Necessary and Sufficient Conditions of Topological Conjugacy Rough Circle's Transformations Product

Iuliana V. Golikova and Olga V. Pochinka

NRU HSE, Nizhniy Novgorod, Russia

02.03.2021

Rough Circle Transformations

Mayer, A. G. (1939). Rough transformation of a circle into a circle. Scientific notes GSU, 12, 215–229 (In Russ.)

- the term of roughness for discrete time systems on circles;
- a rough cascade has only a finite number of periodic points;
- each periodic point is hyperbolic.

Definitions

Definition (Rough diffeomorphism)

Diffeomorphism $f \in \mathit{Diff}(M^n)$ is called rough if it has a C^1-neighborhood $U(f) \subset \mathit{Diff}(M^n)$ such that $\forall f' \in U(f)$ are topologically conjugate to f.

Definition (Morse-Smale diffeomorphisms)

Diffeomorphisms on a connected closed smooth manifold M^n, $n > 1$, are called Morse-Smale diffeomorphisms if the following conditions are true:

- non-wandering set Ω_f consists of finite number of periodic points, for which moduli of eigenvalues of Jacobian are other than unity;
- for any periodic points p and q their stable W^s_p and unstable W^u_q manifolds intersect transversally.

Topological classification of rough circle's transformations

Consider class G of rough orientation-preserving circle's transformations. According to the above-mentioned work of Mayer, a topological classification of diffeomorphisms of class G is given here:

1) $\forall\, f \in G$ a set $Per(f)$ consists of $2n,\ n \in \mathbb{N}$, periodic orbits, each with a period of k.
   Renumber the periodic points of the set $Per(f)$:
   $p_0, p_1, \ldots, p_{2nk-1}, p_{2nk} = p_0$. Then there is an integer l such that $f(p_0) = p_{2nl}$, $l = 0$ if $k = 1$, $l \in \{1, \ldots, k-1\}$ if $k > 1$ and $(k,\ l) = 1$.
2) Diffeomorphisms $f;\ f' \in G$ with parameters $n, k, l;\ n', k', l'$ are topologically conjugate if and only if $n = n'$, $k = k'$ and one of the next conditions is satisfied:
   - $l = l'$ (if $l \neq 0$, then conjugating homeomorphism h is orientation-preserving);
   - $l = k' - l'$ (h changes orientation).

Topological classification of rough circle's transformations

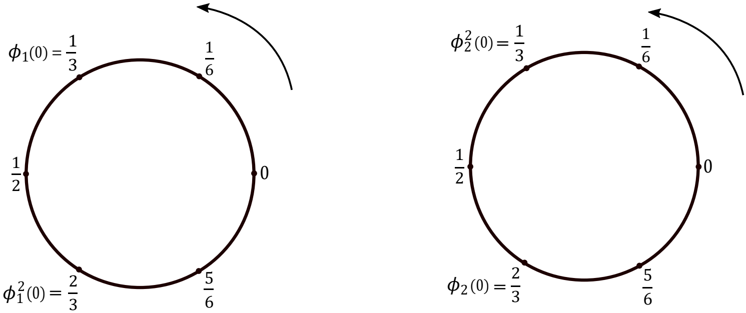


Fig.: Diffeomorphisms $\phi_{1,3,1}$ and $\phi_{1,3,2}$

Topological classification of rough circle's transformations

3) For any triple of integers n, k, l such that $n, k \in \mathbb{N}$, $l = 0$ if $k = 1$, $l \in \{1, \ldots, k-1\}$ and if $k > 1$, $(k,\ l) = 1$, there is a model diffeomorphism $\phi_{n,k,l} \in G$ with given parameters;

$$\phi_{n,k,l} = \pi \Phi_{n,k,l} \pi^{-1} : \mathbb{S}^1 \to \mathbb{S}^1,$$

where π is the universal cover of circle $\mathbb{S}^1$ by real line $\mathbb{R}$:

$$\pi(x) = e^{2\pi i x} : \mathbb{R} \to \mathbb{S}^1,$$

$$\Phi_{n,k,l}(x) = x + \frac{1}{4\pi nk} sin(2\pi nkx) + \frac{l}{k} : \mathbb{R}^1 \to \mathbb{R}^1.$$

Denote by $MG \subset G$ the class of all model diffeomorphisms of a circle.

