Topology of ambient manifolds of nonsingular flows with three nontwisted orbits

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The research was conducted in collaboration with Olga Pochinka

Studied objects: *nonsingular Morse-Smale flows (NMS flows)*, i. e. Morse-Smale flows without fixed points and with three limit cycles.

Definition

The flow f^t is called *Morse-Smale flow* if

- non-wandering set $\Omega(f^t)$ consists of finite number of hyperbolic fixed points and periodic orbits
- stable and unstable manifolds of fixed points and periodic orbits intersect transversally

General theory of dynamical systems implies that ambient manifold M^n is the union of stable and unstable manifolds of periodic orbits.

It is easy to see, that in case of two orbits the ambient manifold of NMS-flow is lens space.

Definition

The 3-manifold is called *lens space* if it consists of two solid tori V_1, V_2 and $V_1 \cap V_2 = \partial V_1 = \partial V_2$.

Examples of lens spaces: $\mathbb{S}^3, \ \mathbb{S}^2 \times \mathbb{S}^1, \ \mathbb{R}P^3$

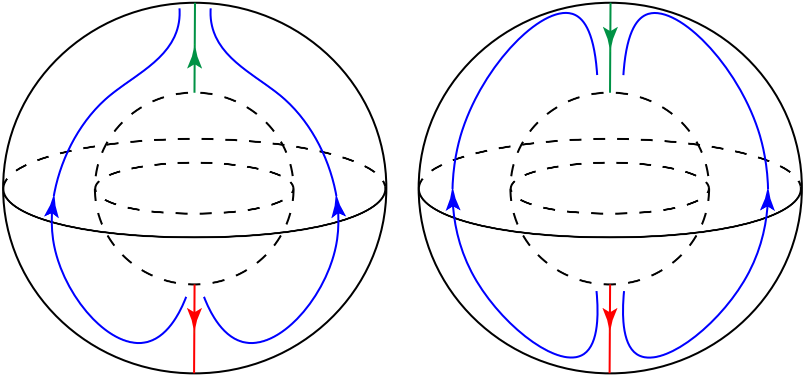


Рис.: Two not topologically equivalent flows on $\mathbb{S}^2 \times \mathbb{S}^1$

In the paper[1] the topology on NMS-flows with 2 and 3 periodic orbits was studied.

Theorem

Let M^3_+ be prime orientable manifold which is ambient manifold for some NMS-flows with 2 or 3 nontwisted periodic orbits, then M^3_+ is a lens space.

Definition

n-manifold is called *prime* if it cannot be expressed as a connected sum of two *n*-manifolds other than $\mathbb{S}^3$

Definition

A periodic orbit of a flow is called nontwisted if its tabular neighbourhood is homeomorphic to solid torus $\mathbb{D}^2 \times \mathbb{S}^1$

[1] B. Campos, A. Cordero, J. Martínez Alfaro, P. Vindel. NMS flows on three-dimensional manifolds with one saddle periodic orbit. Acta Mathematica Sinica. 20 (2004), no. 1, 47–56.

Definition

n-manifold is called *prime* if it cannot be expressed as a connected sum of two *n*-manifolds other than $\mathbb{S}^3$

Definition

n-manifold is called *irreducible* if any embedded $(n-1)$-sphere bounds an embedded *n*-ball.

Theorem

If 3-manifold is irreducible, then it is prime.

But the result by Campos is wrong! We can give countably many counterexamples:

Theorem

There are countably many pairwise non homeomorphic orientable prime 3-manifolds, which admit NMS-flow with 3 periodic orbits and non homeomorphic to any lens space.

