Upper and lower estimates for the speed of convergence of Chernoff approximations of operator semigroups

O. E. Galkin,[1] I. D. Remizov.[2]

Keywords: Chernoff product formula, approxition of C_0-semigroup, speed of convergence, estimates, examples

MSC2010 codes: 47D03, 47D06, 35A35, 41A25

This talk is devoted to the speed (rate) of convergence of Chernoff approximations [3,4] to strongly continuous one-parameter semigroups [1, 2]. We provide simple natural examples for which this convergence: is arbitrary high; is arbitrary slow; holds in the strong operator topology but does not hold in the norm operator topology. We also prove general theorem that gives estimate from above for the speed of decay of the norm of the residual appearing in Chernoff approximations. We provide also supplementary theorems which makes it easier to check the conditions of the main theorem.

This talk is based on the first draft-preprint version [5] of the text e.g. it lacks up-to-date literature overview. We kindly ask authors of relevant research papers to forgive us. We will fill this gap before submitting to a journal.

Usually expressing the C_0-semigroup $\left(e^{tL}\right)_{t\geq 0}$ it terms of variable coefficients of operator L is a hard problem. However, if the so-called Chernoff function [3] is constructed, i.e. an operator-valued function G which satisfies the conditions of the Chernoff theorem (in particular, satisfied $G(t) = I + tL + o(t)$ in the strong operator topology as $t \to +0$), then the semigroup is given by the equality $e^{tL} = \lim_{n\to\infty} G(t/n)^n$. An advantage of this approach arises from the fact that usually it is possible to define G by an explicit and not very long formula which contains coefficients of operator L. Expressions $G(t/n)^n$ are called Chernoff approximations to e^{tL}.

We have constructed [5] the following examples.

Proposition 1. There exists a Banach space $\mathcal{F}$, C_0-semigroup $(e^{tL})_{t\geq 0}$ in $\mathcal{F}$ with generator $(L, D(L))$, and Chernoff function G for operator $(L, D(L))$ such that Chernoff approximations converge on each vector but do not converge in operator norm. More precisely:

1. $\lim_{n\to\infty} \|G(t/n)^n f - e^{tL} f\| = 0$ for all $f \in \mathcal{F}$,
2. $\|e^{tL}\| = \|G(t)\| = 1$,
3. for each $t > 0$ and each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}$ such that $\|f_n\| = 1$ and $\|G(t/n)^n f_n - e^{tL} f_n\| \geq \|f_n\|$ so $\|G(t/n)^n - e^{tL}\| \geq 1 \not\to 0$ as $n \to \infty$.

Proposition 2. There exists C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space $\mathcal{F}$, Chernoff function G and vector $f \in \mathcal{F}$ such that $f \notin D(L)$ but the speed of convergence is arbitrary high. More precisely: for arbitrary chosen non-increasing continuous function $v\colon [0, +\infty) \to [0, +\infty)$ vanishing at infinity at arbitrary high rate (e.g. $v(x) = (1 + x)^{-k}$, $v(x) = e^{-x}$, $v(x) = e^{-e^x}$) and all $T > 0$ we have $\sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL} f\| = Tv(n/T)$ for all $n = 1, 2, 3, \ldots$ such that $Tv(n/T) \leq 1$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = \|f\| = 1$.

Proposition 3. There exists C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space $\mathcal{F}$, Chernoff function G and vector $f \in \mathcal{F}$ such that $f \in \cap_{j=1}^{\infty} D(L^j)$ but the speed of convergence is arbitrary low, i.e. for arbitrary chosen non-increasing continuous function $u\colon [0, +\infty) \to [0, +\infty)$ vanishing at infinity at arbitrary low rate (e.g. $v(x) = (1 + x)^{-1/k}$, $v(x) = 1/\log(x + e)$, $v(x) = 1/\log(\log(x + e^e)))$ and all $T > 0$ we have $\sup_{t\in[0,T]} \|G(t/n)^n f - e^{tL} f\| = Tv(n/T)$ for all $n = 1, 2, 3, \ldots$ such that $Tv(n/T) \leq 1/2$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = \|f\| = 1$.

Our main result is stated as follows.

