflows on compact spaces. In this talk we will present an approach allowing to extend this "Koopmanism" to dynamical systems on locally compact, metric and even more general spaces, which makes it applicable to solutions of many ordinary and partial differential equations. This is a joint work with Balint Farkas (Wuppertal).

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Koopman operator framework for nonlinear identification

A. Mauroy[81]

The semigroup of Koopman operators provides a linear description of nonlinear flows in terms of the evolution of observable functions. We leverage this framework in the context of nonlinear system identification and parameter estimation. Since the infinitesimal generator of the semigroup of Koopman operators is directly connected to the underlying dynamics, we can exploit the key idea that identifying the linear generator is equivalent to identifying the nonlinear dynamics. In this talk, we will first review a method developed in this context for nonlinear ODE identification, along with convergence properties derived from the theory of strongly continuous semigroups. Next we will focus on the extension of the framework to infinite-dimensional dynamics by considering Koopman operator semigroups acting on a space of nonlinear bounded functionals. This will provide a novel method to identify nonlinear PDEs.

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A linear program approach to global attractors

C. Schlosser[82], M. Korda[83]

Keywords: Global attractor, dynamical systems, infinite-dimensional linear programming, sum-of-squares, semidefinite programming, occupation measures

MSC2010 codes: 34D45, 90C22

Global attractors, i.e., sets to which all bounded sets converge asymptotically under the action of the dynamics, are typical and important objects in the (asymptotic) analysis of dynamical systems. Global attractors can be of complex nature so that computing/approximating them is (computationally) challenging. We try to approach the task of computing converging approximations of the global attractor by first embedding it into a linear setting. This is done by using the so-called occupation measures. Historically this idea dates back to the 1970ies when optimal control problems were reformulated as linear programs on measures. For a dynamical system $\dot{x} = f(x)$ with locally Lipschitz dynamics $f : \mathbb{R}^n \to \mathbb{R}^n$ and compact state constraint $X \subset \mathbb{R}^n$ we give the following linear program

$$
\begin{aligned}
p^* := \sup\ & \mu_0(X) \\
\text{s.t.}\ & \mu_0, \hat{\mu}_0, \mu_+, \mu \in M(X) \\
& \int_X \beta v^1 - \nabla v^1 \cdot f \, d\mu_+ = \int_X v^1 \, d\mu_0 \quad \forall v^1 \in C^1(\mathbb{R}^n) \\
& \int_X \beta v^2 + \nabla v^2 \cdot f \, d\mu = \int_X v^2 \, d\mu_0 \quad \forall v^2 \in C^1(\mathbb{R}^n) \\
& \mu_0 + \hat{\mu}_0 = \lambda|_X
\end{aligned}
$$

where $M(X)$ denotes the set of (non-negative) measures on $\mathbb{R}^n$ supported on X and $\lambda|_X$ denotes the Lebesgue measure restricted to X. We show that p^* is given by the Lebesgue measure of the global attractor. The main ingredients for the result are a well known characterization of global attractors in terms of maximal positively invariant sets in combination with a previous result on characterizing maximal positively invariant sets by linear programs on measures.

We can take advantage of the obtained linear structure because it allows to us to consider its dual problem where positivstellensätze from real algebraic geometry can be applied. Together with Lasserre's sum-of-squares hierarchy this leads to a sequence of semidefinite programs for which we show that their solutions provide converging outer approximations of the global attractor.

In this talk we will not go too much into the details of the arguments but we rather give an introduction to the methods used.

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Lévy Laplacian and gauge fields

B. O. Volkov[84]

Keywords: Lévy Laplacian, infinite-dimensional Laplacians, Yang–Mills equations, Yang–Mills heat flow, instantons

MSC2010 codes: 70S15,58B20,58J35,53C07

The Lévy Laplacian Δ_L is an infinite dimensional Laplacian which was originally defined in the following way (see [1]). Let f be a twice Fréchet differentiable function on $L_2([0,1],\mathbb{R})$ and $\{e_n\}$ be an orthonormal basis in $L_2([0,1],\mathbb{R})$. Then

$$\Delta_L f(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\langle f''(x)e_k, e_k\rangle \text{ for all } x\in L_2([0,1],\mathbb{R}).$$

The definition of the Lévy Laplacian depends on the choice of an orthonormal basis.

Let M be a Riemannian manifold and Ω_m be a Hilbert manifold of H^1-curves in M with the fixed origin $m\in M$. The Lévy Laplacian Δ_L^{AGV} on the space of sections in a vector bundle over Ω_m was considered in [2,3,4,5,6]. It turns out that nonlinear differential equations of the Yang–Mills theory are equivalent to linear differential equations containing such Lévy Laplacian.

Le A be a connection in a vector bundle over M and F be the associated curvature. The parallel transport U^A generalized by the connection A can been considered as a section in a vector bundle over Ω_m. The following theorem was proved for $M=\mathbb{R}^4$ by Accardi, Gibilisco and Volovich in [2] and generalized for the Riemannian manifold by Léandre and Volovich in [3].

Theorem 1. A connection A is a solution of the Yang–Mills equations

$$D_A^* F = 0,$$

where D_A^* is the adjoint operator to the exterior covariant derivative generated by A, if and only if the parallel transport U^A is a solution of the Laplace equation for the Lévy Laplacian Δ_L^{AGV}

$$\Delta_L^{AGV} U^A = 0.$$

We show that the similar theorem holds for the Yang–Mills heat flow (see [5]).

Theorem 2. A time-depended connection $[0,T]\ni t\mapsto A(t,\cdot)$ is a solution of the Yang–Mills heat equations

$$\partial_t A = -D_A^* F$$

if and only if the flow of parallel transports $[0,T]\ni t\mapsto U^{A(t,\cdot)}$ is a solution of the heat equation for the Lévy Laplacian Δ_L^{AGV}

$$\partial_t U^A = \Delta_L^{AGV} U^A.$$

Let M be an orientable Riemannian 4-manifold. In this case, there is a connection between Lévy Laplacians and instantons. A connection A is called an instanton (antiinstanton) if it is a solution of the anti-self-duality (self-duality) Yang–Mills equations

$$F = -\star F \; (F = \star F),$$

where $\star$ is the Hodge star on M. The Lévy Laplacian Δ_L^{AGV} is not rotation invariant. A modified Lévy Laplacian Δ_L^W can be obtained from the Lévy Laplacian by the action of an infinite

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dimensional rotation $W \in C^1([0;1];SO(4))$. We show (see [6]) that the Laplace equations for some modified Lévy Laplacians are equivalent to the anti-self-duality (self-duality) Yang–Mills equations, but not to the Yang–Mills equations.

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4. Quantum stochastic evolutions and dynamical semigroups

Stochastic Koopman program, quantum extensions of classical evolutions and a unified approach to classical and quantum dynamical systems

L. Accardi[85]

Abstract. In the first part of the talk I recall the stochastic version of the Koopman program and its quantum extension. In the second part I outline an algebraic approach to dynamical systems with emphasis on the role of 1-cocycles and multiplicative functionals. Then I describe the structure of cocycles and multiplicative functionals that are not differentiable but only stochastically differentiable. The third part of the talk goes in some sense in the converse direction of the first part.

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Long composition of raisings and quantum ergodicity

M. Dubashinskiy[86]

Keywords: quantum unique ergodicity, raising and lowering operators, singularity propagation

MSC2010 codes: 58J51; 11F37, 37D40, 74J20.

The well-known Rudnick–Sarnak conjecture on Quantum Unique Ergodicity (QUE) states the equidistribution of large frequency eigenfunctions of Laplace–Beltrami operator on a compact hyperbolic surface X. This means that quantum system is supposed to be more chaotic then its classical counterpart, that is, the geodesic flow on X.

We construct a quantum homotopy between free quantum particle on X and a certainly chaotic quantum system — the quantization of horocyclic flow on X. This homotopy reaches the quantum horocyclic flow during the infinite time and preserves ergodicity during finite time. The homotopy is given via raising operators on hyperbolic plane. Semiclassical measures of systems are then transformed in a controllable way possessing a geometric description.

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Trace decreasing semigroup for an open quantum system interacting with a repeatedly measured ancilla

S. N. Filippov [87]

Keywords: Zeno e□ect; non-Hermitian Hamiltonian; trace decreasing quantum semigroup.

MSC2010 codes: 81S22

Problem setting. We consider a quantum system dynamics caused by successive selective measurements of an ancilla coupled to the system. This scenario is similar to the Zeno effect because the ancilla remains effectively frozen; however, the system dynamics is non-trivial. For the finite measurement rate ^{1} and the system-probe interaction strength we derive analytical evolution equations for the system in the stroboscopic limit $!\ 0$ and $^{2}\ = \mathrm{const}$. We prove that the induced dynamics of the subnormalized density operator is a semigroup provided the system-ancilla interaction is unitary [1]. Dynamics of the normalized density operator is given by a non-linear equation [1]. The obtained semigroup dynamics can be considered as a deviation from the Zeno subspace dynamics on a longer timescale $T\ \ (\ ^{2}\ )^{1}\ \ \ ^{1}$.

Main result. Importantly, the induced conditional dynamics of the main system is described by the effective non-Hermitian Hamiltonian. By physical examples of multi-spin models we show that the effective non-Hermitian Hamiltonian may drive the system to a maximally entangled stationary state [2]. In addition, we report a new recipe to construct a physical scenario where the quantum dynamics of a physical system represented by a given non-Hermitian Hamiltonian H_{e} may be simulated [2]:

Theorem. The trace decreasing semigroup dynamics with the generator $i[H_{e}\ ;\ ]$ results from the stroboscopic measurements on an ancilla, which is originally prepared in the state $j0_{A}ih0_{A}j$ and interacts with the system through self-adjoint Hamiltonian

$$H = \frac{1}{2}(H_{e} + H_{e}^{y})\quad j0_{A}ih0_{A}j + \sqrt{cI + \frac{i}{}(H_{e}\quad H_{e}^{y})}\quad (j0_{A}ih1_{A}j + j1_{A}ih0_{A}j);$$

where $c = \max(0;\ \ M)$ and M is the minimum eigenvalue of the operator $\frac{i}{}(H_{e}\quad H_{e}^{y})$.

Discussion. The developed formalism can be generalized to the case of time-dependent system-ancilla Hamiltonian and the case of non-selective measurements [3]; however, the semigroup dynamics for the system does not take place in general. We can also consider a situation, in which the measurement basis changes in time, which is illustrated by nonselective measurements in the basis of diabatic states of the Landau-Zener model.

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Trotter-Kato Theorems for Quantum Markov Semi-Groups

J. Gough[88]

We look at the application of Trotter-Kato Theorems for quantum stochastic evolutions. We introduce the notion of perturbations of quantum stochastic models using the series product, and establish the asymptotic convergence of sequences of quantum stochastic models under the assumption that they are related via a right series product perturbation. While the perturbing models converge to the trivial model, we allow that the individual sequences may be divergent corresponding to large model parameter regimes that frequently occur in physical applications.

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Uniform and completely non equilibrium invariant states for weak coupling limit type QMS

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Keywords: quantum Markov semigroup; weak coupling limit type; invariant state; Eulerian cycle.

Abstract

We define the uniform and completely non equilibrium invariant states, which are associated with Eulerian cycles; once we did this, we use the Hierholzer's algorithm to obtain a canonical Euler-Hierholzer cycle, and for it, characterize the invariant state.

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Singular perturbations of quantum dynamical semigroups

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Keywords: quantum dynamical semigroup, singular perturbation, nonstandard generator.

MSC2010 codes: 47D06, 46L53

We describe an approach to strongly continuous quantum dynamical semigroups [2,3,7] via completely positive perturbations of their (in general, unbounded) generators. The semigroup is standard if its generator is "of Lindblad's type" [6], i.e. it is obtained by a completely positive perturbation of a "no-event" generator. Then we consider two cases of dynamical semigroups obtained by singular rank-one perturbations of a standard generator. First, we describe an example which gives a positive answer to a conjecture of Arveson [1] concerning possible triviality of the domain algebra. Second, we consider an improved and simplified construction [4,5] of a nonstandard dynamical semigroup outlined previously in our short communication. This gives answer to an old question on existence of dynamical semigroups with non-Linbladian generators.

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Optimization of coherent and incoherent controls and estimation of reachable sets for an open two-level quantum system

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Keywords: quantum control, coherent control, incoherent control

MSC2010 codes: 81Q93, 34H05, 49Mxx

Control of quantum systems, e.g., individual atoms, molecules is an important direction in modern quantum technologies [1–5]. Typically in experimental situations controlled systems are open, i.e., interacting with the environment. Environment is often considered as having deleterious effects on the dynamics. However, it also can be used for controlling the system. A powerful method of incoherent control was found and studied in [6]. In this case, spectral density of the environment, i.e., distribution of particles of the environment in their momenta and internal degrees of freedom, is used as the control function to manipulate the system. This spectral density is often considered as thermal (Planck distribution), but in general it can be any non-equilibrium non-negative function, possibly depending on time, of momenta and internal degrees of freedom of environmental particles. In [6], general method of incoherent control using this spectral density was obtained, including in combination with coherent control, either subsequent of simultaneous. The method was developed for any multilevel systems. Numerical simulations were performed for an explicit example of four level systems using global search optimization by genetic algorithms. Non-selective quantum measurements were also found to be a powerful tool for incoherent control [7].

Initially for this incoherent method it was not clear to what degree it allows for manipulating the system. In [8], a significant advance was achieved where it was shown that combination of coherent and incoherent controls allows to approximately steer *any* initial density matrix to *any* given target density matrix. This property approximately realizes controllability of open quantum systems in the set of all density matrices — the strongest possible degree of quantum state control. This result has several important features. (1) It is obtained with a physical class of GKSL master equations well known in quantum optics and derived in the weak coupling limit. (2) It was obtained for almost all values of parameters of this class of master equations and for multi-level quantum systems of arbitrary dimension. (3) For incoherent controls in this scheme an explicit analytic solution (not numerical) was obtained. (4) The scheme is robust to variations of the initial state — the optimal control steers simultaneously *all* initial states into the target state, thereby physically realizing all-to-one Kraus maps previously theoretically exploited for quantum control [9].

In [10–14], this method was applied to the example of two-level quantum systems controlled by scalar coherent v and incoherent n controls from various points of view, including different objective criteria and optimization methods, analysis of reachable and controllability sets, use of machine learning. Density matrix of the system $\rho(t) \in \mathbb{C}^{2\times 2}$ is a Hermitian positive semi-definite matrix, $\rho(t) = \rho^\dagger(t) \geq 0$, with unit trace, $\mathrm{Tr}\,\rho(t) = 1$. Following [6], master equation for density matrix is

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[\mathbf{H}_0 + \mathbf{V}v, \rho(t)] + \mathcal{L}_{n(t)}(\rho(t)), \qquad \rho(0) = \rho_0. \tag{1}$$

Here $[A, B]$ denote the commutator $[A, B] = AB - BA$ of operators A, B; $\hbar$ is the Planck's constant. Without loss of generality, $\mathbf{H}_0 = \hbar\omega\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ and $\mathbf{V} = \mu\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, where $\omega > 0$, $\mu \in \mathbb{R}$,

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$\neq 0$. The superoperator of dissipation $L_{n(t)}$ is

$$L_{n(t)}(\rho(t)) = \gamma(n(t)+1)\left(\sigma^- \rho(t)\sigma^+ - \frac{1}{2}\{\sigma^+\sigma^-, \rho(t)\}\right) + \gamma n(t)\left(\sigma^+\rho(t)\sigma^- - \frac{1}{2}\{\sigma^-\sigma^+, \rho(t)\}\right), \quad \gamma > 0$$

It describes the controlled interactions between the quantum system and its environment (reservoir). Here $\{A, B\}$ denotes the anti-commutator $\{A, B\} = AB + BA$ of two operators A, B; matrices $\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ define the transitions between the two energy levels. The following constraint follows from physical meaning of incoherent control as spectral distribution of particles:

$$n(t) \geq 0 \quad \forall t \geq 0.$$

The optional constraints $v(t) \in [v_{\min}, v_{\max}]$, $n(t) \in [0, n_{\max}]$ $\forall t \geq 0$ can be used with some bounds $v_{\min}$, $v_{\max}$, and $n_{\max}$. We denote $u = (v, n)$.