We will construct examples by taking suspension over a surface diffeomorphism with three periodic orbit:

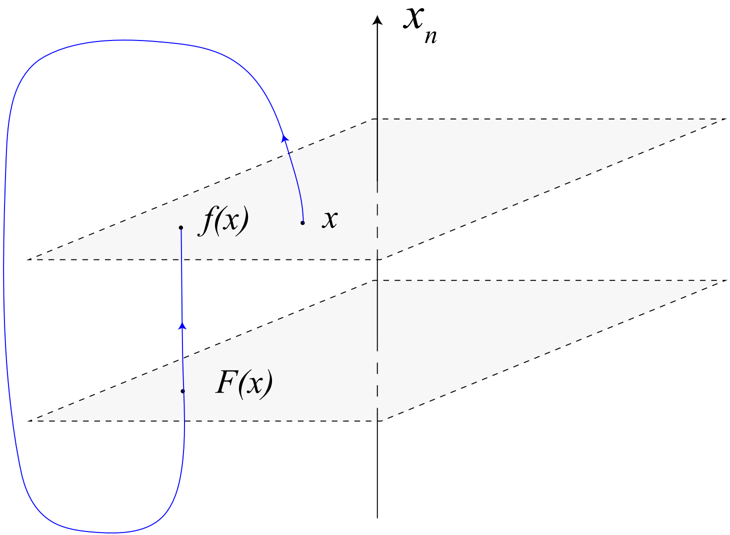
Definition

Consider diffeomorphism $f\colon S \to S$ and construct a diffeomorphism $F\colon S \times \mathbb{R}^1 \to S \times \mathbb{R}^1$ defined by the formula:

$$F(x, r) = (f(x), r - 1).$$

Let $\bar{b}^t(x, r) = (x, r + t)$ be the flow on $S \times \mathbb{R}^1$ and $\Pi_f = (S \times \mathbb{R}^1)/\langle F \rangle$ be quotient space. Let $p_f\colon S \times \mathbb{R}^1 \to \Pi_f$ be natural projection and $b_f^t = p_f \bar{b}^t p_f^{-1}$ be the flow on Π_f.
The flow b_f^t is called *suspension over diffeomorpism f*.

So, now it is enough to find appropriate diffeomorphisms.


x_n
$f(x)$
x
$F(x)$

In the paper[2] diffeomorhisms of orientable surfaces with 3 periodic orbits are considered.
Let g denote genus of the surface, then the diffeomorpism has following periodic data:

- $m_\omega = 1; m_\sigma = 2g; m_\alpha = 1; g > 0$
- $m_\omega = 1; m_\sigma = 2g + 1; m_\alpha = 2; g \geqslant 0$
- $m_\omega = 2; m_\sigma = 2g + 1; m_\alpha = 1; g \geqslant 0$

And, vice versa, there is a gradient-like diffeomorpism of surface with genus g for any set of periodic data listed above.

[2] T. Medvedev, E. Nozdrinova, and O. Pochinka, On Periodic Data of Diffeomorphisms with One Saddle Orbit, Top. Proc. 54 (2019) pp. 49-68.