[1] HSE University, Russian Federation, Nizhny Novgorod city. Email: olegegalkin@yandex.ru

[2] HSE University, Russian Federation, Nizhny Novgorod city. Email: ivremizov@yandex.ru

Theorem 1. Suppose that

1. Number $T > 0$ is given, and C_0-semigroup $(e^{tA})_{t\geqslant 0}$ with generator $(A, D(A))$ in Banach space $\mathcal{F}$ satisfies for some $M_1 \geq 1$ and $w \geq 0$ the condition $\|e^{tA}\| \leqslant M_1 e^{wt}$ for all $t \in [0, T]$.

2. There is a mapping $S\colon (0, T] \to \mathscr{L}(\mathcal{F})$, i.e. $S(t)\colon \mathcal{F} \to \mathcal{F}$ is a linear bounded operator for each $t \in (0, T]$. There exists some constant $M_2 \geq 1$ that $\|S(t)^k\| \leqslant M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k = 1, 2, 3, \ldots$

3. Numbers $m \in \{0, 1, 2, \ldots\}$ and $p \in \{1, 2, 3, \ldots\}$ are fixed. There is a $(e^{tA})_{t\geqslant 0}$-invariant subspace $\mathcal{D} \subset D(A^{m+p}) \subset \mathcal{F}$, i.e. $(e^{tA})(\mathcal{D}) \subset \mathcal{D}$ for any $t \geq 0$ (for example $\mathcal{D}$ may be equal to $D(A^{m+p})$).

4. There exist such functions $K_j\colon (0, T] \to [0, +\infty)$, $j = 0, 1, \ldots, m+p$ that for all $t \in (0, T]$ and all $f \in \mathcal{D}$ we have

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} K_j(t)\|A^j f\|.$$

Then:

1. For all $t > 0$, all integer $n \geq t/T$ and all $f \in \mathcal{D}$ we have

$$\|S(t/n)^n f - e^{tA} f\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n)\|A^j f\|,$$

where $C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$ and $C_j(t) = K_j(t)e^{-wt}$ for $j \neq m+1$.

2. If $\mathcal{D}$ is dense in $\mathcal{F}$ and for all $j = 0, 1, \ldots, m+p$ we have $K_j(t) = o(t^{-m})$ when $t \to +0$, then for all $g \in \mathcal{F}$ and $\mathcal{T} > 0$ the following equality is true:

$$\lim_{\mathcal{T}/T \leq n \to \infty} \sup_{t \in (0, \mathcal{T}]} \left\| S(t/n)^n g - e^{tA} g \right\| = 0.$$

Example 1. Let us consider particular modeling example. Suppose $\|e^{tA}\| \leq e^t$, $\|S(t)\| \leq e^t$, $\|S(t)f - f - tAf - \frac{1}{2}t^2A^2 f\| \leq t^2\sqrt{t}\|A^3 f\|$ for all $f \in D(A^3)$ and $t \in (0; 1]$. Then $\mathcal{D} = D(A^3)$, $m = 2$, $M_1 = M_2 = w = 1$, $K_0(t) = K_1(t) = 0$, $K_2(t) = 1/\sqrt{t}$ for any $t \in (0; 1]$. So estimate in the item of theorem 1 states that for any fixed $t > 0$ the following estimate is true for all $f \in D(A^3)$ and integer $n \geq t$, having the following asymptotic behaviour as $n \to \infty$:

$$\|S(t/n)^n f - e^{tA} f\| \leq \frac{t^3 e^t}{n^2}\left(\frac{1}{\sqrt{t/n}} + \frac{e^{t/n}}{3!}\right)\|A^3 f\| =$$
$$= e^t\left(\frac{t^2\sqrt{t}}{n\sqrt{n}} + \frac{e^{t/n}t^3}{6n^2}\right)\|A^3 f\| = \frac{t^2\sqrt{t}e^t}{n\sqrt{n}}\|A^3 f\| + O\Big(\frac{1}{n^2}\Big).$$