In [10], the optimal control problem for steering the system (1) to a given target density matrix ρ_{target} in minimal final time T was considered with piecewise continuous controls v, n. For this time-minimal control problem, we consider the problems

$$J_i(u) = \|\rho(T_i) - \rho_{\text{target}}\|^2_{\text{HS}} \to \inf, \qquad i = 1, 2, \ldots, N,$$

for a series of decreasing final times $\{T_i\}_{i=1}^N$. "HS" means the Hilbert–Schmidt norm.

In [11], the optimal control problem for maximizing the overlap $\langle \rho(T), \rho_{\text{target}} \rangle_{\text{HS}}$ between the final density matrix $\rho(T)$ and given target density matrix ρ_{target} for the system (1) together with minimization of T was considered with piecewise continuous controls v, n. The problems

$$I_i(u) = \langle \rho(T_i), \rho_{\text{target}} \rangle_{\text{HS}} \to \sup, \qquad i = 1, 2, \ldots, N,$$

are considered for a series of decreasing final times $\{T_i\}_{i=1}^N$.

In [12, 13], the system (1) was considered with piecewise constant controls under various constraints on controls' magnitudes and variations. The article [12] considers a series of time-minimal control problems using Bloch vectors for representing system quantum states, and contains a design of a machine learning scheme based on the kNN method and neural networks (MLPs), which shows useful results for generating suboptimal final time and controls v, n for a given initial density matrix in the numerical experiments. The scheme uses the training dataset formed by the optimization results obtained via the differential evolution and dual annealing methods, implementations of which are available in SciPy. For training MLPs, scikit-learn library was used.

In [13, 14], various tools (support hyperplanes (see [15]), sections (see [16]), etc.) were applied for numerical estimation of reachable and controllability sets in terms of the Bloch parametrization. The work [13] includes also the exact description of reachable and controllability sets for some class of controls. In the framework of the approach using support hyperplanes, [13] notes an optimal control problem, where $v = 0$ and $n = 0$ satisfy the Pontryagin maximum principle [17] but not optimal. In addition, for this optimal control problem some comparative numerical results were obtained using the dual annealing method, Krotov method (see [4, 5, 18]), and Gradient Ascent Pulse Engineering (GRAPE) with one- and two-step gradient projection methods (see [19, 20]) for illustrating, first, how zero controls v, n are far from the optimized controls and, second, the importance of adjusting the methods' parameters.

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Linear dynamical quantum systems: New directions and opportunities

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Keywords: quantum stochastic systems, linear quantum systems, quantum stochastic differential equations, Gaussian quantum systems

MSC2010 codes: 93E03, 94C30, 82C10

Introduction. The state-space representation of linear dynamical systems, deterministic and stochastic, lie at the heart of modern mathematical systems and control theory [1]. The stochastic case is described by linear stochastic differential equations of the form,

$$dx(t) = Ax(t)dt + Bdw(t) + Eu(t)dt;\ dy(t) = Cx(t)dt + Ddv(t) + Fu(t)dt;$$

where x is the state, u is the input signal, y is the output signal, w and v are the system and measurement Wiener noise vectors, respectively, and A, B, C, D are the real *system matrices* (of the appropriate dimensions). This class of models is the basis for the Kalman-Bucy stochastic filtering theory and linear quadratic Gaussian control [2].

In the quantum setting, one encounters a similar class of models, known as linear quantum systems [3]. In the Heisenberg picture of quantum mechanics, the model takes the form of a linear quantum stochastic differential equation (QSDE) [3,4],

$$dx(t) = Ax(t)dt + Bdw(t) + Eu(t)dt;\ dy(t) = Cx(t)dt + Ddw(t) + Fu(t)dt. \qquad (1)$$

In this case, $x(t)$, $y(t)$ and $w(t)$ are vectors consisting of operators on the Hilbert space $\mathfrak{h}_n \otimes \Gamma_s(L^2([0,\infty);\mathbb{C}^m))$, where $\mathfrak{h}_n$ is the Hilbert space for n quantum harmonic oscillators and $\Gamma_s(L^2([0,\infty);\mathbb{C}^m))$ is the boson Fock space over the space of square-integrable complex functions on $[0,\infty)$ taking values in $\mathbb{C}^m$. In particular, $x(t) = (q_1(t), p_1(t), \ldots, q_n(t), p_n(t))^T$, $w(t) = (Q_1(t), P_1(t), \ldots, Q_m(t), P_m(t))^T$ and $y(t) = (Q_1'(t), P_1'(t), \ldots, Q_r'(t), P_r'(t))^T$ $(r \leq m)$, where $q_j(t)$ and $p_j(t)$ are the position and momentum operators of the j-th oscillator, $Q_j(t)$ and $P_j(t)$, and $Q_j'(t)$ and $P_j'(t)$, are the amplitude and phase quadratures of the j-th input and output boson fields (taken to be in a Gaussian state), respectively. In Eq. (1), $u(t)$ is a classical (non-quantum) deterministic or stochastic input signal. Quantum mechanical constraints require that canonical commutation relations be preserved. This leads to algebraic constraints on the A, B, C, D matrices, not encountered in the classical setting, known as the *physical realizability* constraints, of the form [3]:

$$A\mathbb{J}_n + \mathbb{J}_n A^T + B\mathbb{J}_m B^T = 0;\ \mathbb{J}_n C^T + B\mathbb{J}_m D^T = 0;\ D\mathbb{J}_m D^T = \mathbb{J}_r; \qquad (2).$$

where $\mathbb{J}_k = I_k \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Only linear QSDEs (1) with system matrices satisfying (2) correspond to the dynamics of physically valid systems. Note that when the joint initial state of the oscillators is a quantum Gaussian state, then $x(t)$ and $y(t)$ remain in Gaussian states at all times $t \geq 0$.

Physical relevance. Linear quantum systems accurately describe a wide-class of linear quantum devices of interest in many applications, including, for instance, quantum optical cavities, quantum optical parametric oscillators, microwave superconducting resonators, gravitational wave interferometers and optical quantum memories [3]. Since linear quantum systems preserve Gaussian states, they are of interest in Gaussian quantum information processing. The linear structure of the QSDEs of linear quantum systems has motivated the adaptation

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and extension of powerful notions and methods from classical linear systems and control theory [1,2] to address problems for linear quantum systems [3].

Future directions. This talk is intended as an overview talk (rather than a technical talk), on the modelling of linear quantum systems and their applications. In particular, it will discuss some new research directions for this class of systems. This includes the problem of system identification of linear quantum systems [5,6], and developing a theory for infinite-dimensional linear quantum systems, as a quantum analogue of stochastic distributed parameter systems represented by stochastic linear PDEs. The latter is motivated by, for instance, fully quantum modeling of a class of quantum memories called photon-echo memories, such as gradient echo memories [7].

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Quantum control landscapes

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Keywords: quantum control, control landscape, qubit, quantum gates

MSC2010 codes: 81P68, 81Q93, 46N50

Control of quantum systems, e.g., individual atoms, molecules is an important direction in modern quantum technologies [1–5]. Consider coherent control of an N–level quantum system which is isolated from the environment. Its dynamics is described by Schrödinger equation:

$$i\frac{dU_t^f}{dt} = (H_0 + f(t)V)U_t^f; \qquad U_{t=0}^f = \mathbb{I}.$$

Here H_0 and V are the free and interaction Hamiltonians (Hermitian $N \times N$-matrices such that $[H_0, V] \neq 0$), and $f \in L_2([0,T];\mathbb{R})$ is a coherent control.

Let O be a quantum observable (system's Hermitian operator) and $W \in SU(N)$ be a target unitary operator. Typical quantum control objectives correspond to maximization of average value of O and generation of target process W and are characterized by objective functionals

$$J_O[f] = \mathrm{Tr}(O U_T^f \rho_0 U_T^{f\dagger}) \to \max,$$
$$J_W[f] = \frac{1}{4}|\mathrm{Tr}(W^\dagger U_T^f)|^2 \to \max.$$

Globally optimal controls realize global maximum of the objective. Trap is a control which is optimal only locally but not globally. To establish whether traps exist or not for a given control objective is a highly important practical problem, since they determine the level of difficulty for finding globally optimal controls in numerical and laboratory experiment [5-7].

In [5] it was proposed that quantum control objectives are typically free of traps. However, this property was proved only for $N = 2$ [8,9] and for control of transmission (that corresponds to $N = \infty$) [10]. Examples of trapping behavior were found for systems with $N \geq 3$ [6,7].

In [8,9] it was shown that if time T is large enough then the objective functional J_W for a qubit has not traps. To explicitly formulate these results, define the special constant control f_0 and the special time T_0:

$$f_0 := \frac{-\mathrm{Tr}H_0\mathrm{Tr}V + 2\mathrm{Tr}(H_0V)}{(\mathrm{Tr}V)^2 - 2\mathrm{Tr}(V^2)},$$
$$T_0 := \frac{\pi}{\|H_0 - \mathbb{I}\mathrm{Tr}H_0/2 + f_0(V - \mathbb{I}\mathrm{Tr}V/2)\|}.$$

Theorem 1. For $N = 2$, if $\mathrm{Tr}V = 0$ and $T \geq T_0$, then all maxima and minima of the objective functionals $J_O[f]$ and $J_W[f]$ are global. Any control $f \neq f_0$ can not be a trap for any $T > 0$.

In [10] it was proved that control of quantum transmission of a particle with energy E through potential V is free of traps. For fixed E, consider $T_E[V]$ as objective functional of the control potential V. The control goal is to maximize transmission.

Theorem 2. The only extremum of the transmission coefficient $T_E[V]$ is the value $T_E = 1$, i.e.,

$$\frac{\delta T_E}{\delta V} = 0 \iff T_E[V] = 1$$

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In [11,12], small-time control landscapes were studied for the control objective J_W. The following result was proved [11].

Theorem 3. Let $W \in SU(2)$ be a single qubit quantum gate. If $[W, H_0 + f_0 V] \neq 0$ then for any $T > 0$ traps do not exist. If $[W, H_0 + f_0 V] = 0$ then any control, except possibly $f \equiv f_0$, is not trap for any $T > 0$ and the control f_0 is not trap for $T > T_0$.

In [12] it is shown that the control f_0 is not a trap in the case $T \le T_0$ and $[W, H_0 + f_0 V] = 0$. One can show that it is sufficient to consider $H_0 = \sigma_z$, $V = v_x \sigma_x + v_y \sigma_y$ and $W = e^{i\varphi_W \sigma_z}$ without loss of generality. Here $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices, $v_x, v_y \in \mathbb{R}$ such that $\nu = \sqrt{v_x^2 + v_y^2} > 0$, and $\varphi_W \in (0, \pi]$. In this case, the special time is $T_0 = \frac{\pi}{2}$ and the special control is $f_0 = 0$. For fixed φ_W and T the value of the objective evaluated at f_0 is

$$J_W[f_0] = \cos^2(\varphi_W + T). \tag{1}$$

The control $f_0 = 0$ is a critical point, i.e., gradient of the objective evaluated at this control is zero. The Taylor expansion of the functional J_W at f_0 up to the second order has the form:

$$J_W[f_0 + \delta f] = J_W[f_0] + \frac{1}{2}\int_0^T \int_0^T \mathrm{Hess}(t,s)\,\delta f(t)\,\delta f(s)\,dt\,ds + o(\|\delta f\|^2_{L_2}), \quad \delta f \to 0,$$

where the integral kernel of Hessian has the form (see [12]):

$$\mathrm{Hess}(s,t) = -2\nu^2 \cos(\varphi_W + T)\cos(\varphi_W + T - 2|t - s|).$$

We study the spectrum of this integral operator. For this purpose, we consider the following cases:

- (φ_W, T) belongs to the triangle domain
$$D_1 := \left\{(\varphi_W, T) \;:\; 0 < T < \frac{\pi}{2},\; \frac{\pi}{2} \le \varphi_W < \pi - T\right\};$$
- (φ_W, T) belongs to the triangle domain
$$D_2 := \left\{(\varphi_W, T) \;:\; 0 < T \le \frac{\pi}{2},\; \pi - T < \varphi_W \le \pi,\; (\varphi_W, T) \neq (\frac{\pi}{2}, \pi)\right\};$$
- (φ_W, T) belongs to the square domain without the diagonal
$$D_3 := \left\{(\varphi_W, T) \;:\; 0 < T \le \frac{\pi}{2},\; 0 < \varphi_W < \frac{\pi}{2},\; \varphi_W + T \neq \frac{\pi}{2}\right\}.$$

Remark 1. It is easy to see from (1) that if $(\varphi_W, T) \in (0, \frac{\pi}{2}] \times (0, \pi] \setminus (D_1 \cup D_2 \cup D_3)$ then f_0 is a point of global extrema of the objective functional J_W.

Theorem 4. If $(\varphi_W, T) \in D_1 \cup D_2 \cup D_3$ then the Hessian of the objective functional J_W at $f_0 = 0$ is an injective compact operator on $L_2([0, T]; \mathbb{R})$. Moreover,

1. If $(\varphi_W, T) \in D_1$, then Hessian at f_0 is strictly negative.
2. If $(\varphi_W, T) \in D_2 \cup D_3$ then Hessian at f_0 has both negative and positive eigenvalues. In this case, the special control $f_0 = 0$ is a saddle point for the objective functional.

The second case was previously proved using a different method [11]. The first case is a new result of [12], where it was rigorously proved that in this case f_0 is either a global maximum point or a trap. In [12], also numerical optimization methods were used such as Gradient Ascent Pulse Engineering (GRAPE), differential evolution, and dual annealing to show that the special control is a point of global maximum if $('_w;T)\ 2\ D_1$. A rigorous proof of this finding remains an open problem. The numerical results also shown that for $\frac{}{2}$ $'_w$ and $0 < T$ $\frac{}{2}$ achieving the objective functional value 1, i.e., providing exact generation of phase shift gate, requires a final time T being not less than the minimal time $T_{\min} = \ '_w$.

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Gaussian Quantum Markov Semigroups on a One-mode Fock Space: Irreducibility and Normal Invariant States

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The presentation is based on a joint work with J. Agredo and F. Fagnola regarding Gaussian Quantum Markov Semigroup on a One-mode Fock-Space. These are related with gaussian states, or those states under wich position and momentum operators are distributed according to a gaussian distribution. Indeed gaussian QMSs are the only QMSs that preserve "gaussianity" of states. Usually they are considered by physicists via their generator (quadratic in annihiliation and creation operators) without concerns about existance and well-definiteness of the dynamics. On the other hand mathematicians usually consider them via their action on Weyl operators of a regular representation of the CCR, but disregard the generator apart from its formal definition. We tried to get the best of both worlds by rigorously working with a quadratic generator in a generalized GKSL form which produced the usual explicit action on Weyl operators. We successfully linked some properties of gaussian QMSs with properties of the parameters of the generator. This allows one to study a gaussian QMS just with linear algebra tools. In particular our main results are: complete characterization of the irreducibility of the QMSs,characterization of existance and uniqueness of a normal invariant state for the QMS, explicit formulas for such invariant states.