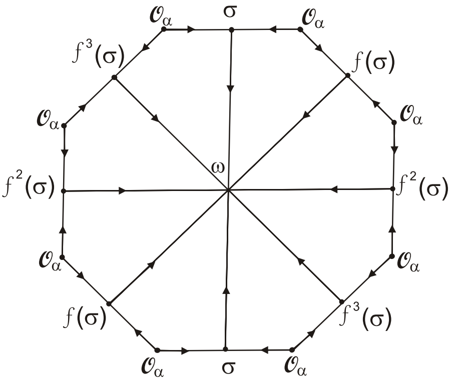


Рис.: $m_\omega = 1, m_\sigma = 4, m_\alpha = 1, g = 2$

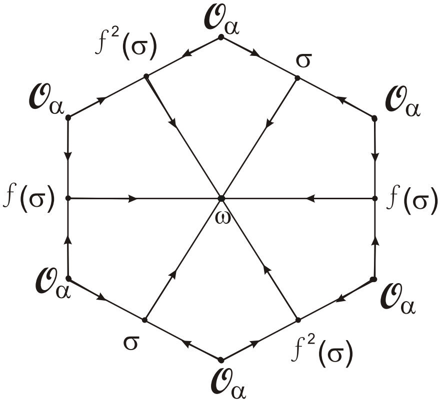


Рис.: $m_\omega = 1, m_\sigma = 3, m_\alpha = 2, g = 1$

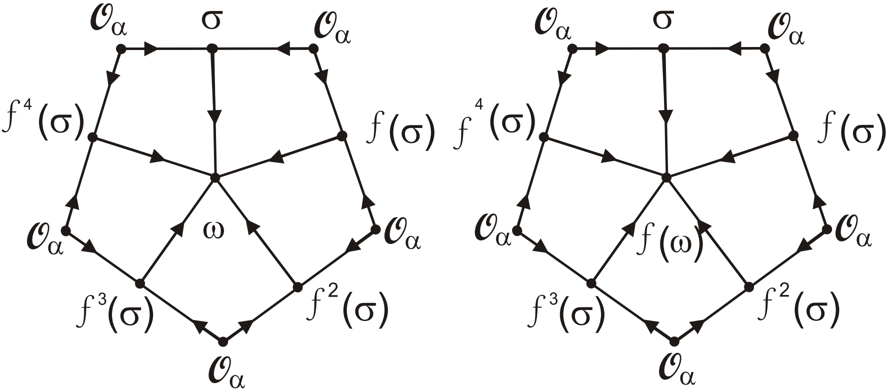


Рис.: $m_\omega = 2, m_\sigma = 5, m_\alpha = 1, g = 2$

Theorem

If 3-manifold is irreducible, then it is prime.

Proposition (Proposition 1.6[3])

If covering space of manifold M is irreducible, then M is irreducible.

It is easy to show, that any M_{f_g} is covered by $\mathbb{R}^3$, since any closed surface is covered either by $\mathbb{R}^2$ or 2-dimentional hyperbolic space. So, M_{f_g} is irreducible, thus it is prime.

[3] Hatcher A. Notes on basic 3-manifold topology. – 2007.

Show, that M_{f_g} are really counterexamples!

Proposition ([4])

Any lens space admits universal covering by $\mathbb{S}^3$.

As we have shown, any M_{f_g} is covered by $\mathbb{R}^3$.
Since the universal covering space is unique for any manifold, M_{f_g} is not homeomorphic to a lens space.

[4] Hatcher A. Notes on basic 3-manifold topology. – 2007.

Any gradient-like flow is topologically conjugated to diffeomorphism of the form $\phi_g\xi_g^1$, where ϕ_g is periodic, coincides with f_g on nonwandering set and its period is a period of saddle separatricies and ξ_g^1 is unit-time shift of gradient-like flow.
So, f_g and ϕ_g are homotopic.

Proposition ([5])

If maps $a, b\colon S \to S$ are homotopis, then M_a and M_b are homeomorphic.

So, further we will consider M_{ϕ_g} instead of M_{f_g}.

[5] Hatcher A. Notes on basic 3-manifold topology. – 2007.

Definition

A model Seifert fibering of $\mathbb{S}^1 \times \mathbb{D}^2$ is a decomposition of $\mathbb{S}^1 \times \mathbb{D}^2$ into disjoint circles, called fibers, constructed as follows. Starting with $[0,1] \times \mathbb{D}^2$ decomposed into the segments $[0,1] \times \{x\}$, identify the disks $\{0\} \times \mathbb{D}^2$ and $\{1\} \times \mathbb{D}^2$ via a $2\pi p/q$ rotation, for $p/q \in \mathbb{Q}$ with p and q relatively prime.

Definition

A Seifert fibering of a 3 manifold M is a decomposition of M into disjoint circles, the fibers, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to a neighborhood of a fiber in some model Seifert fibering of $\mathbb{S}^1 \times \mathbb{D}^2$. A Seifert manifold is one which possesses a Seifert fibering.

The number q in the previous definition is called *multiplicity* jf a fiber.
For the diffeomorphism with periodic data $m_\omega = 1; m_\sigma = 2g; m_\alpha = 1; g > 0$ period of saddle separatricies is $4g$. So, the multiplicities of the fibers corresponding to periodic orbits are $4g, 4g, 2$ and 1 for the remaining fibers.

Obviously, multiplicity is invariant under the Seivert fibering's isomorphism (homeomorphism which preserves fibers).

According to the classical theorem (Theorem 2.3)[6] if Seifert manifold has universal covering space $\mathbb{R}^3$, then all its Seifert fiberings are isomorphic. So, we conclude, that manifolds $M_{\phi g}$ are non homeomorphic for different g

[6]Hatcher A. Notes on basic 3-manifold topology. – 2007.

Theorem

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