One Parameter Semigroups of Operators (OPSO 2021)

TROTTER-KATO THEOREMS FOR QUANTUM MARKOV SEMIGROUPS

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Closed Quantum Evolutions

System has a finite dimensional* Hilbert space $\mathfrak{h}$ with observables being the self-adjoint operators.

Should have put this in the talk! :-(

Schrödinger picture: continuous unitary groups U_t of operators on $\mathfrak{h}$ with generators $-iH$.

Heisenberg picture: continuous group of automorphisms ϕ_t on $\mathcal{B}(\mathfrak{h})$:

$$\phi_t(X) = U_t^* X U_t,$$

with generator

$$\mathcal{L}(\cdot) = -i[\cdot, H].$$

* We will drop this later!

Open Quantum Evolutions

- Example (Wiener Noise)

$U(t) = e^{-iRW(t)}$ where $R = R^*$ and $W(t)$ a Wiener process.

$$dW(t)\,dW(t) = dt.$$

Continuous contraction semigroups $\phi(t)$ $(t \geq 0)$:

$$\phi(t)\,X = \mathbb{E}[U(t)^* X U(t)],$$

with generator

$$\mathcal{L}(\cdot) = -\frac{1}{2}\big[[\cdot, R], R\big].$$

Open Quantum Evolutions

- Example (Poisson Noise)

$U(t) = S^{N(t)}$ where S is unitary and $N(t)$ a Poisson process.

$$dN(t)\, dN(t) = dN(t), \qquad \mathbb{E}[dN(t)] = \nu dt.$$

Continuous contraction semigroups $\phi(t)$ $(t \geq 0)$:

$$\phi(t)\, X = \mathbb{E}[U(t)^* X U(t)],$$

with generator

$$\mathcal{L}(X) = \nu(S^* X S - X).$$

Complete Positivity (CP)

A map $\phi : \mathcal{B} \mapsto \mathcal{B}$ and a positive integer n we define its extension $\phi \otimes id_n$ to the algebra $\mathcal{B} \otimes M_n$, where M_n is the algebra of $n \times n$ matrices, as

$$\phi \otimes id_n \left(\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \right) = \begin{bmatrix} \phi(x_{11}) & \cdots & \phi(x_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(x_{n1}) & \cdots & \phi(x_{nn}) \end{bmatrix}.$$

We say that ϕ is n-**positive** if $\phi \otimes id_n$ is positive.

The map is **completely positive (CP)** if $\phi \otimes id_n$ is positive for all n.

Thanks to JML! :)

Dilations of CP Semigroups

Let $\phi(t) : \mathcal{B} \mapsto \mathcal{B}$ be CP semigroup. A **unitary dilation** is given by finding a second Banach space $\mathcal{A}$ such that (for every $x \in \mathcal{B}$ and $\rho \in \mathcal{B}_\star$)

$$\langle \rho, \phi(t) X \rangle = \langle \rho \otimes \sigma, U_t^* (X \otimes I_{\mathcal{A}}) U_t \rangle,$$

for σ a positive normalized element of $\mathcal{A}$ and U_t a unitary family on $\mathcal{B} \otimes \mathcal{A}$.

If $\mathcal{A}$ is *commutative*, then the most general class of CP semigroups we may obtain is (at the level of generator) Hamiltonian plus Wiener plus Poissonian [Maassen and Kummerer, 1987].

They take $\mathcal{A} = M_n$.

Their result generalizes to general vN algebras – but $\phi(t)$ only pointwise w^*-continuous.

Hudson-Parthasarathy Dilations

We take $\mathcal{A} = \mathcal{B}(\mathfrak{F})$ where $\mathfrak{F}$ is the Fock space over $L^2(\mathbb{R}_+, dt)$.