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Markov Generators of Low Density Limit: the 2-Generic Case

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We will describe the class of low density limit type (LDLT) Markov generator, discuss the locally generic condition, the structure of invariant states in the 2-generic case and some examples exhibiting the breaking of the similarity principle.

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Random linear operators and limit theorems for their compositions

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Keywords: random linear operators; random operator valued process; compositions of random operators; random semigroup; law of large numbers; generalized convergence in distribution

MSC2010 codes: 60B12, 60B20, 47B80, 47H40

Introduction. It is known (see [1]) that the limit properties of distribution of the sum of random variables with values in the topological vector spaces can be described by limit theorems. In particular, the law of large numbers describes the convergence in probability of the sequence of averaged sum of independent identically distributed (iid) random vector valued variables to the limit of the mean value of the sum. The central limit theorem gives the conditions of the convergence in distribution for the sequence of averaged sum of iid random vector valued variables to the Gaussian random vector.

We study the sequence of compositions of iid random variables with values in the Banach algebra of bounded linear operators $B(H)$ acting in the separable Hilbert space H. In the commutative case of operators of an argument shift on a random vector the limit distribution of averaged composition can be described by the limit theorems for the sum of vector valued variables. Some results on the LLN and CLT for the averaged composition of independent random matrices or linear operators was obtained in [2, 3, 4]. We obtain the analogs of LLN and CLT for the sequence of compositions of iid random semigroups or $B(H)$-valued random processes with non-commutative values.

Law of Large numbers for compositions of random semigroups. Let $\mathbf{A}_j,\ j \in \mathbb{N}$, be the sequence of independent identically distributed random variables with the values in Banach space of bounded linear operators $B(H)$ in some Hilbert space H. We are studying the asymptotic behavior of the probability distribution of the averaged random variables

$$\mathbf{A}_n = (\mathbf{A}_n)^{\frac{1}{n}} \circ \ldots \circ (\mathbf{A}_1)^{\frac{1}{n}},$$

when $n \to \infty$. Here the fractional power of the operator is defined by means of spectral decomposition for unitary or self-adjoin operator. The fractional power for the operator $\mathbf{U}(t)$ belonging to the set of semigroup values $\{\mathbf{U}(t),\ t \geq 0\}$ is defined according to its dynamical sense: $(\mathbf{U}(t))^{\frac{1}{n}} = \mathbf{U}(\frac{t}{n})$.

Let $Y_s(H)$ be the topological vector space of the maps $[0, +\infty) \to B(H)$ which is continuous in the strong operator topology. The topology τ_s of the space $Y_s(H)$ is generated by the family of seminorms $\phi_{T,v},\ T \geq 0,\ v \in H$: $\phi_{T,v}(\mathbf{U}) = \sup\limits_{t \in [0,T]} \|\mathbf{U}(t)v\|_H,\ \mathbf{U} \in Y_s$.

The random semigroup is defined as the measurable mapping $\mathbf{U}: \Omega \to Y_s(H)$ of the probability space $(\Omega, \mathcal{A}, \mu)$ into the measurable space $(Y_s, \mathcal{B}_s)$ such that the values of this map are C_0-semigroups. Here $\mathcal{B}_s$ is the Borel σ-algebra of subsets of the topological space $(Y_s(H), \tau_s)$.

Theorem 1. Let $\mathbf{A}$ be the random variable with the values in the set of self-adjoint operators in the space H and let $\mathbf{U}(t) = \exp(i\mathbf{A}t),\ t \geq 0,$ be the corresponding random semigroup. Let D be the dense linear manifold of the space H such that $\int \|\mathbf{A}(\omega)u\|_H d\mu(\omega) < \infty\ \forall u \in D$. Let operator $\mathbf{A}u = \int \mathbf{A}(\omega)u\, d\mu(\omega),\ u \in D$ be essentially self-adjoint. Let $\{\mathbf{U}_n\}$ be the sequence of independent identically distributed random semigroups such that any of them has the same

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distributions. Then the sequence $\{\mathbf{U}_n \circ \ldots \circ \mathbf{U}_1\}$ of its compositions satisfies the LLN in the strong operator topology:

$$\lim_{n\to\infty}[\sup_{t\in[0,T]} P(\|(\mathbf{U}_n^{\frac{1}{n}} \circ \ldots \circ \mathbf{U}_1^{\frac{1}{n}} - M[\mathbf{U}_n^{\frac{1}{n}} \circ \ldots \circ \mathbf{U}_1^{\frac{1}{n}}])x\|_H > \epsilon)] = 0 \quad \forall x \in H;\ \forall \epsilon > 0.$$

Generalization weak convergence and convergence in distribution for compositions of operator valued random processes. Let E be a Hilbert space. Let $B(E)$ be a Banach space of bounded linear operators in the space E endowed with some operator topology. Let $ca(B(E); \mathcal{B}(B(E)))$ be a Banach space of Borel measure with bounded variation oh the measurable space $(B(E); \mathcal{B}(B(E)))$. Let X be a locally convex space of complex valued functions on the space E which is invariant with respect to argument shift on any vector of the space E. Let $L(X)$ be a locally convex space of linear operators acting in the space X.

Definition. A sequence of measures $\{\mu_n\} : \mathbb{N} \to ca(B(E); \mathcal{B}(B(E)))$ converges $L(X)$-weakly to the measure $\mu \in ca(B(E); \mathcal{B}(B(E)))$ if the sequence of operators $\{\Phi_n\}$ where $\Phi_n u(x) = \int\limits_{B(E)} u(\mathbf{A}x) d\mu_n(\mathbf{A});\ u \in X;\ x \in E;$ converges in the space $L(X)$ to the operator $\Phi : \Phi u(x) = \int\limits_{B(E)} u(\mathbf{A}x) d\mu(\mathbf{A});\ u \in X;\ x \in E.$

Definition. The sequence $\{\xi_n\}$ of random variables with values in the space $B(E)$ converges in the distribution $L(X)$-weakly to the random variable ξ if the sequence of Borel measures $\{\mu_n\}$: $\mu_n(A) = P(\xi_n^{-1}(A));\ A \in \mathcal{B}(B(E));\ n \in \mathbb{N}$, converges $L(X)$-weakly to the measure μ : $\mu(A) = P(\xi^{-1}(A));\ A \in \mathcal{B}(B(E))$.

Theorem 2. Let $\xi(t);\ t \geq 0;$ be a random process with values in the space $B(E)$. Let $\{\xi_n\}$ be a sequence of iid random processes such that any of them has the same distribution. Let X be a Banach space of functions $u : E \to \mathbb{C}$ such that for any $t \geq 0$ the linear operator $u \to \mathbf{F}(t)u = \mathsf{M}(u(\xi(t)))$ is satisfied on the space X. If the function $\mathbf{F}(t);\ t \geq 0;$ satisfies the conditions of Chernoff theorem then the sequence of random processes $\{\eta_n(t);\ t \geq 0\}$, where $\eta_n(t) = \xi_n(\frac{t}{n}) \circ \ldots \circ \xi_1(\frac{t}{n})$, converges in distribution with respect to the space $(B(X); \tau_{sot})$ to the Markov random processes corresponding to the semigroup $\exp(\mathbf{F}'(0)t);\ t \geq 0.$

Remark 1. The CLT for the sum of iid random vectors in Euclidean space $\mathbb{R}^d$ gives the convergence in distribution with respect to topological vector space $C_b(\mathbb{R}^d)$ of bounded continuous function endowed with the pointwise convergence topology.

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Exponential Stability for Port-Hamiltonian Systems

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MSC2000 codes: 82C10, 47D06, 46L55

The decoherence-free subalgebra $N(T)$ is the maximal sub-algebra where a open quantum system, described by a uniformely continuous Quantum Markov Semigroup (QMS), behaves a a closed system. Many information about the QMS can be deduced by the structure of it. In particular, where $N(T)$ is atomic, it induces a block diagonal representation for the generator of the semigroup providing a natural decomposition of it. We extend this result by using the decomposition of the decoherence-free subalgebra in direct integrals of factors, obtaining a structure theorem for every uniformly continuous QMSs. Moreover we prove that, when there exists a faithful normal invariant state, f$N(T)$ has to be atomic and decoherence takes place.

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Master equations in all orders of perturbation theory with Bogolubov-van Hove scaling

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Keywords: non-Markovian quantum dynamics; open system; master equation. (3-8 keywords recommended)

MSC2010 codes: 81S22, 81Q05, 81Q15

We consider the dynamics of the reduced density matrix for the spin-boson model in the rotating wave approximation with the reservoir at zero temperature. We show that if one considers the perturbation theory with Bogolubov-van Hove scaling, then the dynamics of the perturbative part of the reduced density matrix is described by the Gorini – Kossakowski – Sudarshan – Lindblad equation with constant coefficients. So, it is simpler than the usual time-convolutionless master equation derived by usual perturbation methods (without time scaling). We also show that the initial conditions for the exact reduced density matrix and for its perturbative part generally do not coincide. Moreover, under certain resonance conditions the initial condition for the asymptotic equation should be not a density matrix to provide the given asymptotic precision outside the reservoir correlation time. The talk is based on [1], both further details and references could be found here.

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Adiabatic control of quantum-mechanical systems

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Abstract. We show that a periodic emergence and destruction of an additional quantum number leads to an exponential growth of energy of a quantum-mechanical system subject to a slow cyclic variation of parameters. We also show how a state of a quantum-mechanical system can be controlled by an arbitrarily slow variation of parameters.

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5. Further applications of semigroups in mathematical physics

Lie-algebraic approach to one-dimensional translationally invariant free-fermionic dissipative systems

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Keywords: condensed matter; fermionic systems; open quantum dynamics; Lie algebras.

MSC2010 codes: 81Q80, 17B81, 47D06

Translationally-invariant quadratic fermionic chain.

Let us consider the one-dimensional translationally-invariant quadratic fermionic chains with dissipation. The Markovian evolution of open quantum systems is described by the GKSL equation [3]

$$\frac{d}{dt} = \quad i[H; \;] + D[\;] \quad L \; : \tag{1}$$

where is the density matrix, H is the Hamiltonian, D is the dissipator, and L is the Liouvillian superoperator. The most generic Hamiltonian for the case we are interested in is

$$H = \sum_j \sum_{n \; 1} \left(t_n c_j^y c_{j+n} + \;_n c_j c_{j+n} + \text{H.c.} \right) \quad \sum_j c_j^y c_j ; \tag{2}$$

where c_j and c_j^y are fermionic annihilation and creation operators, is the chemical potential, and the complex parameters t_n and $_n$ are the hopping and p-wave pairing amplitudes, correspondingly. The dissipator has the form

$$D[\;] = \sum_j \left(L_j \; L_j^y \quad \frac{1}{2} \left\{ L_j^y L_j ; \; \right\} \right); \quad L_j = \sum_n \left(u_{j \; n} c_n + v_{j \; n} c_n^y \right); \tag{3}$$

where u_x and v_x are arbitrary functions. We assume that all parameters can be time dependent.

Due to translational invariance, under the periodic boundary conditions the generator of time evolution in momentum space reads as

$$L = \sum_{k 2 \mathrm{BZ}} L_k ; \quad L_k = \quad i[H_k; \;] + D_k[\;]; \tag{4}$$

where the summation is over momenta inside the Brillouin zone, k . The Hamiltonian and the dissipator for each momentum mode are of the form

$$H_k = \left(\;_k c_k^y c_k + i \;_k c^y{}_k c_k^y \quad i \;_k c_k c \;_k \right); \tag{5}$$

$$D_k[\;] = L_k \; L_k^y \quad \frac{1}{2} \left\{ L_k^y L_k ; \; \right\}; \quad L_k = u_k c_k + v_k c^y{}_k ; \tag{6}$$

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where $_k$ and $_k$ are functions of $_n$ and $_n$, whereas u_k and v_k are the Fourier components of u_x and v_x. All L_k commute with each other, and the equation (1) can be solved separately for each momentum k.

Lie-algebra of superoperators.

Note that the L_k consists of the following superoperators

$$\begin{array}{llll} X_{1;3} & = \left(n_{\ k} \quad \frac{1}{2}\right) \ ; & X_{2;4} & = \left(n_{\ k} \quad \frac{1}{2}\right); \\ X_5 & = c_k^y c^y_{\ k} \ ; & X_6 & = \ c_k^y c^y_{\ k}; \\ X_7 & = c_{\ k} c_k \ ; & X_8 & = \ c_{\ k} c_k; \\ X_9 & = c_k \quad c_k^y; & X_{10} & = c_{\ k} \quad c^y_{\ k}; \\ X_{11} & = c_k \quad c_{\ k}; & X_{12} & = c^y_{\ k} \quad c_k^y; \\ X_{13} & = c_{\ k} \quad c_k; & X_{14} & = c_k^y \quad c^y_{\ k}; \\ X_{15} & = c^y_{\ k} \quad c_{\ k}; & X_{16} & = c_k^y \quad c_k: \end{array} \tag{7}$$

One can easily check that these superoperators forms a finite-dimensional Lie algebra with respect to composition, therefore, we can solve the equation for each k using the selebrated Wei-Norman method [1]. In fact, and this Lie algebra is equivalent to $\mathrm{u}(1) \quad \mathrm{sl}(4; \mathrm{C})$, where the element of the radical $\mathrm{u}(1)$ is $X_1 \quad X_2 \quad X_3 + X_4$.

Therefore, linear operator equation (1) can be replaced by a system of nonlinear scalar equations with the help of the celebrated Wei-Norman method [1]. In our case, one obtains the system of 16 coupled Riccati equations [2].

Liouvillian spectrum and the closure of dissipative gap.

It can be shown that the spectrum of L_k contains 16 eigenvalues, which can be found explicitely. There always exists a zero eigenvalue corresponding to the steady state solution. The dissipative gap $_d$ of the Liouvillian is the maximal negative number from the real parts of the eigenvalues. Dissipative gap is an important quantity since it's closure signals a dissipative phase transition. It was found that for the model with parameters

$$_k = \quad + \sum_{n=1}^{+1} \frac{2\cos kn}{n^{3=2}}; \quad L_j = {}^p\overline{g}\left(c_j + c_j^y\right) \tag{8}$$

the dissipative gap behaves as $_d \quad \frac{4}{g} jkj$. This linear in momentum behaviour resembles the situation at the point of a continuous phase transition in closed (non-dissipative) systems, where a conformal field theory description is available. The presence of a linear mode in the dissipative system (8) may indicate that it is possible to construct a (non-unitary) conformal field theory describing the system at the point of a dissipative phase transition.