The main result

Theorem

Diffeomorphisms $\phi_{n_1,k_1,l_1} \times \phi_{n_2,k_2,l_2}$, $\phi_{n'_1,k'_1,l'_1} \times \phi_{n'_2,k'_2,l'_2}$ are topologically conjugate if and only if $n_1k_1 = n'_1k'_1, n_2k_2 = n'_2k'_2$, and $[k_1, k_2] = [k'_1, k'_2]$.

Dynamics of rough circle's transformations product

Consider class G^2 of diffeomorphisms which are the Cartesian product of rough circle's transormations. Then $MG^2 \subset G^2$ is a subset of model diffeomorphisms product and it is also called the class of model diffeomorphisms.

Proposition

Diffeomorphism $f \in G^2$, which is the rough circle's transformations $f_1, f_2 \in G$ product with parameters $n_1, k_1, l_1;\ n_2, k_2, l_2$, respectively, is topologically conjugate to the model diffeomorphism $\phi_{n_1,k_1,l_1} \times \phi_{n_2,k_2,l_2} \in MG^2$.

Proof

There are $h : \mathbb{S}^1 \to \mathbb{S}^1$, $h' : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h_1 f_1 = \phi_{n_1,k_1,l_1} h_1$ and $h_2 f_2 = \phi_{n_2,k_2,l_2} h_2$, and the following diagrams are commutative:

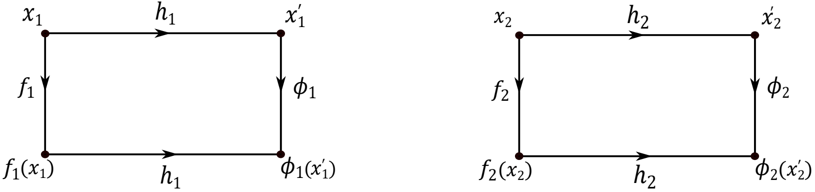


Fig.: The action of conjugating homeomorphisms h_1 and h_2

For simplicity, let us denote diffeomorphism ϕ_{n_1,k_1,l_1} by ϕ_1 and ϕ_{n_2,k_2,l_2} by ϕ_2.

Proof

Define the diffeomorphisms as follows:

$$f,\ \phi : \mathbb{T}^2 \to \mathbb{T}^2 :$$

$$f : (x_1, x_2) \longmapsto (f_1(x_1), f_2(x_2))$$

and

$$\phi : (x'_1, x'_2) \longmapsto (\phi_1(x'_1), \phi_2(x'_2)).$$

Consider a homeomorphism

$$h : \mathbb{T}^2 \to \mathbb{T}^2,\ (h_1(x_1), h_1(x_2)) = (x'_1, x'_2)$$

and show that h is the homeomorphism conjugating diffeomorphisms f и ϕ.

$$hf(x_1, x_2) = (h_1 f_1(x_1), h_2 f_2(x_2)) = (\phi_1 h_1(x_1), \phi_2 h_2(x_2)) = \phi h(x_1, x_2).$$

Proof

Consequently,

$$hf = \phi h,$$

and the following commutative diagram is.

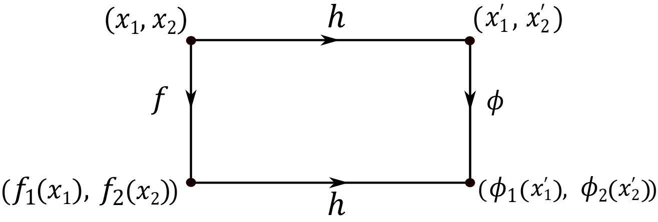


Fig.: The action of conjugating homeomorphism h

Dynamics of rough circle's transformations product

Proposition

Diffeomorphism $\phi_{n_1,k_1,l_1} \times \phi_{n_2,k_2,l_2}$ is gradient-like diffeomorphism of 2-dimentional torus $\mathbb{T}^2$ and its non-wandering set cosists of $4n_1n_2k_1k_2$ periodic points, and they are all the same period

$$k = LCM(k_1, k_2).$$

Proof

Consider cover $p : \mathbb{R}^2 \to \mathbb{T}^2$:

$$p(x_1, x_2) = (e^{2\pi i x_1}, e^{2\pi i x_2}).$$

Then an automorphism $g_{m,n}$ of the cover p acts as follows:

$$g_{m,n} : \ \mathbb{R}^2 \to \mathbb{R}^2, \ g_{m,n}(x_1, x_2) = (x_1 + m, x_2 + n), \ m, n \in \mathbb{Z},$$

and automorphism group $Aut(p)$ is $g_{m,n}$.