The **exponential vectors** on $\mathfrak{F}$ are defined by

$$|e^f\rangle = |\Omega\rangle \oplus \bigoplus_{n=1}^{\infty} \left(\frac{1}{\sqrt{n!}} f^{\otimes n} \right), \qquad f \in L^2(\mathbb{R}_+, dt).$$

We have $\langle e^f | e^g \rangle = e^{\langle f, g \rangle}$. They are total in $\mathfrak{F}$. The **annihilation process** is defined by

$$B_t |e^f\rangle = \left(\int_0^t f(\tau) d\tau \right) |e^f\rangle.$$

*separable Hilbert space possibly infinite dimensional.

Hudson-Parthasarathy Dilations

Quantum Ito table:

$$dB_t dB_t^* = dt,\ dB_t d\Lambda_t = dB_t,\ d\Lambda_t dB_t^* = dB_t^*,\ d\Lambda_t d\Lambda_t = d\Lambda_t.$$

Triple (S, L, H) in $\mathcal{B}(\mathfrak{h})$ with* $H = H^*$ and S unitary. The QSDE

$$dU_t = \Big\{(S - I) \otimes d\Lambda_t + L \otimes dB_t^* - L^* S \otimes dB_t - (\frac{1}{2} L^* L + iH) \otimes dt\Big\} U(t),$$

$U(t) = I$, admits a unique solution, which is a strongly continuous one-parameter **co-cycle** with respect to the time shift on the Fock space.

Hudson-Parthasarathy Dilations

With the choice state σ corresponding to the Fock vacuum vector, we get the strongly continuous CP semigroup with generator

$$\mathcal{L}(\cdot) = \frac{1}{2}[L^*, \cdot]L + \frac{1}{2}L^*[\cdot, L] - i[\cdot, H].$$

This is the Lindblad-Gorini-Kossakowski-Sudarshan generator.

This includes the essentially classical cases as special cases. Indeed,

$$B_t + B_t^* \equiv W_t, \qquad \Lambda_t + B_t^* + B_t + t \equiv N_t.$$

Damped Cavity Mode

Cavity mode a, CCR $[a, a^*] = 1$.

Open model on $\mathfrak{h}_{\text{cavity}} \otimes \mathfrak{F}$ with

$$S = I, \quad L = \sqrt{\gamma}a, \quad H = \omega a^* a,$$

quantum OU-process!

Dilated evolution $a_t = U_t^*(a \otimes I_{\mathfrak{F}})U_t$

$$da_t = -(\gamma/2 + i\omega)a_t dt + \sqrt{\gamma} dB_t.$$

We have $[a_t, a_t^*] = 1$, but $\Phi_t(a) = e^{-(\gamma/2+i\omega)t}a$.

Singular Perturbations (Classical/Linear)

x_s - slow, x_f - fast!

$$\begin{aligned} \dot{x}_s &= A_{ss}x_s + A_{sf}x_f + B_s u \\ \epsilon\dot{x}_f &= A_{fs}x_s + A_{ff}x_f + B_f u \end{aligned}$$

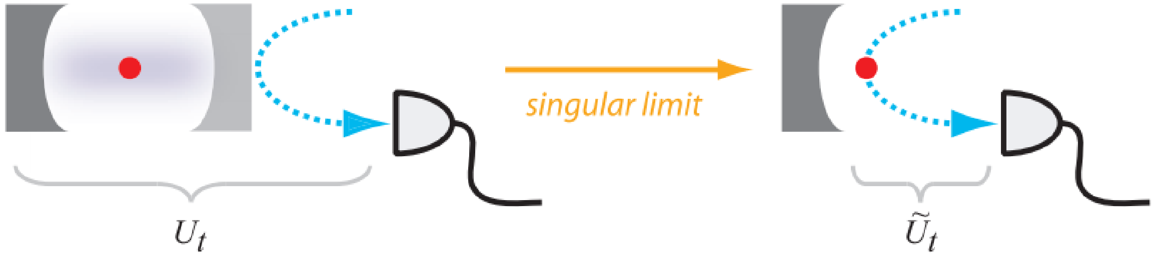
$$y = C_s x_s + C_f x_f + Du.$$

$$\epsilon \to 0, \qquad x_f = -\frac{1}{A_{ff}}(A_{fs}x_s + B_f u).$$

$$\dot{x}_s = (A_{ss} - A_{sf}\frac{1}{A_{ff}}A_{fs})x_s + (B_s - A_{sf}\frac{1}{A_{ff}}B_f)u$$