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Analog of the Cauchy problem for a loaded equation of the second order

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Let

$$D_1 = \{(x;y): 0 < x < +\infty;\ 0 < y < +\infty\};$$
$$D_2 = \{(x;y): -\infty < x < 0;\ 0 < y < +\infty\};$$
$$J = \{(x;y): x = 0;\ 0 < y < +\infty\};$$

$D = D_1 \cup J \cup D_2.$

We consider the following linear loaded [1] integro-differential equation.

$$0 = \begin{cases} u_{xx} + a_1 u_x + b_1 u_y + c_1 u - \sum_{i=1}^{n} m_i(x;y) D_{oy}^{\ i} u(x;y) & D_1; \\ u_{xx} - u_{yy} + a_2 u_x + b_2 u_y + c_2 u - \sum_{i=1}^{n} n_i(x;y) D_{oy}^{\ i} u(0;y) & D_2; \end{cases} \quad (1)$$

where $a_k; b_k; c_k$ $(k = 1;2)$ real numbers, moreover $b_1 < 0$; $D_{oy}^{\ i}$ ($D_{oy}^{\ i}$) is the Riemann-Liouville fractional integral operator of order $\ _i$ ($\ _i$), if $f \in L(a;b)$; $(a < b < +\infty)$ for a function $f(y)$, fractional operator of order $\ _i < 0$($\ _i < 0$), is defined as

$$D_{oy}^{\ i} f(y) = \frac{sgn(y - a)}{\Gamma(\ _i)} \int_0^y (y - t)^{\ _i - 1} f(t)dt; \qquad y \in (a;b).$$

The function $D_{ay}^{\ i} f(y)$ is defined almost everywhere on $(a;b)$ and belongs to the class $L(a;b)$. $D_{ay}^0 f(y) = f(y)$; at $\ _i = 0$, and we assume that functions $a_i(x;y)$; $b_i(x;y)$ are continuously differentiable by Hölder into the closure $\overline{D}$ of the domain D, where $c = const \geq 0$ is given real parameter. In this work, we will study the Cauchy problem in the domain D.

Cauchy problem. Find a a function $u(x;y)$; satisfying the following conditions:

1) $u(x;y)$ is continuous in $D_i (i = 1;2)$;

2) partial derivatives u_x and u_y are continuous in $D_i \cup J (i = 1;2)$;

3) $u(x;y)$ is a regular solution of Equation (1) in the domains $D_i (i = 1;2)$

4) the gluing conditions

$$\ _1(y) = \ (y)\ _2(y) + \ (y);$$
$$\ _1(y) = \ (y)\ _2(y) + \ (y)\ _2(y) + \ (y);$$

are satisfied on J, where $\ (y);\ (y);\ (y);\ (y);\ (y)$ are given real-valued functions such that, $\ (y);\ (y) \in C^2$; $\ (y);\ (y);\ (y) \in C^1$; and for any $y \in J$; $\ (y)\ (y) \neq 0$; and

$$\ _1(y) = u(+0;y); \qquad \ _2(y) = u(-0;y);$$
$$\ _1(y) = u_x(+0;y); \qquad \ _2(y) = u_x(-0;y);$$

5) $u(x;0)$ satisfies the conditions

$$u(x;0) = \ '_1(x); \qquad 0 \leq x < +\infty;$$
$$u(x;0) = \ '_2(x); \qquad -\infty < x \leq 0;$$

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$$u_y(x,0) = \ (x), \qquad -\infty < x < 0,$$

where $\varphi_1, \varphi_2, \ $ are given real-valued functions such that functions $\varphi_1, \varphi_2 \in C^2, \ \in C^1$ moreover

$$\varphi_1(0) = \ (0)\,\varphi_2(0) + \ (0),$$

$$\varphi_1''(0) = \ (0)\,\varphi_2''(0) + \ (0)\,\varphi_2(0) + \ (0).$$

Theorem 1. If $\varphi_1(x), \varphi_2(x) \in C^2, \ (x) \in C^1,$ and $\varphi_1(0) = \ (0)\,\varphi_2(0) + \ (0),\ \varphi_1''(0) = \ (0)\,\varphi_2''(0) + \ (0)\,\varphi_2(0) + \ (0)$, then there exists a unique solution to the problem Cauchy in the domain D.

The unique solvability of the problem is proved using the theory of integral equations [2].

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Stochastic solutions of generalized time-fractional evolution equations

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Keywords: time-fractional evolution equations, fractional calculus, randomly scaled Lévy processes, linear fractional Lévy motion, generalized grey Brownian motion, inverse subordinators, Saigo-Maeda generalized fractional operators, Appell functions, three parameter Mittag-Leffler function, Feynman-Kac formulae, anomalous diffusion

MSC2020 codes: 35R11, 35S10, 45D05, 45K05, 60G18, 60G22,60G51, 60G52, 60H30

We consider a general class of integro-differential evolution equations which includes the governing equation of the generalized grey Brownian motion and the time- and space-fractional heat equation:

$$u(t,x) = u_0(x) + \int_0^t k(t,s)Lu(s,x)ds, \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{1}$$

where L is a pseudo-differential operator associated to a Lévy process and $k(t,s)$, $0 < s < t < \infty$, is a general kernel.

We present a general relation between the parameters of the equation and the distribution of any stochastic process, which provides a stochastic solution of Feynman-Kac type. More precisely, we derive a series representation in terms of the time kernel k and the symbol ψ of the pseudodifferential operator L for the characteristic function of the one-dimensional marginals of any stochastic solution. We explain how this series simplifies in the important case of homogeneous kernels which includes the kernel $k(t,s) = (t-s)^{\beta-1}/\Gamma(\beta)$ for time-fractional evolution equations and, more generally, kernels corresponding to Saigo-Maeda fractional diffintegration operators. The connection between Saigo-Maeda fractional diffintegration operators and positive random variables with Laplace transform given by Prabhakar's three parameter generalization of the Mittag-Leffler function is established. These results yield a stochastic representation for (1) with a Saigo-Maeda kernel in terms of a randomly slowed down Lévy process $(Y_{At^\beta})_{t\geq 0}$, where Y is a Lévy process with infinitesimal generator L, A is an independent random variable with Laplace transform given by the three-parameter Mittag-Leffler function, and β corresponds to the degree of homogeneity of the kernel. If Y has a stable distribution (e.g., in the case of a symmetric fractional Laplacian in space), the randomly slowed down Lévy process can be replaced by a randomly scaled linear fractional stable motion, providing a stochastic solution in terms of a self-similar process with stationary increments.

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Order structures and semigroups of isotone resolving operators

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Keywords: nonlinear problems of mathematical physics; semigroups of operators; isotone operator; operator lattice.

MSC2010 codes: 47H07, 47H20, 45K05

Order structures naturally arise in the study of various problems in mathematical physics. Ordered functional spaces with the order relation generated by the cone of nonnegative functions [1-3] can be used to study problems of mathematical physics, solutions of which are nonnegative in their physical meaning (theory of diffusion processes, theory of heat transfer, theory of neutron and radiation transfer, kinetic theory of gases). It should be noted here, in particular, the classical results of the mathematical theory of nuclear reactors [4,5].

We consider in the work some classes of nonlinear boundary value and initial boundary value problems for systems of integro-differential equations of the theory of radiation transfer, the theory of neutrino transport, the theory of diffusion and heat transfer. For these problems, questions of correctness were studied using the Tarski theorem on fixed points of isotone operators acting in complete lattices [6]. The order properties of semigroups of resolving operators acting in complete and conditionally complete lattices are studied, on the basis of which the problems of order and metric stabilization of solutions of the corresponding nonlinear problems are investigated [7-10].

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Inverse problem for Sobolev type equation of the second order

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Keywords: Sobolev type equation; inverse problem; method of successive approximations; polynomial boundedness of operator pencils.

MSC2010 codes: 34A55, 65M32, 65L09, 35G05

Introduction. Let $U;F;Y$ be Banach spaces, operators $B_1;B_0 \in Cl(U;F)$; $A \in L(U;F)$; $\ker A \neq \{0\}$; $C \in L(U;Y)$; the functions $\Psi:[0;T]\to L(Y;F)$; $f:[0;T]\to F$; $\psi:[0;T]\to Y$. Consider the following problem

$$Av''(t)=B_1v'(t)+B_0v(t)+\Psi(t)q(t)+f(t);\quad t\in[0;T]; \tag{1}$$

$$v(0)=v_0;\quad v'(0)=v_1; \tag{2}$$

$$Cv(t)=\psi(t). \tag{3}$$

The problem of finding a pair of functions $v(t)\in C^2([0;T];U)$ and $q(t)\in C^1([0;T];Y)$ from relations (1) – (3) is called the inverse problem.

Existence of solutions. Let the pencil $\vec{B}=(B_0;B_1)$ be polynomially A-bounded and condition

$$\int_\gamma R^A_\mu(\vec{B})d\mu \equiv \mathbb{O}; \tag{A}$$

where $\gamma=\{\mu\in\mathbb{C}:|\mu|=r>a\}$, be fulfilled, then $v(t)=Pv(t)+(\mathbb{I}-P)v(t)=u(t)+\omega(t)$. Here P is the relatively spectral projector in U. Put $U^0=\ker P$; $U^1=\operatorname{im} P$. Suppose that $U^0\subset \ker C$. Then, by virtue of [3], problem (1) – (3) is equivalent to the problem of finding the functions $u\in C^2([0;T];U^1)$; $\omega\in C^2([0;T];U^0)$; $q\in C^1([0;T];Y)$ from the relations

$$u''(t)=S_1u'(t)+S_0u(t)+(A^1)^{-1}Q\Psi(t)q(t)+(A^1)^{-1}Qf(t); \tag{4}$$

$$u(0)=u_0;\ u'(0)=u_1; \tag{5}$$

$$Cu(t)=\psi(t)-C\omega(t); \tag{6}$$

$$H_0\omega''(t)=H_1\omega'(t)+\omega(t)+(B_0^0)^{-1}(\mathbb{I}-Q)\Psi(t)q(t)+(B_0^0)^{-1}(\mathbb{I}-Q)f(t); \tag{7}$$

$$\omega(0)=\omega_0;\quad \omega'(0)=\omega_1; \tag{8}$$

where $S_1=(A^1)^{-1}B_1^1$; $S_0=(A^1)^{-1}B_0^1$; $u_0=Pv_0$; $u_1=Pv_1$; $\omega_0=(\mathbb{I}-P)v_0$; $\omega_1=(\mathbb{I}-P)v_1$; $H_0=(B_0^0)^{-1}A^0$; $H_1=(B_0^0)^{-1}B_1^0$; $t\in[0;T]$.

Theorem 1. Let the pencil $\vec{B}$ be polynomially A-bounded and condition (A) be fulfilled, moreover, the ∞ be a pole of order $p\in\mathbb{N}_0$ of the A-resolvent of the pencil $\vec{B}$; $U^0\subset\ker C$; $\Psi\in C^{p+2}([0;T];L(Y;F))$; $f\in C^{p+2}([0;T];F)$; $\psi\in C^{p+4}([0;T];Y)$; for any $t\in[0;T]$ operator $C(A^1)^{-1}Q\Psi$ be invertible, with $(C(A^1)^{-1}Q\Psi)^{-1}\in C^{p+2}([0;T];L(Y))$ the condition $Cu_1=\psi'(0)$ be satisfied at some initial value $v_1\in U^1$; and the initial values $w_k=(\mathbb{I}-P)v_k\in U^0$ satisfy

$$w_k=-\sum_{j=0}^{p}K_j^2(B_0^0)^{-1}\frac{d^{j+k}}{dt^{j+k}}\Big[(\mathbb{I}-Q)(\Psi(0)q(0)+f(0))\Big];\quad k=0;1.$$

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Then there exists a unique solution $(v; q)$ of inverse problem (1) – (3), where $q \in C^{p+2}([0;T];Y)$, $v = u + w;$ whence $u \in C^2([0;T];U^1)$ is a solution of (4) – (6) and the function $w \in C^2([0;T];U^0)$ is a solution of (7) – (8) given by

$$w(t) = \sum_{j=0}^{p} K_j^2 (B_0^0)^{-1} \frac{d^j}{dt^j}\Big[(I - Q)(\ (t)q(t) + f(t))\Big].$$

Applications. Let $\subset \mathbb{R}^n$ be a bounded domain with a boundary ∂ of class C^∞. In the cylinder $\times [0;T]$ consider the Boussinesq – Love equation

$$(\quad)v_{tt} = (\quad ')v_t + (\quad '')v + fq; \tag{9}$$

with initial conditions

$$v(x;0) = v_0(x);\ v_t(x;0) = v_1(x); \tag{10}$$

boundary condition

$$v(x;t)|_{\partial} = 0 \tag{11}$$

and overdetermination condition

$$\int v(x;t)K(x)dx = \ (t); \tag{12}$$

where $K(x)$ is a given function in $L_2(\ )$ and $f(x;t) = \ (t)$.

Equation (9) describes the longitudinal vibration in the elastic rod, taking into account the inertia and under external load. Conditions (10) set the initial displacement and initial speed, respectively, and (11) sets the value at the boundaries. The overdetermination condition (12) arises when, in addition to finding the function u, one needs to restore part of external load q. Problem (9) – (12) can be reduced to the second-order Sobolev type equation (1) with conditions (2), (3).

Theorem 2. Let one of the conditions

$$\notin (\ ) \text{ or } (\ \in (\ )) \wedge (\ = \ ') \wedge (\ \neq \ '')$$

be fulfilled. Moreover, $K;\ u_0;\ u_1 \in U^1$, $f \in C^2([0;T];L(Y;F))$, $\in C^4([0;T];Y);$ $\sum_{\neq k} \frac{<f(\ ;t);K>}{k} \neq 0$, the condition $\int v_1(x)K(x)dx = \ '(0)$ be satisfied at some initial value $v_1 \in U^1;$ and the initial values $w_k = (I - P)v_k \in U^0$ satisfy

$$< v_0 + \frac{f(\ ;0)q(0)}{(\ _k \ '')}; \varphi_k >= 0 \ for\ k: \ _k = \ ;$$

$$< v_1 + \frac{f_t(\ ;0)q(0) + f(\ ;0)q'(0)}{(\ _k \ '')}; \varphi_k >= 0 \ for\ k: \ _k = \ :$$

Then there exists a unique solution $(v; q)$ of inverse problem (9) – (12), where $q \in C^2([0;T];Y)$, $v = u + w;$ whence $u \in C^2([0;T];U^1)$ is the solution of (4) – (6) and the function $w \in C^2([0;T];U^0)$ is a solution of (7), (8) given by

$$w(t) = \sum_{= \ k} < \frac{f(\ ;t)q(t)}{(\ _k \ '')}; \varphi_k > \varphi_k.$$

Computational experiment. Let

$$= 1;\ ' = 1;\ '' = 2;\ = 2;\ = 2;\ " = 4;\ T = 1;\ l = \ ;$$

$$f(x;t) = \cos(x); \quad v_0(x) = \sin(2x); \quad v_1(x) = \sin(2x); \quad K(x) = \cos(x); \quad F(t) = \frac{4}{3}\sin(t).$$

Consequently, the Boussinesq – Love equation (9) takes the form

$$(\ \ 1 \qquad)v_{tt} = 2(\quad + 1)v_t \quad 2(\quad + 2)v + \cos(x)q(t);$$

conditions (10) have the form

$$v(x;0) = \sin(2x); \quad v_t(x;0) = \sin(2x);$$

and the overdetermination condition

$$\int_0^1 v(x;t)\cos(x)dx = \frac{4}{3}\sin(t).$$

Therefore, all conditions of Theorem 2 are satisfied. The function q was obtained by the method of successive approximations.

$$q(t) = \frac{(\ \ 24\sqrt{21} + 56)e^{\frac{(\ \ 3+\sqrt{21})t}{3}} + (24\sqrt{21} + 56)e^{\ \ \frac{(3+\sqrt{21})t}{3}} \qquad 168\sin(t)}{21}.$$

It is an approximate solution of the problem posed, reaching admissible error $1.944964447 < \varepsilon$ at the first approximation step.