$$\phi : \mathbb{T}^2 \to \mathbb{T}^2,$$

$$\phi(x_1, x_2) = p(\Phi_{n_1,k_1,l_1} \times \Phi_{n_2,k_2,l_2})p^{-1}.$$

The lift $\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \ \Phi = \Phi_{n_1,k_1,l_1} \times \Phi_{n_2,k_2,l_2}$ of diffeomorphism ϕ is defined by the formula:

$$\Phi(x_1, x_2) = \left(x_1 + \frac{1}{4\pi n_1 k_1} sin(2\pi n_1 k_1 x_1) + \frac{l_1}{k_1}, \ x_2 + \frac{1}{4\pi n_2 k_2} sin(2\pi n_2 k_2 x_2) + \frac{l_2}{k_2} \right)$$

Proof

A point on the torus $\mathbb{T}^2$ is periodic iff there exists a period q and integers $\mu,\ \nu$ such that

$$\Phi^q(x_1, x_2) = (x_1 + \mu, x_2 + \nu).$$

Find periodic points of diffeomorphisms ϕ_1 and ϕ_2. In order to do this, fixed points of the maps

$$\Phi_{n_1,k_1,l_1} - \frac{l_1}{k_1},\ \Phi_{n_2,k_2,l_2} - \frac{l_2}{k_2}$$

have to be found.

Proof

$$x_1 + \frac{1}{4\pi n_1 k_1} sin(2\pi n_1 k_1 x_1) = x_1$$

$$sin(2\pi n_1 k_1 x_1) = 0$$

$$x_1^i = \frac{i}{2n_1 k_1}, i = 0, \ldots, 2n_1 k_1 - 1.$$

Similarly, $x_2^j = \dfrac{j}{2n_2 k_2},\ j = 0, \ldots, 2n_2 k_2 - 1.$

Then $\Phi(x_1^i, x_2^j) = \left(x_1^i + \dfrac{l_1}{k_1},\ x_2^j + \dfrac{l_2}{k_2} \right).$

Proof

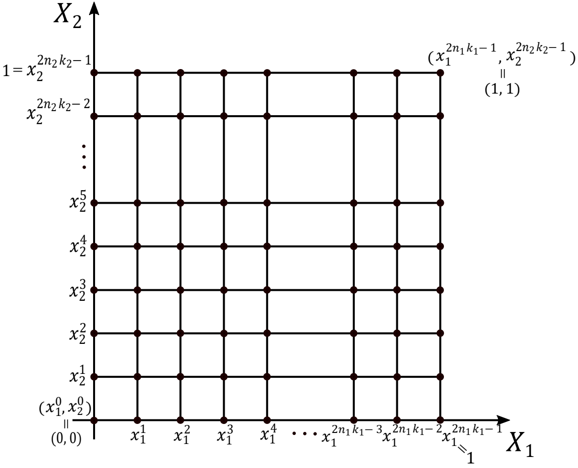


Fig.: Periodic points (x_1^i, x_2^j) on the surface of a torus $\mathbb{T}^2$

Proof

$$\Phi^q(x_1^i, x_2^j) = \left(x_1^i + \kappa\frac{l_1}{k_1},\ x_2^j + \kappa\frac{l_2}{k_2} \right).$$

For points (x_1^i, x_2^j) to be periodic of period q, the following condition must be met:

$$q\frac{l_1}{k_1} = \mu,\ q\frac{l_2}{k_2} = \nu,\ \mu, \nu \in \mathbb{Z} \Longleftrightarrow$$

$$\begin{cases} q\dfrac{l_1}{k_1} = \mu, \\ q\dfrac{l_2}{k_2} = \nu; \end{cases}$$

$$q\left(\frac{l_1}{k_1} + \frac{l_2}{k_2}\right) = \mu + \nu,$$

$$q\left(\frac{l_2 k_1 + l_1 k_2}{[k_1, k_2]}\right) = \mu + \nu \Longrightarrow q = [k_1, k_2],$$

where LCM of indegers a, b is denoted by $[a, b]$.

