# Robin Hudson

(1940– 2021)

Canonical Pairs $\qquad [q, p] = i\hbar \qquad (\hbar = 2)$

Stone–v.Neumann Theorem $\qquad$ Unique $\sim (\hat{x}, -2i\partial)$ on $L^2(\mathbb{R}, dx)$

Schrödinger Rep.

$$\begin{cases} \hat{x} f(x) = x f(x) \\ \partial f(x) = f'(x). \end{cases}$$

Creation / Annihilation $\qquad [a, a^*] = 1$

$$q = a + a^*$$

$$p = \frac{1}{i}(a - a^*)$$

Number $\qquad N = a^* a \geq 0$

spectrum $N = \{0, 1, 2, 3, \dots\}$.

$$\mathbb{E}[\,f(q,p)\,] \equiv \mathrm{tr}\left[\rho\, f(\hat{x}, -2i\partial)\right] \text{ on } L^2(\mathbb{R})$$

In the Schrödinger rep.

$$|n\rangle_{\text{number}} \;—\; n^{\text{th}} \text{ Hermite function}$$

$$N|n\rangle_{\text{number}} = n\,|n\rangle_{\text{number}}$$

$$a^*|n\rangle_{\text{number}} = \sqrt{n+1}\,|n+1\rangle_{\text{number}}$$

$$a\,|n\rangle_{\text{number}} = \begin{cases} \sqrt{n}\,|n-1\rangle_{\text{number}} & (n \geq 1); \\ 0, & n = 0. \end{cases}$$

Exponential vectors $(\alpha \in \mathbb{C})$

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle_{\text{number}},$$

$$\langle\alpha|\beta\rangle = e^{\alpha^*\beta}.$$

$$a|\alpha\rangle = \alpha|\alpha\rangle.$$

In the groundstate $|0\rangle = |0\rangle_{\text{number}}$

* $q, p$ are Gaussian $(\mu = 0, \sigma = 1)$.

In the state $\mathbb{E}_\alpha[\,\cdot\,] = \dfrac{\langle \alpha | \cdot | \alpha \rangle}{\langle \alpha | \alpha \rangle}$

* $q, p$ are Gaussian $(\mu = 2\mathrm{Re}/\mathrm{Im}\,\alpha, \sigma = 1)$

* $N$ is Poissonian (intensity $|\alpha|^2$)

$$e^{i\theta N} |\alpha\rangle = \sum_n e^{i\theta n} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_{\text{number}}$$

$$= |e^{i\theta} \alpha\rangle,$$

so $$\mathbb{E}[e^{i\theta N}] = \frac{e^{\alpha^*(e^{i\theta}\alpha)}}{e^{\alpha^*\alpha}} = e^{|\alpha|^2(e^{i\theta} - 1)}.$$

! The most important prob. laws (continuous + discrete) are already there in undergraduate quantum physics !

Wigner's Quasi-probability: $\mathcal{W}_\psi$

$$\langle \psi | e^{iuq + ivp} | \psi \rangle$$

$$\equiv \int_{\mathbb{R}^2} e^{iux + ivy} \, \mathcal{W}_\psi(x,y) \, dx \, dy$$

Hudson's Theorem (1974)

$$\mathcal{W}_\psi \geq 0 \iff \psi \text{ is Gaussian.}$$

i.e. $\psi(x) = e^{-ax^2 + bx + c}$

for some $a, b, c \in \mathbb{C}$,

$\operatorname{Re} a > 0$.

# Quantum Central Limit Theorem

Let $(q_k, p_k)_{k=1}^{\infty}$ be an i.i.d. sequence of canonical pairs.