Further, the required function $v(x;t)$ was found using the algorithms developed for the direct problem [2]

$$v(x;t) = \frac{\sqrt{2}\sin(\sqrt{\ }4x)}{\sqrt{\ }}\Bigg(\frac{64\sqrt{2}\cos(t)}{85\ \ ^{\frac{3}{2}}} + \frac{224\sqrt{2}\sin(t)}{255\ \ ^{\frac{3}{2}}} +$$

$$+\frac{\sqrt{2}e^{\ \ t}}{224910\ \ ^{\frac{3}{2}}}\Bigg(\quad 63\cosh\left(\frac{t\sqrt{21}}{3}\right)(\quad 1785\ \ ^2 + 5440t + 2688) +$$

$$+2\sinh\left(\frac{t\sqrt{21}}{3}\right)\sqrt{21}(16065\ \ ^2 + 19040t \quad 1728)\Bigg)\Bigg).$$

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High order heat-type equations and random walks on the complex plane

S. Mazzucchi[117]

The celebrated Feynman-Kac formula for the solution of the heat equation is the first (and most famous) example of an extensively developed theory connecting stochastic processes with the solution of parabolic equations associated to second order elliptic operators. However, this theory cannot be applied to higher order PDEs such as, for instance, high-order heat type equations of the form:

$$\frac{\partial}{\partial t}u(t,x) = a\frac{\partial^N}{\partial x^N}u(t,x), \qquad u(0,x) = f(x), \tag{1}$$

where $t\in\mathbb{R}^+, x\in\mathbb{R}, a\in\mathbb{R}$ and $N\in\mathbb{N}$, $N>2$. In fact, the lack of a maximum principle for Eq. (1) forbids a probabilistic representation of its solution of the form

$$u(t,x) = \mathbb{E}[f(x+X_t)],$$

in terms of the expectation with respect to the distribution of a *real valued* stochastic process $\{X_t\}_{t\in\mathbb{R}^+}$. This problem has been extensively studied, e.g. by Krylov (1960), Hochberg(1978), Funaki (1979), Burzdy (1995), Orsingher (1999), Levin and Lyons (2009).

Recently an alternative technique has been proposed in [1]. A sequence $\{W_n^N(t)\}$ of scaled random walks on the complex plane is constructed as

$$W_n^N(t) = \frac{1}{n^{1/N}}\sum_{j=1}^{\lfloor nt\rfloor}\xi_j, \tag{2}$$

where $\{\xi_j\}_{j\in\mathbb{N}}$ are independent identically distributed complex random variables, uniformly distributed on the set of $N-th$ roots of unit. If $N>2$, the particular scaling exponent $1/N$ appearing in (2) does not allow the weak convergence of $W_n^N(t)$. Nevertheless, the expectation of particular functionals admit a limit for $n\to\infty$, in particular the following result holds

$$\lim_{n\to+\infty}\mathbb{E}[\exp(i\lambda W_n^N(t))] = \exp\left(\frac{i^N}{N!}\lambda^N t\right) \tag{3}$$

and allows to interpret in a weak sense the limit of $W_n^N(t)$ as an N-stable random variable. The convergence in Eq. (3) allows the proof of the following probabilistic representation formula for the solution of (1)

$$u(t,x) = \lim_{n\to+\infty}\mathbb{E}[f(x+W_n^N(t))],$$

for a suitable class of analytic initial data f.

The talk will provide an overview of these results.

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On L^1 asymptotics of collisionless transport semigroups with stochastic boundary operators

M. Mokhtar-Kharroubi[118]

This work deals with collisionless transport equations in bounded open domains $\Omega \subset \mathbb{R}^d$ with C^1 boundary $\partial\Omega$, a velocity measure $m(dv)$ with support $V \subset \mathbb{R}^d$ and a stochastic (i.e. mass-preserving on the positive cone) boundary operator H relating the outgoing and incoming fluxes. Such equations are governed by stochastic C_0-semigroups $(U_H(t))_{t \geq 0}$ on $L^1(\Omega \times V; dx \otimes m(dv))$. Under very natural assumptions, we show that if an invariant density exists then $(U_H(t))_{t \geq 0}$ converges strongly (not simply in Cesarò means) to its ergodic projection. We show also that if no invariant density exists then $(U_H(t))_{t \geq 0}$ is sweeping in the sense that, for any density φ, the total mass of $U_H(t)\varphi$ concentrates near suitable sets of zero measure as $t \to +\infty$. Finally, we show a general weak compactness theorem of interest for the existence of invariant densities; this theorem is based on several results on smoothness and transversality of the dynamical flow associated to $(U_H(t))_{t \geq 0}$.

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An operational construction of the sum of two non-commuting observables in quantum theory and related constructions

V. Moretti[119] (joint work N. Drago and S. Mazzucchi)

Keywords: Quantum Theory; Spectral Theory; Trotter Formula; Jordan Product; Feynman Integral

MSC2010 codes: 81Q10, 46N50, 46T12, 81S40

Introduction. The existence of a real linear-space structure on the set of observables of a quantum system – i.e., the requirement that the linear combination of two generally non-commuting observables A, B is an observable as well – is a fundamental postulate of the quantum theory yet before introducing any structure of algebra. However, it is by no means clear how to choose the measuring instrument of a general observable of the form $aA + bB$ ($a, b \in \mathbb{R}$) if such measuring instruments are given for the addends observables A and B when they are incompatible observables. A mathematical version of this dilemma is how to construct the spectral measure of $f(aA + bB)$ out of the spectral measures of A and B. We present such a construction with a formula which is valid for general unbounded selfadjoint operators A and B, whose spectral measures may not commute, and a wide class of functions $f : \mathbb{R} \to \mathbb{C}$:

$$f\left(\overline{aA + bB}\right) = \text{s-}\lim_{N \to +\infty} \int_{\mathbb{R}^{2N}} f\left(\frac{1}{N}\sum_{n=1}^{N}(a\lambda_n + b\mu_n)\right) dP^{(A)}_{\lambda_1} dP^{(B)}_{\mu_1} \cdots dP^{(A)}_{\lambda_N} dP^{(B)}_{\mu_N} : \quad (3)$$

The proof relies essentially on the Lie-Trotter-Kato formula. In the bounded case, we prove that the non associative Jordan product

$$A \circ B := \frac{1}{2}(AB + BA)$$

of A and B (and suitably symmetrized polynomials of A and B) can be constructed with the same procedure out of the spectral measures of A and B.

$$A \circ B = \text{s-}\lim_{N \to +\infty} \int_{\mathbb{R}^{2N}} \left(\sum_{n=1}^{N} \frac{\lambda_n}{N}\right)\left(\sum_{n=1}^{N} \frac{\mu_n}{N}\right) dP^{(A)}_{\lambda_1} dP^{(B)}_{\mu_1} \cdots dP^{(A)}_{\lambda_N} dP^{(B)}_{\mu_N} ; \quad (4)$$

Formula (3) turns out to have an interesting operational interpretation and, in particular cases, a nice interplay with the theory of Feynman path integration and the Feynman-Kac formula.

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Pointwise convergence of integral kernels for Feynman-Trotter path integrals

S. I. Trapasso[120]

Keywords: Feynman path integrals; Trotter formula; pointwise convergence; time-frequency analysis.

MSC2010 codes: 81S40, 81S30, 35S05, 42B35, 47L10

This contribution is based on joint work with F. Nicola (Polytechnic University of Turin).

Introduction. The Feynman path integral formulation of quantum mechanics is universally recognized as a milestone of modern theoretical physics. Roughly speaking, the core principle of this picture provides that the integral kernel of the time-evolution operator shall be expressed as a "sum over all possible histories of the system". This phrase entails a sort of integral on the infinite-dimensional space of suitable paths, to be interpreted in some sense as the limit of a finite-dimensional short-time approximation procedure. In spite of the suggestive heuristic insight, the quest for a rigorous theory of Feynman path integrals is far from over, as evidenced by the wide variety of mathematical approaches developed over the last seventy years - cf. [1] and the references therein for a broad introductory account.

Lagrangian formulation via the Trotter formula. Among the several proposed frameworks, the closest one to Feynman's original intuition is probably the time-slicing approximation due to E. Nelson [4]. In short, if $U(t)$ is the Schrödinger time evolution operator with Hamiltonian $H = H_0 + V$ (free particle plus a suitable potential perturbation), then the Trotter product formula holds for all $f \in L^2(\mathbb{R}^d)$:

$$U(t)f = e^{-\frac{i}{\hbar}t(H_0+V)}f = \lim_{n\to\infty} E_n(t)f, \quad E_n(t) = \left(e^{-\frac{i}{\hbar}\frac{t}{n}H_0}e^{-\frac{i}{\hbar}\frac{t}{n}V}\right)^n.$$

Integral representations for the approximate propagators $E_n(t)$ can be derived, so that the Trotter formula allows one to give a precise meaning to path integrals by means of a sequence of integral operators.

The problem of pointwise convergence. Notwithstanding the convergence results in suitable operator topologies, a closer inspection of Feynman's writings suggests that his original intuition underlay the much more difficult and widely open problem of the pointwise convergence of the integral kernels of the approximation operators $E_n(t)$ to that of $U(t)$. In the recent paper [5] we addressed this problem by means of function spaces and techniques arising in the context of time-frequency analysis. The toolkit of Gabor analysis has been fruitfully applied to the study of path integrals only in recent times, leading to promising outcomes [6,7,8].

Main results. With reference to the notation above, we consider a setting where H_0 is the Weyl quantization of a real quadratic form, hence covering fundamental examples such as the free particle or the harmonic oscillator. In addition, we introduce a bounded potential perturbation V whose regularity is characterized in terms of the decay in phase space of its windowed Fourier transform (such levels of regularity are encoded by the so-called modulation spaces). This setting covers, and in fact extends, a case that is often met in the literature on mathematical path integrals - namely, the harmonic oscillator plus a bounded perturbation which is the Fourier transform of a complex (finite) measure (see for instance the pioneering works by K. Itô and the line of research developed by S. Albeverio, R. Høegh-Krohn and S. Mazzucchi).

We exploit techniques of Gabor analysis of pseudodifferential operators to prove that the problem of pointwise convergence has a positive answer under the previous assumptions. Precisely, we prove stronger convergence results which imply uniform convergence on compact subsets for the integral kernels in the Trotter formula.

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Our results hold for any fixed value of t 2 RnE, where E is a discrete set of exceptional times - in that case the integral kernels are genuine distributions. In the recent contribution [2] we characterized the properties of such distribution kernels (precisely, they are "mild distributions" in the sense of Feichtinger's Banach-Gelfand fundamental triple of harmonic analysis, cf. e.g. [3]) and we derived weaker convergence results in the sense of modulation spaces even for t 2 E.

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Optimal control problem for linear Sobolev type mathematical models

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Keywords: Sobolev type equations; strong solutions; optimal control; phase space.

MSC2010 codes: 35K70, 49K20

The report presents a review of the work of the Chelyabinsk mathematical school on Sobolev type equations in studying the optimal control problems for linear Sobolev type models with initial Cauchy (Showalter–Sidorov) conditions or initial-final conditions. To identify the nonemptiness of the set of solutions to the control problem we use the phase space method, which has already proved itself in solving Sobolev type equations. The method reduces the singular equation to a regular one defined on some subspace of the original space and applies the theory of degenerate (semi)groups of operators to the case of relatively bounded, sectorial and radial operators. Here mathematical models are reduced to initial (initial-final) problems for an abstract Sobolev type equation. Abstract results are applied to the study of control problems for the Barenblatt–Zheltov–Kochina mathematical model, which describes fluid filtration in a fractured-porous medium, the Hoff model on a graph simulating the dynamics of I-beam bulging in a construction, and the Boussinesq–Löve model describing longitudinal vibrations in a thin elastic rod, taking into account inertia and under external load, or the propagation of waves in shallow water. The mathematical models under consideration belong to a wide class of Sobolev type models (i.e., models based on Sobolev type equations).

The mentioned mathematical models with one or another initial (initial-final) conditions in suitable Hilbert spaces can be reduced to the corresponding problems for a linear Sobolev type equation

$$Ax^{(n)} = Bx + y + Cu, \tag{1}$$

where operators $A \in \mathcal{L}(\mathfrak{X};\mathfrak{Y}),$ $B \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y}),$ $C \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$ functions $u : \mathfrak{I} \to \mathfrak{U},$ $y : \mathfrak{I} \to \mathfrak{Y}$ $(\mathfrak{I} \subset \mathbb{R}),$ and $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ are Hilbert spaces. To select the only process under study, the mathematical models under consideration and their abstract interpretation (1) are supplemented by one of the following conditions:

– the Cauchy conditions

$$x^{(m)}(0) = x_m, \ m = 0, \dots, n-1, \tag{2}$$

– the Showalter–Sidorov conditions

$$P\left(x^{(m)}(0) - x_m\right) = 0, \ m = 0, \dots, n-1, \tag{3}$$

– the initial-final conditions

$$P_{in}\left(x^{(m)}(0) - x_m^0\right) = 0, \ P_{fin}\left(x^{(m)}(\tau) - x_m\right) = 0, \ m = 0, \dots, n-1, \tag{4}$$

where P, P_{in}, P_{fin} are some spectral projectors in the space $\mathfrak{X}.$ Condition (4) differs from the initial conditions in that one projection of the solution is specified at the initial moment of time, and the other at the final moment of the considered time interval. The initial-final condition is a generalization of the Showalter–Sidorov condition, which in turn is a generalization of the classical Cauchy condition. As it is well known, the Cauchy problem for the Sobolev type

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equation (1) (in case $\ker A \neq \{0\}$) is not solvable for arbitrary initial values x_m. To overcome this difficulty, G.A. Sviridyuk proposed the phase space method. The foundations of this concept were laid down in [1]. Another approach to overcome the difficulties associated with non-existence of the solution to (1), (2) is to consider the initial Showalter–Sidorov condition (3) and a more general initial-final condition (4) instead of the initial Cauchy condition (2). We are interested in solving the optimal control problem, which consists in finding a pair $(\hat{x}, \hat{u})$, for which the relation

$$J(\hat{x}, \hat{u}) = \min_{(x,u) \in \mathfrak{X} \times \mathfrak{U}_{ad}} J(x, u), \qquad (5)$$

holds. Here the pairs (x, u) satisfy the Cauchy problem (1), (2) or the Showalter–Sidorov problem (1), (3), or the initial-final problem (1), (4) and $J(x, u)$ is some specially constructed quality functional, $\mathfrak{U}_{ad}$ is some closed and convex set in the control space $\mathfrak{U}$.