Topological classification of model diffeomorphisms

Denote $q=[k_1,k_2],\ q'=[k'_1,k'_2]$.

Theorem

Diffeomorphisms $\phi_{n_1,k_1,l_1}\times\phi_{n_2,k_2,l_2}$, $\phi_{n'_1,k'_1,l'_1}\times\phi_{n'_2,k'_2,l'_2}$ are topologically conjugate if and only if

- $q=q'$
- $n_1k_1=n'_1k'_1,\ n_2k_2=n'_2k'_2$.

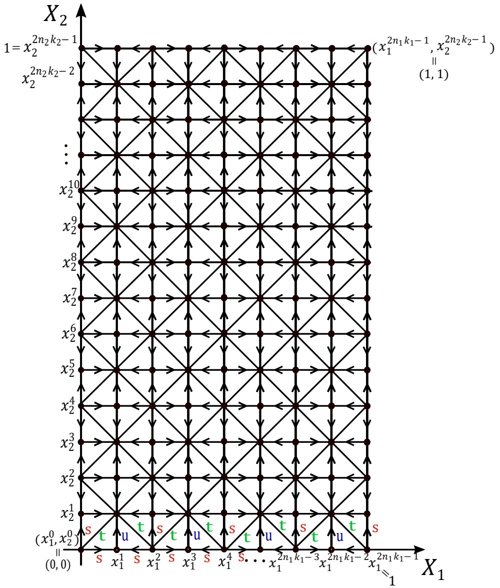
Necessity

Grines, V. Z., Kapkaeva, S. K., Pochinka, O. V. (2014). A three-colored graph as a complete topological invariant for gradient-like diffeomorphisms of surfaces. Sbornik: Mathematics, 10, p. 20-46 (In Russ).

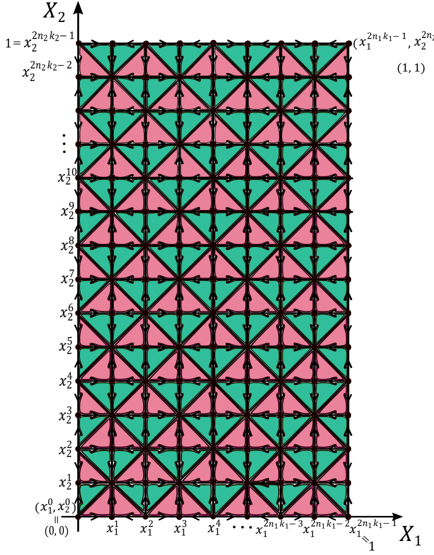
Proposition

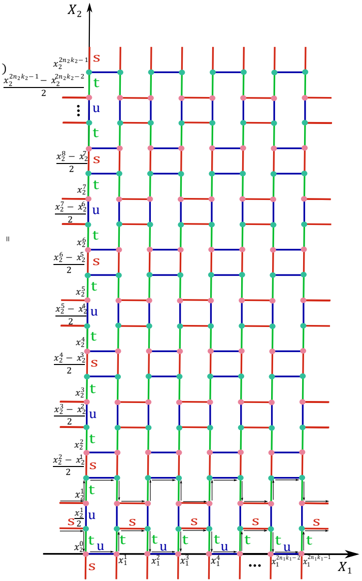
Diffeomorphisms f, f' are topologically conjugate if and only if their three-colored graphs (T_f, P_f), $(T_{f'}, P_{f'})$ are isomorphic.