Independent?

$$\mathbb{E}[f(q_1, p_1, q_2, p_2)] = \mathrm{tr}\left[\rho_{12} \; f(x_1, -2i\partial_1, x_2, -2i\partial_2)\right]$$

then $\rho_{12} = \rho_1 \otimes \rho_2$ on $\underline{L^2(\mathbb{R}, dx_1)} \otimes \underline{L^2(\mathbb{R}, dx_2)}$

Let
$$Q_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} q_k, \quad P_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} p_k$$

then
$$[Q_n, P_n] = 2i.$$

Suppose $\mathbb{E}[q_n] = \mathbb{E}[p_n] = 0$,

$\mathbb{E}[q_n^2] = \mathbb{E}[p_n^2] = \sigma^2$, $\mathbb{E}[q_n p_n + p_n q_n] = 0.$

Theorem (Cushen + Hudson, 1971)

$$\lim_{n\to\infty} \mathbb{E}\left[e^{iuQ_n + ivP_n}\right] = \tilde{\mathbb{E}}\left[e^{iu\tilde{Q}+iv\tilde{P}}\right]$$

* $[\tilde{Q}, \tilde{P}] = 1$

* $\sigma = 1$, take Schr. rep on $L^2(\mathbb{R})$
  $\tilde{\mathbb{E}}$ – groundstate $|0\rangle$

* $\sigma > 1$, take $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$

$$\tilde{Q} = \sqrt{\frac{\sigma^2+1}{2}}\, q \otimes 1 + \sqrt{\frac{\sigma^2-1}{2}}\, 1 \otimes q$$
$$\tilde{P} = \sqrt{\frac{\sigma^2+1}{2}}\, p \otimes 1 + \sqrt{\frac{\sigma^2-1}{2}}\, 1 \otimes p$$

and $\tilde{\mathbb{E}}$ – $|0\rangle \otimes |0\rangle$.

N.B. $\sigma^2 = \Delta q\, \Delta p \geq 1$ (HUP).

# Quantum Itō Calculus

(Фок)
Fockspace

Bosons!

$$\Gamma(h) = (\mathbb{C}|\Omega\rangle) \oplus h \oplus \Big(\bigotimes_{\text{symm.}}^{2} h\Big) \oplus \cdots$$

$$|f\rangle = \Omega \oplus (f) \oplus \left(\tfrac{1}{\sqrt{2!}} f \otimes f\right) \oplus \cdots$$

* Annihilator

$$A(g)|f\rangle = \langle g|f\rangle \; |f\rangle$$

* Creator

$$A(g)^{*}|f\rangle = \frac{\partial}{\partial u} |f + ug\rangle \Big|_{u=0}$$

* Conservor/ Gauge/ Number

$$\Lambda(H)|f\rangle = \frac{1}{i}\frac{\partial}{\partial u}\left| e^{iHu} f\right\rangle\Big|_{u=0}$$

$$[A(f), A^*(g)] = \langle f | g \rangle$$

Let $h = L^2(\mathbb{R}_+, dt)$

$$A_t = A(1_{[0,t]})$$
$$A_t^* = A^*(1_{[0,t]})$$
$$\Lambda_t = \Lambda(\mathbb{I}_{[0,t]})$$

Quantum Itō Table

$$dA_t \, dA_t^* = dt$$
$$dA_t \, d\Lambda_t = dA_t$$
$$d\Lambda_t \, dA_t^* = dA_t^*$$
$$d\Lambda_t \, d\Lambda_t = d\Lambda_t$$

Hudson + Parthasarathy (1984)

on $h \otimes \Gamma(L^2(\mathbb{R}_+, dt))$

$$dU_t = \Big[ (S-1) \otimes d\Lambda_t + L \otimes dA_t^* - LS^* \otimes dA_t - (\tfrac{1}{2} L^* L + iH) \otimes dt \Big] U_t$$

$$U_0 = I \otimes I .$$

$(S, L, H)$ in $\mathcal{B}(h)$

$S^{-1} = S^*$, $H = H^*$.

$(U_t)_{t \geq 0}$ extends to a unitary

$$\langle u \otimes \Omega | \, U_t^* (X \otimes 1) U_t \, | v \otimes \Omega \rangle$$

$$\equiv \langle u | \, \Phi_t(X) \, v \rangle , \quad \forall u, v \in h$$

$\Phi_t$ a CP semigroup with generator

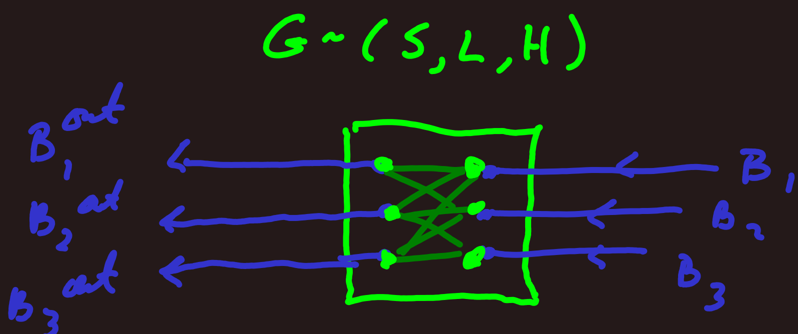
$$\mathcal{L}(X) = \tfrac{1}{2} [L^*, X] L + \tfrac{1}{2} L^* [X, L] - i [X, H] .$$