The report provides an overview of the results developed in the framework of the direction headed by G.A. Sviridyuk on the optimal control of the solutions to the initial-final problem and, in particular, the Showalter–Sidorov and Cauchy problems for linear Sobolev type equations. The first who began to study the controllability problems and the optimal control problem for linear Sobolev type equations with the Cauchy condition were G.A. Sviridyuk and A.A. Efremov [2]. In these papers, the optimal control problem with a quadratic quality functional was studied in case $n = 1$ with (A, p)-bounded or (A, p)-sectorial operator B and the Cauchy condition, the necessary and sufficient conditions for the existence and uniqueness of a solution were obtained. G.A. Sviridyuk suggested moving from considering the classical solution $x \in C^1(\mathfrak{J}; \mathfrak{X})$ of (1), (2) to the strong solution $x \in H^{p+1}(\mathfrak{X})$ of this problem, which allowed to set the optimal control problem (1), (2), (5) and to use the technique of Hilbert spaces for its research. These studies formed the basis of a number of works by G.A. Sviridyuk's disciples and followers on the study of optimal control problems for linear Sobolev type equations based on the theory of degenerate resolving (semi)groups of operators. When considering the classical Cauchy condition, due to the degeneracy of the equation, it was necessary [1-2] to reconcile the initial data with the control action. Then G.A. Sviridyuk suggested an idea to use more general initial Showalter–Sidorov condition (initial-final condition), which made it possible to remove the restriction on the set of optimal controls and opened the way to a whole class of problems on this subject. In [3] the necessary and sufficient conditions for the existence and uniqueness of the solution of optimal control problems for high-order Sobolev type equations with an initial-finite condition were obtained. The ideas and methods developed by G.A. Sviridyuk and A.A. Efremov on controllability of linear abstract Sobolev type equation opened the way to the study of more general controllability problems.

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6. Related topics

Lefschetz trace formula on stratified spaces

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Keywords: stratified space; Lefschetz trace formula; Witten's deformation.

MSC2010 codes: 58A14, 32S60

We show a version of the Lefschetz trace formula for stratified maps between stratified spaces with isolated singularities, using the intersection homology. It is different from the version of that formula shown by Goresky and McPherson. Our main technical tool is a version of the Witten's deformation of the de Rham complex, involving the Dunkl harmonic oscillator.

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On the Liouville and strong Liouville properties for a class of non-local operators

D. Berger[126], R. L. Schilling[127]

Keywords: Characteristic exponent; Lévy generator; Liouville property; strong Liouville property.

MSC2010 codes: *Primary:* 60G51, 35B53. *Secondary:* 31C05, 35B10, 35R09, 60J35.

Introduction. A C^2-function $f : \mathbb{R}^n \to \mathbb{R}$ is called *harmonic*, if $\Delta f = 0$ for the Laplace Operator Δ. It is kown that any bounded harmonic function is constant. Often it is helpful to understand Δf as a Schwartz distribution in $D'(\mathbb{R}^n)$ and to re-formulate the Liouville problem in the following way: The operator Δ enjoys the **Liouville property** if

$$f \in L^\infty(\mathbb{R}^n) \text{ and } \forall \varphi \in C_c^\infty(\mathbb{R}^n) : \langle \Delta f, \varphi\rangle := \langle f, \Delta\varphi\rangle = 0 \implies f \equiv \text{const} \tag{1}$$

holds; $\langle \cdot, \cdot \rangle$ denotes the (real) dual pairing used in the theory of distributions. An excellent account on the history and the importance of the Liouville property can be found in the paper [1] by Alibaud *et al.* If the condition '$f \in L^\infty(\mathbb{R}^n)$' in (1) can be replaced by '$f \geq 0$', we speak of the **strong Liouville property**. In this talk we prove a necessary and sufficient condition for the Liouville and strong Liouville properties of the infinitesimal generator of a Lévy process and subordinate Lévy processes. The talk is based on the preprint [2].

Main result. A Lévy process is a stochastic process with independent, stationary increments and right-continuous paths with finite left-hand limits. It is well-known that the infinitesimal generator L_ψ of a Lévy process is a **pseudo-differential operator**

$$\widehat{L_\psi u}(\xi) = -\psi(\xi)\hat{u}(\xi), \quad u \in S(\mathbb{R}^n)$$

where $\hat{u}(\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}u(x)\,dx$ is the Fourier transform and $S(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing smooth functions. The **symbol** $\psi : \mathbb{R}^n \to \mathbb{C}$ is a **continuous and negative definite** function which is uniquely characterized by its **Lévy–Khintchine representation**

$$\psi(\xi) = -ib\cdot\xi + \frac{1}{2}Q\xi\cdot\xi + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{i\xi\cdot x} + i\xi\cdot x \mathbb{1}_{(0,1)}(|x|)\right)\nu(dx);$$

where $b \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ (a positive semidefinite matrix) and ν (a Radon measure on $\mathbb{R}^n \setminus \{0\}$ such that $\int_{\mathbb{R}^n\setminus\{0\}} \min\{|x|^2, 1\}\,\nu(dx) < \infty$) uniquely describe ψ. Our main result states the following:

Theorem 1. Let L_ψ be the generator of a Lévy process with symbol ψ. The operator L_ψ has the Liouville property if, and only if, the zero-set of the symbol satisfies $\{\psi = 0\} = \{0\}$.

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Convergence the solution of initial-boundary value problem for one partial differential equation in the *-weak sense

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Keywords: modified Boussinesq equation; Sobolev type equation; initial-boundary value problem; Galerkin method; *-weak convergence.

MSC2010 codes: 35C09, 35Q35

Introduction. Let $\subset \mathbb{R}^n$ be a domain with the boundary ∂ of class C^∞ and $T \in \mathbb{R}_+$. In the cylinder $C = \times (0;T)$, consider the modified Boussinesq equation

$$(\quad)u_{tt} \quad ^2 u + u^3 = 0, \quad (x,t) \in \quad \times (0;T) \tag{1}$$

with Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial \quad \times (0;T) \tag{2}$$

and Cauchy conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \quad ; \tag{3}$$

where $\ ,\ \in \mathbb{R}$. The equation has many applications in various fields of natural science. For example, it simulates wave propagation in shallow water, taking into account capillary effects. In this case, the function $u = u(x,t)$ determines the wave height. In monograph [1] a linear mathematical model of wave propagation in shallow water is constructed. A (modified) mathematical model of wave propagation in shallow water in a one-dimensional region was investigated in [2] and a soliton solution of equation (1) was obtained. The existence of a unique global solution to the Cauchy problem for equation (1) was proved [3] for $\ = 1,\ = 1$. In [4], a similar solution was obtained for describing the interaction of shock waves. In all the works listed above, an essential condition is the continuous invertibility of the operator at the highest derivative with respect to t. However, the operator can be degenerate. Equations that are not solvable with respect to the highest time derivative, according to [5] are called Sobolev type equations.

Using the theory of p-bounded operators developed by G.A. Sviridyuk and his disciples [6, 7], it was shown in [8] that in appropriately chosen spaces the problem (1)–(3) can be reduced to the initial value problem

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \tag{4}$$

for an abstract semilinear second-order Sobolev type equation

$$L\ddot{u} \quad Mu + N(u) = 0, \tag{5}$$

where $\dot{u}, \ddot{u}$ are the first and the second derivatives with respect to t, $L = \quad ,M = \quad ^2 \quad ,N(u) = u^3$. Then, using the phase space method, a theorem on the existence of a unique local solution was proved.

Definition 1. The set $\mathfrak{P}$ is called the phase space of equation (5) if
1) for any $(u_0,u_1) \in T\mathfrak{P}$ ($T\mathfrak{P}$ is the tangent bundle of $\mathfrak{P}$) there is a unique solution to problem (4), (5);
2) any solution $u = u(t)$ of equation (5) lies in $\mathfrak{P}$ as a trajectory.
Moreover, the notation $(u_0,u_1) \in T_{u_0}\mathfrak{P}$ should be understood as $u_0 \in \mathfrak{P}$ and $u_1 \in T_{u_0}\mathfrak{P}$.

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Let $\ker L \neq \{0\}$ and the operator M be $(L,0)$ bounded, then, by the splitting theorem [7], equation (7) can be reduced to an equivalent system of equations

$$\begin{cases} 0 = (\mathbb{I} - Q)(M + N)(u); \\ \dot{u}^1 = L_1^{-1} Q(M + N)(u); \end{cases}$$

where $u^1 = Pu$, P is some projector along $\ker L$. Then the phase space $\mathfrak{P}$ of equation (5) is the set [8]

$$\mathfrak{P} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(M + N)(u) = 0\}.$$

It was also noted that in the case of monotonicity of the operator N, the phase space would be a simple manifold.

Main result. Let us formulate and prove a theorem that answers the question on how to find a solution to (1) – (3).

Let the operator $L : H^1(\Omega) \to H^{-1}(\Omega)$ be given by formula

$$\langle Lu, v\rangle = \int_\Omega (\nabla u \nabla v + \lambda uv)dx.$$

Denote $B = L^4(\Omega) \cap H_0^1(\Omega)$ and $D = H^1(\Omega) \cap \operatorname{coim} L$ (where $\operatorname{coim} L = H^1(\Omega) \ominus \ker L$).

In addition, define spaces of distributions (functions with values in a Banach space) $L^\infty(0,T;B)$ and $L^\infty(0,T;L^2(\Omega))$. Construct the conjugate spaces using the Dunford – Pettis theorem: $(L^\infty(0,T;B))' \simeq L^1(0,T;L^{\frac{4}{3}}(\Omega) \cup H^{-1}(\Omega))$ and $(L^\infty(0,T;D))' \simeq L^1(0,T;D^*)$.

Let λ_k be the eigenvalues of the homogeneous Dirichlet problem (2) for the Laplace operator, numbered nonincreasingly taking into account their multiplicity, and φ_k be the corresponding eigenfunctions. Note that the linear span of $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ for $m \to \infty$ is dense in B and orthonormal (in the sense of the inner product in $L^2(\Omega)$).

Theorem 1. Let $\lambda \in [\lambda_1, +\infty)$, $u_0 \in B$ and $u_1 \in D$ and $(u_0, u_1) \in T_{u_0}\mathfrak{P}$. Then there exists a solution $u = u(x,t)$ to problem (1)–(3) such that $u \in L^\infty(0,T;B)$ and $\dot{u} \in L^\infty(0,T;D)$.

Theorem 2. Under the conditions of Theorem 1 and Ω being such that embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact, the solution to problem (1)–(3) is unique.

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Operator splitting for abstract Cauchy problems with dynamical boundary conditions

P. Csomós[129]

Keywords: operator splitting; Lie and Strang splitting; Trotter product; abstract dynamical boundary problems; error bound.

MSC2010 codes: 47D06, 47N40, 34G10, 65J08, 65M12, 65M15

In the present work we focus on the abstract setting of coupled Cauchy problems, where one of the sub-problems provides a extra condition, of boundary type, to the other. We consider equations of the form:

$$\begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0; \quad u(0) = u_0 \in E; \\ \dot{v}(t) = Bv(t) & \text{for } t \geq 0; \quad v(0) = v_0 \in F; \\ Lu(t) = v(t) & \text{for } t \geq 0; \end{cases}$$

where E and F are Banach spaces over the complex field $\mathbb{C}$, A and B are (unbounded) linear operators on E and F, respectively. The coupling of the two problems involves the unbounded linear operator L acting between E and F. Moreover, the coupling is such that one of the problems prescribes a "boundary type" extra condition for the other one.

In order to give an approximate solution to this problem, we study operator splitting methods. The theory of one-sided coupled operator matrices (see [1], [3], [4]) provides an excellent framework to study the well-posedness of such problems. We show that with this machinery even operator splitting methods can be treated conveniently and rather efficiently. We consider three specific examples: the Lie (sequential), the Strang, and the weighted splitting, and prove the convergence of these methods along with error bounds under fairly general assumptions. Simple numerical examples show that the obtained theoretical bounds can be computationally realised.

The talk is based on the joint work [2] with B. Farkas and M. Ehrhardt (Wuppertal).

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On a Toeplitz algebra generated by a directed graph

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Keywords: Toeplitz algebras, graph algebras, representations, C^*-algebras

MSC2010 codes: 46L05

Introduction. This report focuses on constructing and using properties of Toeplitz algebras generated by infinite directed graphs. The simplest examples of an infinite directed graph are graphs $(X; \varphi)$ generated by mappings $\varphi: X \to X$ of a countable set on itself. In papers [1],[2] properties of C^*-algebras generated by such graphs were described. In this paper we will show that many of statements from mentioned papers still hold in general case.

Let $E = (E^0; E^1; s; r)$ be a directed graph, where E^0 is a set of graph vertices, E_1 is a set of edges and $s; r: E^1 \to E^0$ are source and range maps respectively. An edge sequence $\mu = e_1 \dots e_n$ is called a path from $a = s(e_1)$ to $b = r(e_n)$ if $r(e_j) = s(e_{j+1})$ for all $1 \le j \le n-1.$ A path is called a cycle if $r(e_n) = s(e_1).$ We assume that a given directed graph doesn't contain cycles and loops. Let $\mu = e_1; \dots; e_n$ be a path. We call n the length of the path μ and we write that as $d(\mu) = n.$

Let $l^2(E^0)$ be the Hilbert space of functions on E^0 with a scalar product:

$$\langle f; g \rangle = \sum_{a \in E^0} \overline{g}(a) f(a)$$

A family of functions $\{e_a\}_{a \in E^0}$ on $E^0;$ where $e_a(b) = \delta_{a;b}$ is the Kronecker delta, forms an orthonormal basis of $l^2(E^0).$ Let $l(E^0)$ be a linear subspace of $l^2(E^0)$ generated by finite linear combinations of functions from $\{e_a\}_{a \in E^0}$. A restriction of a scalar product to $l(E^0)$ is still a scalar product. Define two operators $T: l(E^0) \to l(E^0)$ and $T': l(E^0) \to l(E^0)$ as:

$$Te_a = \sum_{b \in (r(s^{-1}(a)))} e_b \text{ and } T'e_a = \sum_{b \in (s(r^{-1}(a)))} e_b$$

respectively.

Proposition 1. For any $e_a; e_b \in \{e_a\}_{a \in E^0}$ the following holds:

$$(T'e_a; e_b) = (e_a; Te_b).$$

Let $\ \ = \sup_{E_0}(\text{card}(s(r^{-1}(a))) + \text{card}(r(s^{-1}(a)))).$

Theorem 2. The operator $T: l(E^0) \to l(E^0)$ extends to a bounded linear operator $\mathbf{T}$ on $l^2(E^0)$ if and only if $\ \ < \infty$

Corollary 3. If $\mathbf{T}$ is an extension of an operator $T;$ then an adjoint operator $\mathbf{T}^*$ is an expansion of T'.

Definition 4. A C^*-algebra generated by operators $\mathbf{T}$ and $\mathbf{T}^*$ is called the Toeplitz algebra $\mathcal{T}_E$ generated by a directed graph $E = (E^0; E^1; s; r):$

We will show some of properties of the algebra $\mathcal{T}_E$.

Let S be a semigroup with a zero generated by finite products of operators $\mathbf{T}$ and $\mathbf{T}^*$. Define $\text{ind}L (L \in S);$ such that $\text{ind}\mathbf{T} = 1; \text{ind}\mathbf{T}^* = -1$ and if $L = V_1 \cdots V_n;$ then $\text{ind}L = \sum_1^n \text{ind}V_i;$ where $V_j = \mathbf{T}$ or $V_j = \mathbf{T}^*$.

Let B_n be a closed subspace in $\mathcal{T}_E$ generated by linear operators from $S_n = \{L \in S : \text{ind}L = n\}.$

Theorem 5. The following statements hold:

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a) $B_n \quad B_m \qquad B_{n+m}$;

b) $T_E = \bigoplus_Z B_n$;

c) B_0 is C -algebra;

d) There exists conditional expectation $F_0\colon T_E \ ! \ B_0$;

e) There exists representation $\ : S^1 \ ! \ \mathrm{Aut} T_E$ of an unit circle S^1 to an automorphism group T_E:

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Graded semigroup C^*-algebras and non-commutative Fourier coefficients

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Keywords: contractive operator; grading; semigroup of operators; reduced semigroup C^*-algebra; σ-index of a monomial; topologically graded C^*-algebra.