$$y = (C_s - CA_f\frac{1}{A_{ff}}A_{fs})x_s + (D - C_s\frac{1}{A_{ff}}B_f)u.$$

A Physical Model



System $=$ Atom $+$ Cavity mode

Cavity mode a, CCR $[a, a^*] = 1$.

$$H_{\text{atom/cavity}} = G \otimes a^* a - i(F \otimes a^* - F^* \otimes a)$$

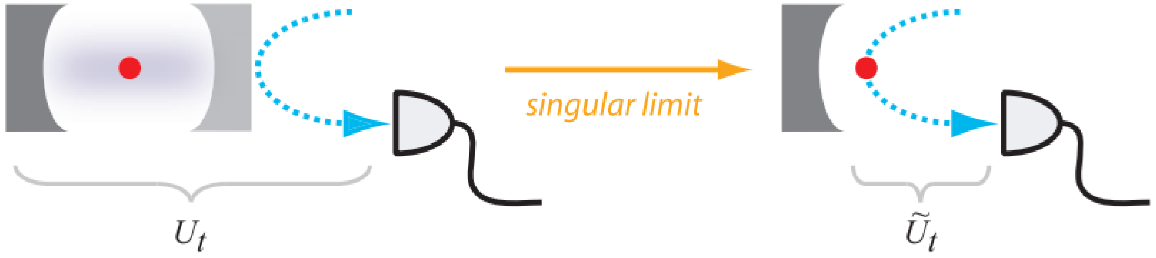
$L = I \otimes \sqrt{\gamma} a$ (only the cavity mode couples to the environment)

$$da_t = -\frac{1}{2}\gamma a_t dt - iG_t a_t dt + F_t dt + \sqrt{\gamma} dB_t$$

Singular limit $da_t = 0$, rearrange

$$a_t dt = \frac{1}{\gamma/2 + iG}\left(F_t dt - \sqrt{\gamma} dB_t\right).$$

A Physical Model



$$S^{(n)} = I \otimes I$$

$$L^{(n)} = n\, I \otimes \sqrt{\gamma} a$$

$$H^{(n)}_{\text{atom/cavity}} = n^2 E_{11} \otimes a^* a + n E_{10} \otimes a^* + n E_{01} \otimes a + E_{00}$$

As $n \to \infty$ the unitary process/Heisenberg evolution converges to the reduced model on $\mathfrak{h}_{\text{atom}} \otimes \mathfrak{F}$ with

$$\tilde{S} = \frac{\gamma/2 - iE_{11}}{\gamma/2 + iE_{11}}, \quad \tilde{L} = \frac{i\sqrt{\gamma}}{\gamma/2 + iE_{11}} E_{01}, \quad \tilde{H} = E_{00} + \text{Im} E_{01} \frac{1}{\gamma/2 + iE_{11}} E_{10}.$$

JG, R. van Handel, J. Stat. Phys. 127, 575-607 (2007)

Trotter-Kato Theorem (Hilbert Space)

Strongly continuous contraction semigroups $V_n(t)$ $(t \geq 0)$ on Hilbert space $\mathfrak{h}$ with generators $\mathcal{L}_n$.

Strongly continuous contraction semigroup $V(t)$ $(t \geq 0)$ on closed Hilbert subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ with generator $\mathcal{L}$ with core $\mathcal{D}_0$.