Constructing the diffeomorphism graph

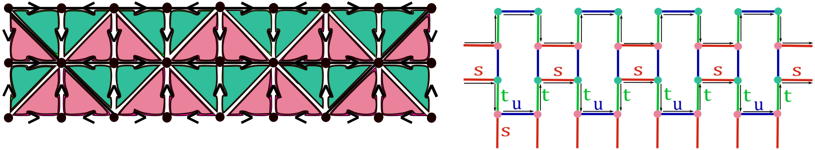


Constructing the diffeomorphism graph





Constructing the diffeomorphism graph



Sufficiency

Let $P_\phi,\ P_{\phi'}$ are automorphisms of graphs $T_\phi,\ T_{\phi'}$ respectively. Graphs (T_ϕ, P_ϕ) and $(T_{\phi'}, P_{\phi'})$ are isomorphic if there exists a one-to-one correspondence ξ:

$$\xi : (T_\phi, P_\phi) \to (T_{\phi'}, P_{\phi'})$$

and

$$\xi P_\phi = P_{\phi'} \xi.$$

Sufficiency

E. Kurenkov, K. Ryazanova, "On periodic translations on n-torus", Dinamicheskie Sistemy, 7(35):2 (2017), 113-118.

Proposition

If two periodic homeomorphisms of n-torus $g : \mathbb{T}^n \to \mathbb{T}^n$, $g' : \mathbb{T}^n \to \mathbb{T}^n$ of a form

$$g(x) = x + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} mod 1, \;\; g'(x) = x + \begin{pmatrix} \gamma'_1 \\ \gamma'_2 \\ \vdots \\ \gamma'_n \end{pmatrix} mod 1,$$

where $\gamma_i = \dfrac{p_i}{q}, \; \gamma'_i = \dfrac{p'_i}{q'}, \; p_i, p'_i, q, q' \in \mathbb{Z}, \; i = 1, \ldots, n$, have the same periods ($q = q'$), then they are topologically conjugate by means of a homeomorphism $h : \mathbb{T}^n \to \mathbb{T}^n$, induced by the unimodular $n \times n$ matrix A.

Sufficiency

Let $p_i = l_i[k_1, k_2]$, $p'_i = l'_i[k'_1, k'_2]$ for $i = 1, 2$, then

$$g(x) = x + \begin{pmatrix} \frac{p_1}{q} \\ \frac{p_2}{q} \end{pmatrix} mod 1, \; g'(x) = x + \begin{pmatrix} \frac{p'_1}{q'} \\ \frac{p'_2}{q'} \end{pmatrix} mod 1.$$

For pink vertices there exists an unimodular matrix:

$$A = \left(\begin{array}{cc} p_1 d' - p'_2 b & -p_1 b' + p'_1 b \\ p_2 d' - p'_2 d & -p_2 b' + p'_1 d \end{array} \right),$$

where $(b, d) = (b', d') = 1$, $p_1 d - p_2 b = p'_1 d' - p'_2 b' = 1$.

$$fA = \left(\begin{array}{cc} p_1 d' - p'_2 b & -p_1 b' + p'_1 b \\ p_2 d' - p'_2 d & -p_2 b' + p'_1 d \end{array} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} mod 1 + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{p_1}{q} \\ \frac{p_2}{q} \end{pmatrix} \; mod 1$$

Sufficiency

$$Af' = \begin{pmatrix} p_1d' - p_2'b & -p_1b' + p_1'b \\ p_2d' - p_2'd & -p_2b' + p_1'd \end{pmatrix} \begin{pmatrix} x_1 + \dfrac{p_1'}{q'} \\ x_2 + \dfrac{p_2'}{q'} \end{pmatrix} mod 1.$$

According to the previuos conditions:

$$fA = Af'.$$

As for green points:

$$A' = A \begin{pmatrix} x_1 \\ x_2 - \dfrac{1}{4n_2k_2} \end{pmatrix} mod 1 + \begin{pmatrix} 0 \\ \dfrac{1}{4n_2k_2} \end{pmatrix}$$

and

$$fA' = A'f'$$

is also true.

References

- А. Г. Майер, “Грубое преобразование окружности в окружность”, Учёные записки Горьк. гос. ун-та., 14 (1939), 215-229.
- В. З. Гринес, С. Х. Капкаева, О. В. Починка “Трехцветный граф как полный топологический инвариант для градиантно-подобных диффеоморфизмов поверхностей”, Матем. сб., 205:10 (2014), 19-46.
- E. Kurenkov, K. Ryazanova, “On periodic translations on n-torus”, Dinamicheskie Sistemy, 7(35):2 (2017), 113-118.