MSC2010 codes: 46L05, 47L40

Introduction. This report presents a construction that allows us to set a topological grading on the reduced semigroup C^*-algebra over an arbitrary group.

The reduced semigroup C^*-algebra is the algebra which is generated by the left regular representation of a semigroup with the cancellation property. The study of such C^*-algebras was started by Coburn in 1960s. It has been further developed in the papers by a number of authors (see, for example, [1]).

A grading for an object of a category allows us to investigate the structure of this object. In the category of C^*-algebras, one considers the gradings that are also called the C^*-bundles, or the Fell bundles [2]. The notion of the topologically graded C^*-algebra was introduced in [3] with the aim to extend the concepts of harmonic analysis to the non-commutative case. An important property of a such grading is the existence of special operators, namely, the conditional expectation and the Fourier coefficients.

We studied the semigroup C^*-algebras and their gradings in [4–9].

The report presents the results of [7, 9]. In [7], we considered the topological gradings of the semigroup C^*-algebras over the group $\mathbb{Z}_n$ of integers modulo n.

Fell bundle for semigroup C^*-algebra. Throughout S is a discrete semigroup with the cancellation property and the unit e.

Let us consider the Hilbert space $l^2(S)$ of all square summable complex-valued functions defined on the semigroup S. The reduced semigroup C^*-algebra $C^*_r(S)$ is the C^*-subalgebra generated by the set of isometries $\{T_a \mid a \in S\}$ in the algebra of all bounded operators on $l^2(S)$. Here the operator T_a is defined as follows: $T_a(e_b) = e_{ab},\ a, b \in S$, where $\{e_a \mid a \in S\}$ is the canonical orthonormal basis in the space $l^2(S)$.

Further, in the C^*-algebra $C^*_r(S)$, we consider the involutive subsemigroup Mon consisting of operators of the form

$$V = T_{a_1}^{i_1} T_{a_2}^{i_2} \cdots T_{a_k}^{i_k},$$

where $a_j \in S,\ i_j \in \{-1, 1\}$, $j = 1, \ldots, k$, $k \in \mathbb{N}$, and $T_{a_j}^{1} := T_{a_j}$, $T_{a_j}^{-1} := T^*_{a_j}$. These operators are called the monomials.

Let G be an arbitrary group. We assume that there is a surjective semigroup homomorphism

$$\sigma : S \longrightarrow G.$$

It was shown in [9] that the formula

$$\operatorname{ind} V = \sigma(a_1)^{i_1} \sigma(a_2)^{i_2} \cdots \sigma(a_k)^{i_k}$$

defines an involutive surjective homomorphism of semigroups $\operatorname{ind} : Mon \longrightarrow G.$

The value $\operatorname{ind} V$ is called the σ-index of the monomial V. This concept was introduced in [7] for the case $G = \mathbb{Z}_n$.

Monomials with the σ-index e constitute an involutive subsemigroup of operators in the monomial semigroup Mon. In the C^*-algebra $C^*_r(S)$ we consider the C^*-subalgebra $\mathfrak{A}_e$ generated by this subsemigroup of operators.

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For each $g \in G$, let $\mathcal{A}_g$ be the closure of the linear span for all operators of the σ-index g.

The semigroup C^*-algebra $C_r^*(S)$ is said to be G-graded if it contains a family of Banach subspaces satisfying the conditions listed in the following theorem.

Theorem 1. For the family of Banach spaces $\{\mathcal{A}_g \mid g \in G\}$ the following statements hold:

1) $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$;
2) $\mathcal{A}_g^* = \mathcal{A}_{g^{-1}}$ for every $g \in G$;
3) the family $\{\mathcal{A}_g \mid g \in G\}$ consists of the linearly independent subspaces of $C_r^*(S)$;
4) $C_r^*(S) = \overline{\bigoplus_{g \in G} \mathcal{A}_g}$.

So the family of subspaces $\{\mathcal{A}_g \mid g \in G\}$ constitutes a Fell bundle for the semigroup C^*-algebra $C_r^*(S)$ over the group G.

Topological grading of semigroup C^*-algebra. The G-graded C^*-algebra $C_r^*(S)$ is said to be topologically graded if there exists an operator with the properties formulated in the following theorem.

Theorem 2. There exists a contractive linear operator

$$F : C_r^*(S) \longrightarrow \mathcal{A}_e$$

which is the identity mapping on $\mathcal{A}_e$ and vanishes on each subspace $\mathcal{A}_g$, $g \in G$, $g \neq e$.

Thus, in addition, the family $\{\mathcal{A}_g \mid g \in G\}$ forms the topological G-grading of $C_r^*(S)$.

As a corollary, for each $g \in G$, there is a contractive linear operator

$$F_g : C_r^*(S) \longrightarrow \mathcal{A}_g$$

which is the identity mapping on $\mathcal{A}_g$ and vanishes on every subspace $\mathcal{A}_h$, $h \in G$, $h \neq g$.

As is known, the operators F_g are non-commutative analogs of the Fourier coefficients.

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Modeling a Symbolic Image of a Dynamic System

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Keywords: symbolic image, dynamical systems, trajectories of the systems, path on the graph.

MSC2010 codes: 37A05, 37B10

Introduction. In the study of dynamic systems, methods based on the construction of a symbolic image, which is a directed graph obtained with aid, are applied final phase space cover [1, 2, 7].

The trajectories of the system are encoded by paths on the graph [1, 6] and can be found using route algorithms in the graph by many agents, and closed paths by many salesmen. The structure of the graph changes, and vertices and edge arise as a result of algorithmic procedures of numerical resolution of the corresponding dynamic systems. There are a number of problems associated with numerical implementation of symbolic image algorithms on large-dimensional graphs [3-5].

1. Symbolic image of a dynamical system. Let $f : M \to M$ be a homeomorphism of compact manifold M generating a discrete dynamical system and $p(x; y)$ by a distance on M. Let $C = \{M(1); \dots; M(n)\}$ be a finite closed covering of manifold M. The set $M(i)$ is called cell with index i.

Definition 1 [1, 7]. Symbolic image of the dynamical system $x_{n+1} = f(x_n)$ for covering C is a directed graph G with vertices $\{i\}$ corresponding to cells $\{M(i)\}$. The vertices i and j are connected by the edge $i \to j$ if $f(M(i)) \bigcap M(j) \neq 0.$

Symbolic image is a tool for a space discretization and graphic representation about the global structure of the system dynamics. Symbolic image depends on a covering C. In other words, an edge $i \to j$ is the trace of the mapping $x \to f(x)$, where $x \in M(i)$, $f(x) \in M(j)$. If there isn't an edge $i \to j$ on G then there are not the points $x \in M(i)$ such that $f(x) \in M(j)$.

Definition 2 [1, 7]. A vertex of a symbolic image G is said to be recurrent if there is a periodic path passing through it. The set of recurrent vertices is denoted by RV. The recurrent vertices i and j are called equivalent if there exists a periodic path passing through i and j.

Thus, the set of recurrent vertices RV is split into equivalence classes $\{H_k\}$. In the graph theory such classes are called strong connectivity component.

2. Program implementation of the symbolic image construction algorithm. Consider the following algorithm for constructing a symbolic image.

Algorithm for the construction of a symbolic image

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1. For the initial set M_0, in which the values of a discrete dynamical system lie, we construct a cell cover C. In this particular case, the cells have the shape of a square with a predetermined edge size d_0 (cells can have either an arbitrary shape or an arbitrary size).

2. To cover C, we construct a graph G, such that it is a symbolic image.

3. Using one of the algorithms for finding strongly connected vertices, for example, Tarjan's algorithms [3, 4] or the algorithm for finding strongly connected vertices based on the path search [5], we allocate strongly connected vertices $\{i_k\}$. If $\{i_k\} = \emptyset$, then there is no attractor in M_0 and the localizeable chain-recurrent set is empty and the process of its localization stops [2]. Otherwise, we remove the irrevocable vertices from the graph and move from the M_0 region to the new M_1 region, such that $M_1 = \{x \in M_0^{i_k} : i_k \in G\}$.

4. We build for M_1 a coverage of C_1, so that the edge size of the new cell $d_1 = \frac{d_0}{2}$. Go to step 2 if $d_1 > \varepsilon$, where ε is a predefined limit cell size.

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The problem of software implementation of the algorithm for constructing a symbolic image is based on the theory proposed by G. S. Osipenko [1, 2]. For the implementation of this project, the Python programming language was chosen, which is justified by a wide choice of libraries that are convenient for working with graphs and other mathematical structures, its stability and portability. This language is well suited for the implementation of the task. The OpenGL graphics environment is selected to display the graphics and render the images needed to render the symbolic image. OpenGL (Open Graphics Library) – a specification that defines a platform-independent programming interface for writing applications that use two-dimensional and three-dimensional computer graphics. The OpenGL GLUT library is used. OpenGL Utility Toolkit (GLUT) is a library of utilities for OpenGL applications that is mainly responsible for the system level of I/o operations when working with the operating system. The following functions are used: window creation, window management, monitoring of keyboard input and mouse events. The NetworkX library created specifically for the selected Python programming language is connected. The NetworkX package is designed to create and analyze the study of structures, dynamics and functions of complex networks, and therefore directed graphs. The library includes special data structures for graphs, both directed and non-directed, a lot of standard algorithms on graphs, generators of classical and random graphs, flexible work with vertices and edges, which can be almost any object and have any parameters and open code, which allows you to freely use any algorithms from the library.

To work with arrays of data, the NumPy library was connected. This library provides ample opportunities for working with two-dimensional arrays: a lot of convenient methods for working with arrays, the ability to create large arrays in one line, and more.

Was also used by SciPy. The part relating to differential equations was imported from it. The SciPy package allows you to solve both simple systems and systems of differential equations with parameters. Anaconda was chosen as the development environment. The convenience of working with Anaconda is that most of the widely used Python libraries are already installed in distributions, which greatly simplifies their connection.

As test cases built display Hanona, Zaslavsky, attractors of Peter de Jon etc.

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Operator systems defined via one-parameter unitary groups in the quantum error correction theory

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Keywords: operator systems; non-commutative operator graph; quantum error correcting code; POVM; Jaynes-Cummings model.

MSC2010 codes: 81P73, 47L05

Introduction. The operator system V is the linear subspace of some algebra with involution that is closed under conjugation $A \in V \Rightarrow A^* \in V$ and satisfying $I \in V$. That type of spaces considered a quantum analog of the so-called confusability graphs of communication channels, so these spaces also called non-commutative operator graphs. The main condition defining operator systems interesting for a given application is really tiny, we are interested in operator systems that have the orthogonal projection P of $\mathrm{rank} P \geq 2$ such that $\dim PVP = 1$, given condition is called the Knill-Laflamme condition, projection P and the subspace $\mathrm{Im} P$ are called the quantum anticlique and the quantum error correcting code respectively. **Discussion.** We are interested in the question of how should be generated the graph V allow us to check the Knill-Laflamme condition in a more easy manner. Appears, that in the finite-dimensional case it is possible to give the sufficient condition of existing of quantum error-correcting code for the graphs generated by covariant positive operator-valued measure (POVM), these graphs have the following form

$$V = \mathrm{span}\left\{U_g Q U_g^* \; g \in G\right\}; \tag{1}$$

where U_g is the projective unitary representation of the compact group G and Q is the positive operator.

This technique could be extended on the infinite-dimensional case, it is possible to introduce several examples of the non-commutative operator graphs that generated by unitary representations in the spirit of (1) possessing quantum anticliques, in most of that examples instead of the group $U_g;\ g \in G$ we take the group of operators $U_t = e^{-itH}; t \in \mathbb{R}$ defining the dynamics of some interesting quantum system with the Hamiltonian H. Such examples are given for the two-mode quantum oscillator and for the Jaynes-Cummings model of light-matter interaction, also introduced the example in the one-particle bosonic Fock space.

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On Abstract Friedrichs Systems and their Use in Complex Media

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At the outset we shall discuss a particular problem class, which is closely linked to the classical concept of Friedrichs systems. We shall re-consider Friedrichs systems from a operator theoretic perspective by initially studying operator equations of the form

$$(1 + A)\,U = F;$$

where A is maximal accretive, i.e. $A; A$ accretive. Starting out with the structural assumption that A is an extension of a skew-symmetric operator A, we are interested in describing maximal accretive extensions of A. Complex materials can be addressed by combining such maximal accretive extensions with suitable material laws, which allows to go beyond the classical Friedrichs type systems framework. Such materials are distinguished by resulting in maximal accretive space-time equations. We shall illustrate the setting by inspecting its utility in the context of electrodynamics in metamaterials.

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Remark on the autorepresentation of bounded functions

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Keywords: the Chernoff function; shift operator; the Cauchy problem; the Chebyshev norm; the linear transfer equation; the binomial coefficients; the Fourier transform.

MSC2010 codes: 35A35, 35C99, 35K15, 35K30

Introduction. C_0-semigroup theory is considered to demonstrate hidden connections between different branches of mathematics (see [1] and references therein). For instance in work [2] using semigroup of shifts in $C[0,\infty)$ the following relation has been proved:

$$f(x+t)=\lim_{n\to\infty}\sum_{k=0}^{n}C_n^k t^k(1-t)^{n-k}f(x+k/n),\quad t\in[0,1],\quad x\in[0,\infty),\quad f\in C[0,\infty). \quad (1)$$

This formula is fulfilled uniformly both with respect to x on $[0,\infty)$ and with respect to $t\in[0,1]$. In particular inserting $x=0$ into (1) one obtains the well-known Bernstein's theorem [2].

In the report presented a number of results similar to formula (1) is discussed.

Main result. Let $UC_b(\mathbb{R}^m)$ is the space of all uniformly continuous bounded functions $f:\mathbb{R}^m\to\mathbb{R}$ with the Chebyshev norm $\|f\|=\sup_{\vec{x}\in\mathbb{R}^m}|f(\vec{x})|$ then the following statement is valid:

Theorem. If R is arbitrary positive number and $f\in UC_b(\mathbb{R}^m)$ then:

$$\lim_{n\to\infty}\sup_{|\vec{\xi}|\leq R}\|f-S_n(\vec{\xi})f\|=0, \quad (2)$$

where $|\vec{\xi}|=\sqrt{\xi_1^2+\xi_2^2+\ldots+\xi_m^2}$, $S_n(\vec{\xi})$ is a linear operator acting on functions from $UC_b(\mathbb{R}^m)$ in accordance with the next rule:

$$(S_n(\vec{\xi})f)(\vec{x})=\frac{1}{2^n}\sum_{k=0}^{n}C_n^k f\left(\vec{x}+\frac{2k-n}{n}\vec{\xi}\right), \quad (3)$$

and C_n^k are the binomial coefficients.

Proof. Let one consider the Cauchy problem for the linear transport equation with constant vector of coefficients $\vec{v}\in\mathbb{R}^m$:

$$\frac{\partial u}{\partial t}+\vec{v}\cdot\nabla u=0,\qquad u(\vec{x},0)=u_0(\vec{x}),\qquad \vec{x}\in\mathbb{R}^m, \quad (4)$$

where $\vec{v}\cdot\nabla\equiv v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}+\ldots+v_m\frac{\partial}{\partial x_m}$.