The following are equivalent:

- $\forall \phi \in \mathfrak{h}_0, T > 0, \qquad \lim_{n\to\infty} \sup_{0\leq t\leq T} \|V_n(t)\phi - V(t)\phi\| = 0.$
- $\forall \phi \in \mathcal{D}_0,$ there exists $\phi_n \in \mathrm{dom}(\mathcal{L}_n)$, such that $\phi_n \to \phi, \quad \mathcal{L}_n\phi_n \to \mathcal{L}\phi.$

Trotter-Kato Theorem (Banach Space)

Strongly continuous contraction semigroups $V_n(t)$ $(t \geq 0)$ on Banach space $\mathfrak{B}$ with generators $\mathcal{L}_n$.

Strongly continuous contraction semigroup $V(t)$ $(t \geq 0)$ on closed Banach subspace $\mathfrak{B}_0 \subset \mathfrak{B}$ with generator $\mathcal{L}$ with core $\mathcal{D}_0$.

The following are equivalent:

- $\forall X \in \mathfrak{B}_0, T > 0, \qquad \lim_{n\to\infty} \sup_{0\leq t\leq T} \|V_n(t)X - V(t)X\| = 0.$
- $\forall X \in \mathcal{D}_0,$ there exists $X_n \in \mathrm{dom}(\mathcal{L}_n)$, such that $X_n \to X, \quad \mathcal{L}_n X_n \to \mathcal{L}X.$

Trotter-Kato Applied to Quantum Stochastic Models

- **T. Kurtz.** A limit theorem for perturbed operator semigroups with applications to random evolutions. *J. Funct. Anal., 12:55–67, (1973)*

- **L. Bouten, A. Silberfarb,** *Commun. Math. Phys. 283, 491-505 (2008)*
- **L. Bouten, R. van Handel, A. Silberfarb,** *Journal of Functional Analysis 254 3123-3147 (2008)*
- **L. Bouten, R. van Handel**, *J. Math. Phys. 49, 102109 (2008).*
- **JG, M.R. James,** *Commun. Math. Phys., 287 (3), 1109-1132 (2009)*
- **JG, H.I. Nurdin,** *Phil. Trans. Roral Soc. (2012)*
- **L. Bouten, R. Gohm, JG, H. Nurdin,** *European Journal of Quantum Technology, Vol.2, 9 (2015)*
- **C. Catana, L. Bouten, M. Guta,** *Journal of Physics A: Math. and Theor., 48(36), (2015)*
- **L. Bouten, JG,** *Journal of Mathematical Physics 60, 043501 (2019)*

Trotter-Kato (QSDE)

Set $V = U^*$, and

$$dV(t) = V(t)\left\{G_{11} \otimes d\Lambda_t + G_{10} \otimes dB_t^* + G_{01} \otimes dB_t + G_{00} \otimes dt\right\}, \quad V(0) = I,$$

and assume that there is a unique cocycle as solution extending to a unitary on $\mathfrak{h} \otimes \mathfrak{F}$.

For $\alpha, \beta \in \mathbb{C}$, set

$$K^{\alpha,\beta} = \alpha^*(S^* - I)\beta + L^*\beta - \alpha^* S^* L - (\frac{1}{2}L^*L - iH).$$

Trotter-Kato (QSDE)

L. Bouten, R. van Handel, A. Silberfarb,
Journal of Functional Analysis 254 3123-3147 (2008)

Strongly continuous unitary cocycles $V_n(t)$ $(t \geq 0)$ on $\mathfrak{h} \otimes \mathfrak{F}$.

Strongly continuous unitary cocycle $V(t)$ $(t \geq 0)$ on $\mathfrak{h}_0 \otimes \mathfrak{F}$, with $\mathcal{D}_0$ a core for the $K_{\alpha,\beta}$.