# Applications

$$h \otimes \Gamma(\mathbb{C}^n \otimes L^2(\mathbb{R}_+))$$

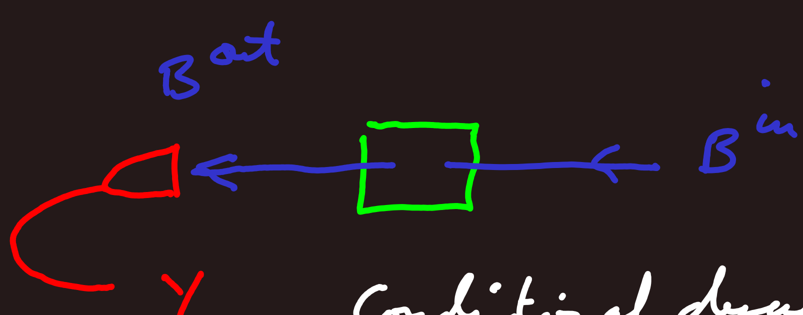
$$B(t) = \begin{bmatrix} B_1(t) \\ \vdots \\ B_n(t) \end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} \Lambda_{11}(t) & \cdots & \Lambda_{1n}(t) \\ \vdots & & \vdots \\ \Lambda_{n1}(t) & \cdots & \Lambda_{nn}(t) \end{bmatrix}$$

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, \quad H = H^*.$$

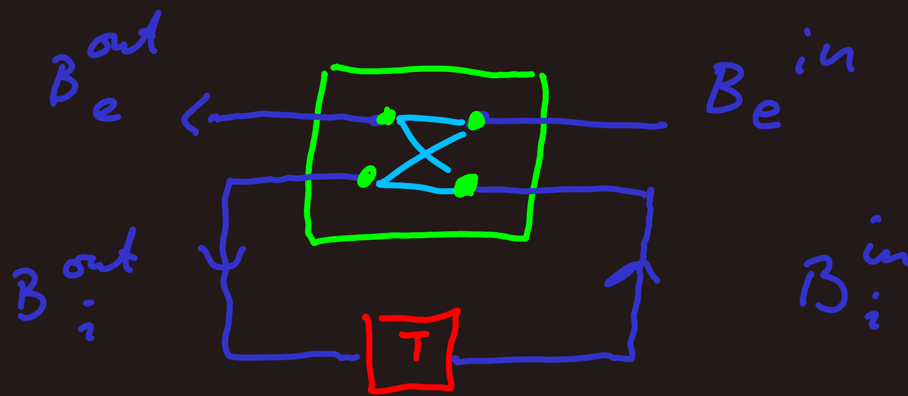



Input - Output.

$$B_k^{out}(t) = U_t^*(I \otimes B_k(t))\, U_t$$




Conditional dynamics of the system due to output measurement




$$S = \begin{bmatrix} S_{ee} & S_{ei} \\ S_{ie} & S_{ii} \end{bmatrix}$$

$$L = \begin{bmatrix} L_e \\ L_i \end{bmatrix}$$

$$\begin{cases} S_{fb} = S_{ee} + S_{ei}\, T\, (1 - S_{ii} T)^{-1} S_{ie} \\ L_{fb} = L_e + S_{ei}\, T (1 - S_{ii} T)^{-1} L_i \\ H_{fb} = H + \sum\limits_{\alpha = e, i} \operatorname{Im} L_\alpha^* S_{\alpha i} T (1 - S_{ii} T)^{-1} L_i \end{cases}$$

→ Complete Formalism for open quantum input-output models

* interconnection
* feedback
* conditional evolution (filtering)
* Control / System Ident.