On the one hand it is well-known that exact solution of the problem (4) is equal to:

$$u(\vec{x},t)=\exp(-t\vec{v}\cdot\nabla)\,u_0(\vec{x})\equiv u_0(\vec{x}-\vec{v}\,t). \quad (5)$$

On the other hand if initial condition $u_0\in UC_b(\mathbb{R}^m)$ then all results of theorem 3 of article [3] are true, therefore exact solution of equation (4) can be expressed via the Chernoff function $G(\vec{v}\,t)$ as follows:

$$u(\vec{x},t)=\lim_{n\to\infty}G^n(\vec{v}\,t/n)\,u_0(\vec{x}), \quad (6)$$

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where in correspondence with theorem 2 from article [3] :

$$(G(\sqrt{t})u_0)(x) = \frac{u_0(x) + u_0(x - 2\sqrt{t})}{2}. \tag{7}$$

Using notations from (5) it is easy to extract operator $G(\sqrt{t})$ from formula (7) explicitly:

$$G(\sqrt{t}) = \frac{1 + \exp(-2t\sqrt{(\cdot)}\, r)}{2}. \tag{8}$$

Inserting expression (8) into (6) and comparing the result of this substitution with (5) it is not difficult to derive that:

$$u_0(x - \sqrt{t}) = \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{n} C_n^k u_0\left(x - \frac{2k}{n}\sqrt{t}\right). \tag{9}$$

At last one can achieve the statement of the theorem by renaming variables and initial condition as follows: $x - \sqrt{t} \mapsto x$, $u_0(x) \mapsto f(x)$ and $\sqrt{t} \mapsto \xi$.

Setting $x = 0$ in formula (9) and renaming after that in this formula $-\sqrt{t}$ as ξ one can establish that the theorem possesses by the next

Corollary. Let $f \in UC_b(\mathbb{R}^m)$ then:

$$\lim_{n \to \infty} \| f - \Lambda_n f \| = 0, \tag{10}$$

where Λ_n is a linear operator acting on functions from $UC_b(\mathbb{R}^m)$ as follows:

$$(\Lambda_n f)(x) = \frac{1}{2^n} \sum_{k=0}^{n} C_n^k f\left(\frac{2kx}{n}\right). \tag{11}$$

Discussion. The proved theorem means that any function $f \in UC_b(\mathbb{R}^m)$ can be represented as linear combination of quite large number of some shifts of itself with preassigned accuracy. For example if $f(x) = (1 + |x|^2)^{-1}$ then formulas (2) and (3) are reduced to:

$$\frac{1}{1 + |x|^2} = \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{n} \frac{n^2 C_n^k}{n^2 + |nx + (2k - n)\xi|^2}. \tag{12}$$

The corollary means that any function $f \in UC_b(\mathbb{R}^m)$ can be represented as linear combination of quite large number of its stretched copies. In particular formulas (10) and (11) applied to the same function give one:

$$\frac{1}{1 + |x|^2} = \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{n} \frac{n^2 C_n^k}{n^2 + 4k^2|x|^2}. \tag{13}$$

Remark 1. If $\omega \in \mathbb{R}^m$ then it is easy to check by direct calculation that functions $f_1(x) = \cos(\omega \cdot x)$ and $f_2(x) = \sin(\omega \cdot x)$ obey both the theorem and the corollary. Moreover these functions are eigenfunctions of operator $S_n(\xi)$ with twice degenerate eigenvalue $(\cos(\omega \cdot \xi/n))^n$.

Remark 2. If function $f \in UC_b(\mathbb{R}^m) \bigcap L_1(\mathbb{R}^m)$ then one can consider its Fourier transform:

$$\mathrm{F}[f](\vec{q}) \equiv \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp(-i\vec{q} \cdot x) f(x)\, d^m x, \tag{14}$$

where $\vec{q} \cdot x \equiv q_1x_1 + q_2x_2 + \ldots + q_nx_n$.

Applying integral transform (14) to formula (3) one can find that:

$$\mathcal{F}[S_n(\tau)f](\xi) = \left(\cos\frac{\xi\,\tau}{n}\right)^n \mathcal{F}[f](\xi). \tag{15}$$

If $f(0) = 0$ then the same operation applied to formula (11) gives one that:

$$\mathcal{F}[\Phi_n f](\xi) = \frac{n^m}{2^{n+m}} \sum_{k=1}^{n} \frac{C_n^k}{k^m} \mathcal{F}[f]\left(\frac{n\xi}{2k}\right). \tag{16}$$

It is obvious that any consequences from formula (15) are trivial. But there are nontrivial consequences from formula (16).

Application of other kinds of integral transforms to formulas (3) and (11) is very perspective too.

Remark 3. There is the following relation between the Chernoff function and the linear operator (3):

$$S_n(\tau) = \exp(\tau r)\, G^n(\tau/n).$$

Remark 4. It is interesting to estimate rate of decreasing of limits in formulas (2) and (10) in the framework of technique developed in theorem 1 of paper [4].

Remark 5. It is clear that all results of this work can be extended on the space $UC_b(\mathbb{R}^m;\mathbb{C})$ of all uniformly continuous bounded functions $f : \mathbb{R}^m \to \mathbb{C}$ with the Chebyshev norm.

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On final dynamics of semilinear parabolic equations

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Keywords: reaction–diffusion–convection equations, finite-dimensional dynamics on attractor.

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The question on selecting a finite number of "determining" degrees of freedom of an infinitely dimensional dynamical system is of a great interest. We consider [1] the class of one-dimensional dissipative reaction-diffusion-convection systems and obtain conditions under which the final (at large times) phase dynamics of a system can be described by an ODE with Lipschitz vector field in R^N. Precisely in this class, the first example [2] of a parabolic problem of mathematical physics without the indicated property was recently constructed.

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N-soliton solutions in the problem of Rogue waves

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Keywords: solitons; breathers; inverse scattering transform; extreme events; rogue waves.

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The rogue wave problem was originally related to abnormally high oceanic waves which occurred suddenly on the sea surface, seemingly without clear precursors, and had resulted in a number of fatalities including ship sinking, damages of off-shore platforms, coastal constructions and large ships; people deaths [1]. Shortly later rogue waves became a hot topic of research in optics. These waves represent electromagnetic fields of extreme intensity which can breakdown the fiber in data transmission lines, when occur accidentally [2]. In the recent time, kindred researches commenced in application to other physical realms [3].

The rogue wave studies in diverse physical fields have much in common. A significant part of the research is developing *nonlinear* rogue wave models assuming that the corresponding dynamics is due to fast unstable nonlinear effects, when the wave intensity / energy gets concentrated within some location and time interval. For example, the nonlinear cubic Schrödinger (NLS) equation is the first-order approximation to the wave processes on the surface of a deep sea, and also to the waves of light in optical fibers. The quadratic in nonlinearity Korteweg – de Vries (KdV) equation describes water waves under the shallow water condition, while its modified version with the cubic nonlinearity is applicable to the nonlinear optical fibers.

Remarkably, the mentioned above equations, the nonlinear Schrödinger equation, and the two kinds of the Korteweg – de Vries equations (and also the extended, quadratic-cubic KdV, called the Gardner equation), are completely integrable by the Inverse Scattering Transform (IST) nonlinear partial differential equations [4,5]. They all possess soliton solutions which correspond to the nonlinear limit of stationary waves (or wave groups) of the permanent shape. Due to the nonlinear nature, the solitons may exhibit drastically different dynamical and probabilistic properties compared to linear waves, hence they have received much attention in the context of the rogue wave studies, and have inspired a new branch of the research in the field of mathematics. Conventionally, *rogue wave solutions* in mathematics are the ones which demonstrate fast localized growth of the solution; they often appear as a result of small perturbation of a uniform solution, but behave qualitatively similar within some range of the perturbation parameters (the issue of robustness [6]).

Within the IST solitons correspond to the discrete spectrum of the associated scattering problem, while the continuous spectrum specifies the other (small-amplitude) waves. Since the associated scattering problem is isospectral, the spectrum may be used to characterize the wave system at any instant. The IST may be understood as a nonlinear generalization of the Fourier transform. The limit of purely discrete spectrum is often considered as a reasonable problem statement. Fortunately, in this case the machinery of the IST is significantly simpler and allows some analytic results.

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The concept of *soliton gas*, i.e., ensembles of solitons with irregular parameters (such as amplitudes and locations) represents the counterpart of the traditional paradigm of independent linear waves, which leads to the Gaussian statistics. In particular, the stochastic sea wave dynamics in many cases may be represented as a linear superposition of linear Fourier modes (the Gaussian sea), or of nonlinear (Stokes) modes. However, the recent advances in the rogue wave problem suggest that nonlinear interactions between the modes and coherent soliton-like structures may be the main reason of occurrence of abnormally high waves in the ocean.

Since we are interested in the statistical properties of the waves, the application of kinetic equations for solitons [7], which describe the transport of the soliton density, is limited. The direct numerical simulation of soliton ensembles is a popular efficient tool to consider the problem when solitons interact occasionally. However, the simulation of simultaneous interaction between many soliton is a strongly nonlinear process, even within formally weakly nonlinear frameworks. It represents a heavy problem to the direct numerical simulation [8]. These very rare events may correspond to the appearance of extremely large waves. Exact analytic solutions to integrable equations help to investigate the problem systematically.

The general form of a multi-soliton / multi-breather solution of the integrable KdV-type equations, which is even with respect to any variable when one of the variables is put equal to zero, was derived and analyzed in [9,10]. Physically, this solution corresponds to the most synchronized wave sequence (focusing of solitons). It was shown that the focused wave increases in amplitude only when sign-alternating solitons interact. Hence, unipolar KdV-type solitons do not exhibit extreme dynamics. This conclusion was also confirmed in direct numerical simulations, e.g. [11].

Let us focus on the example of the classic Korteweg – de Vries (KdV) equation, which may be presented in the standard dimensionless form as follows,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where the solution $u(x,t) \in \Re$ describes the perturbation, $x \in (-\infty, \infty)$ has the meaning of the spatial coordinate, and $t \in (-\infty, \infty)$ is the time. The dynamics of irregular waves within (1) (waves of the discrete and continuous spectra) was simulated numerically in [12], where the regimes of strongly non-Gaussian statistics were found. The effect of two-soliton interactions on the statistical properties of the solution was analyzed analytically in [13]. The evolution of rarefied soliton gas was modeled numerically in a number of publications, e.g. [11].

The multisoliton solution of (1) can be obtained with the help of the Darboux transformation

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \ln W_N(\psi_1, \psi_2, \ldots, \psi_N), \tag{2}$$

where W_N is the Wronskian of N so-called seed functions $\psi_j(x,t)$, $j = 1, \ldots, N$ (see e.g. in [10]),

$$\psi_j = \cosh k_j(x - 4k_j^2 t - x_j), \quad \text{if } j \text{ is odd}, \tag{3}$$

$$\psi_j = \sinh k_j(x - 4k_j^2 t - x_j), \quad \text{if } j \text{ is even}.$$

The parameters $k_j > 0$ specify N solitons with amplitudes $A_j = 2k_j^2 > 0$; x_j are constants which determine locations of the solitons. Though the solution (2) is explicit, its computation is technically difficult when N is large.

The use of extra-high precision of the numerical subroutine (similar to [14]) helps to construct the multisoliton solutions with larger N. We have implemented this approach to calculate the solution (2) to the KdV equation (1), using typically the mantissa of the length of 100 digits. The solution (2) may be used for constructing dense soliton gas within a finite interval. The generated gas states may possess the property of uniformity within some shorter interval if the density is not too large [15]. We also suggest the physically consistent way to calculate the soliton density function explicitly using the Darboux transformation approach. Then the spectral moments $\mu_n = \langle u^n \rangle$, $n = 1, 2, \ldots$, where the angle brackets mean ensemble averaging, may be calculated.

We particularly consider the situation of synchronous collisions of KdV solitons, which corresponds to the choice $x_j = 0$, $j = 1, \ldots, N$ in (3). Then the solution (2) is characterized by the symmetries $u(x, 0) = u(-x, 0)$ and $u(0, t) = u(0, -t)$. We choose for certainty the particular distribution of the soliton amplitudes $A_j = 1/d^{j-1}$, $j = 1, \ldots, N$, and vary the constant $d > 1$. The appearance of the evolving solution, and the evolution of the third and the fourth statistical moments, μ_3, μ_4, have been examined. The value $d = 3$ corresponds to the marginal situation in the case $N = 2$, when the exchange and overtaking scenarios of the soliton interaction change; however it does not correspond to a peculiar solution when $N > 2$.

Though we consider solutions when the number N is always limited, the obtained solution is able to approximate the limit $N \to \infty$ in some range of parameters, what allows us direct examination of this ultimate limit of soliton interactions.

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Derivations on Murrey-von Neumann algebras

F. Sukochev[144]

This talk presents an account of joint work with Aleksey Ber, Jinghao Huang and Karimbergen Kudaybergenov, whose large part can be found in the paper, *Notes on derivations of Murray–von Neumann algebras*, *Journal of Functional Analysis,* 279 (2020), no. 5, 108589, 26 pp (by A. Ber, K. Kudaybergenov, F. Sukochev).

Recall, that a fat Cantor set E in $[0,1]$, is nowhere dense (in particular it contains no intervals), yet has positive Lebesgue measure. Further, it is well-known that its characteristic function χ_E is approximately differentiable but nowhere differentiable function. Firstly, we explain the description of the algebra of all approximately differentiable functions on $(0,1)$ (which is probably well known to the experts) and then introduce the algebra of all approximately differentiable operators affiliated with the hyperfinite II_1 factor (introduced by von Neumann in 1930's). To this end, I shall briefly explain major results/notions concerning derivations on algebras of unbounded operators.

Next, I shall explain that the classical differential operator $\partial : D(0,1) \mapsto S(0,1)$ acting from the algebra of all differentiable functions on $(0,1)$ into the algebra of all Lebesgue measurable functions allows an extension to the differential operator on the algebra $S(0,1)$. However, any such extension cannot be translation-invariant, which is in the strong contrast with the properties of ∂.

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Central Limit Theorem for generalized measures in a multidimensional space

M. O. Uskov[145]

One of the important generalized measures is the so called Feynman measure. Using this measure mathematicians defined the Feynman path integral. This generalized measure can not be identified with any sigma-additive Borel measure. Below we regard an approach to approximation of the Feynman type measure which was invented by O. G. Smolyanov. In the case of locally convex topological vector space the Feynman measure can be approximated by sigma-additive complex measures as shown in [1] without a detailed proof. In this talk we'll provide a sequence of sigma-additive complex measures which converges in some sense to a Feynman type measure on a Euclidean space. The formula is similar to the Remizov's one with operators.

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Homogenisation and the Weak Operator Topology

M. Waurick[146]

It is well-known that the description of one-dimensional elliptic homogenisation problems is parallel to the study of weak* convergence for the considered inverses of the L_1-coefficients. In an operator framework this is equivalent to the convergence of the inverses in the weak operator topology. In the talk we provide a higher-dimensional version of this perspective. More precisely, we characterise the convergence of the coefficients for elliptic homogenisation problems in terms of certain weak-type topologies. The approach is based on complexes of densely defined and closed operators in Hilbert spaces. The thus introduced topology is not comparable to the weak operator topology but strictly weaker than the strong operator or norm topology.

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