The following are equivalent:

- $\forall \Phi \in \mathfrak{h}_0 \otimes \mathfrak{F}, T > 0, \qquad \lim_{n\to\infty} \sup_{0\leq t\leq T} \|V_n(t)\Phi - V(t)\Phi\| = 0.$
- $\forall \phi \in \mathcal{D}_0, \alpha, \beta \in \mathbb{C},$ there exists $\phi_n \in \mathrm{dom}(K^{(n)}_{\alpha,\beta})$ such that

$$\phi_n \to \phi, \quad K^{(n)}_{\alpha,\beta}\phi_n \to K^{(n)}_{\alpha,\beta}\phi.$$

Adiabatic Elimination

$$\mathfrak{h} = \mathfrak{h}_{\mathrm{slow}} \oplus \mathfrak{h}_{\mathrm{fast}} \qquad X = \begin{bmatrix} X_{\mathsf{ss}} & X_{\mathsf{sf}} \\ X_{\mathsf{fs}} & X_{\mathsf{ff}} \end{bmatrix}.$$

$$S(n) = S; \qquad L(n) = n \begin{bmatrix} 0 & L_{\mathsf{sf}}^{(1)} \\ 0 & L_{\mathsf{ff}}^{(1)} \end{bmatrix} + L^{(0)};$$

$$H(n) = \begin{bmatrix} H_{\mathsf{ss}}^{(0)} & H_{\mathsf{sf}}^{(0)} + nH_{\mathsf{sf}}^{(1)} \\ H_{\mathsf{fs}}^{(0)} + nH_{\mathsf{fs}}^{(1)} & H_{\mathsf{ff}}^{(0)} + nH_{\mathsf{ff}}^{(1)} + n^2 H_{\mathsf{ff}}^{(2)} \end{bmatrix};$$

$$-\frac{1}{2}L(n)^* L(n) - iH(n) \equiv n^2 \begin{bmatrix} 0 & 0 \\ 0 & A_{\mathsf{ff}} \end{bmatrix} + nZ + R, \qquad \text{with } A_{\mathsf{ff}} \text{ invertible on } \mathfrak{h}_{\mathsf{f}}.$$

Adiabatic Elimination

$(\widehat{S}, \widehat{L}, \widehat{H})$ are defined by

$$\widehat{S} = \begin{bmatrix} \widehat{S}_{\mathsf{ss}} & \widehat{S}_{\mathsf{sf}} \\ \widehat{S}_{\mathsf{fs}} & \widehat{S}_{\mathsf{ff}} \end{bmatrix}, \; \widehat{L} = \begin{bmatrix} \widehat{L}_{\mathsf{s}} & 0 \\ \widehat{L}_{\mathsf{f}} & 0 \end{bmatrix}, \; \widehat{H} = \begin{bmatrix} \widehat{H}_{\mathsf{ss}} & 0 \\ 0 & 0 \end{bmatrix},$$

with

$$\begin{aligned} \widehat{S}_{\mathsf{ab}} &\triangleq \left(\delta_{\mathsf{ac}} + L_{\mathsf{af}}^{(1)} \frac{1}{A_{\mathsf{ff}}} L_{\mathsf{cf}}^{(1)*} \right) S_{\mathsf{cb}}, \\ \widehat{L}_{\mathsf{a}} &\triangleq L_{\mathsf{as}}^{(0)} - L_{\mathsf{af}}^{(1)} \frac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}}, \\ \widehat{H}_{\mathsf{ss}} &\triangleq H_{\mathsf{ss}}^{(0)} + \mathrm{Im} \left\{ Z_{\mathsf{sf}} \frac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}} \right\}. \end{aligned}$$

$$\widehat{L}_{\mathsf{f}} = \widehat{S}_{\mathsf{sf}} = \widehat{S}_{\mathsf{fs}} = 0$$

Then $U_n(t)P_{\mathsf{s}}$ converges uniformly to $\hat{U}(t)P_{\mathsf{s}}$.

Спасибо!

Sova + Kuntz!