Lemma 1. Suppose that $n \in \{0, 1, 2, \ldots\}$, suppose that functions $a, b, c\colon \mathbb{R} \to \mathbb{R}$ are differentiable $2\lfloor (n-1)/2 \rfloor$ times and the inequality $\inf_{x\in\mathbb{R}} |a(x)| > 0$ holds. Suppose, in addition, that operator L maps each twice differentiable function $u\colon \mathbb{R} \to \mathbb{R}$ to the function $Lu = au'' + bu' + cu$. Then there are nonnegative constants $C_0, C_1, \ldots, C_{\lfloor (n+1)/2 \rfloor}$, such that for any $2\lfloor (n+1)/2 \rfloor$ times differentiable function $v\colon \mathbb{R} \to \mathbb{R}$, the following inequality is true:

$$\|v^{(n)}\| \leqslant \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} C_k \|L^k v\|.$$

Remark 1. Let us use the symbol $UC_b(\mathbb{R})$ for the space of all bounded, uniformly continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with the norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Let us use symbol $HC_b(\mathbb{R})$ for the space of all Hölder continuous functions $u\colon \mathbb{R} \to \mathbb{R}$, and for each $n \in \{1, 2, 3, \ldots\}$ let us denote by $HC_b^n(\mathbb{R})$ the space of all such functions $u \in HC_b(\mathbb{R})$, that $u', \ldots, u^{(n)} \in HC_b(\mathbb{R})$. It is clear that $HC_b^n(\mathbb{R}) \subset UC_b(\mathbb{R})$ and $HC_b^n(\mathbb{R})$ is dense in $UC_b(\mathbb{R})$ for all $n \in \{1, 2, 3, \ldots\}$. Similarly, the space $UC_b^n(\mathbb{R})$ is defined. The following result is an example of usage of theorem 1 and lemma 1.

Theorem 2. Suppose that

1. Numbers $m, q \in \{1, 2, 3, \ldots\}$ are fixed, and $\hat{q} = 2\lfloor (q+1)/2 \rfloor$. Functions a, b, c from the class $HC_b^{2m+\hat{q}-2}(\mathbb{R})$ are given, and $\inf_{x\in\mathbb{R}} a(x) > 0$. Operator L on $UC_b(\mathbb{R})$ with domain $D(L) = HC_b^2(\mathbb{R})$ is defined by the formula

$$Lu = au'' + bu' + cu.$$

2. Numbers $T > 0$, $M \geq 1$ and $\sigma \geq 0$ are given. For any $t \in (0, T]$ a bounded linear operator $S(t)$ on $UC_b(\mathbb{R})$ is defined such that $\|S(t)^k\| \leqslant Me^{k\sigma t}$ for all $k = 1, 2, 3, \ldots$.

3. There exist nonnegative constants $K_0, K_1, \ldots, K_{2m+q}$ such that for all $t \in (0, T]$ and all $f \in UC_b^{2m+q}(\mathbb{R})$ we have

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k L^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{2m+q} K_j \|f^{(j)}\|.$$

Then:

1. The closure $\overline{L}$ of operator L is a generator of C_0-semigroup $(e^{t\overline{L}})_{t\geqslant 0}$ in Banach space $UC_b(\mathbb{R})$, and the condition $\|e^{t\overline{L}}\| \leqslant e^{\gamma t}$ for all $t \geq 0$ is satisfied, where $\gamma = \max(0, \sup_{x\in\mathbb{R}} c(x))$.

2. For all $t > 0$, all integer $n \geq t/T$ and all $f \in UC_b^{2m+\hat{q}}(\mathbb{R})$ we have

$$\|S(t/n)^n f - e^{t\overline{L}} f\| \leq \frac{Mt^{m+1}e^{wt}}{n^m} \sum_{j=0}^{2m+\hat{q}} C_j \|f^{(j)}\|,$$

where $w = max(\sigma, \gamma)$, $\hat{q} = 2\lfloor (q+1)/2 \rfloor$ and $C_0, C_1, \ldots, C_{2m+\hat{q}}$ are nonnegative constants that are independent of t and n.

3. For all $g \in UC_b(\mathbb{R})$ and all $\mathcal{T} > 0$ the following equality is true:

$$\lim_{\mathcal{T}/T \leq n \to \infty} \sup_{t\in(0,\mathcal{T}]} \left\| S(t/n)^n g - e^{t\overline{L}} g \right\| = 0.$$

Acknowledgements. Authors are partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of science and higher education of the RF grant ag. No 075-15-2019-1931. Authors are thankful to the members of research group Evolution semigroups and applications for discussion of the research presented.

References:

[1] E. Hille, R.S. Phillips: Functional Analysis and Semi-Groups. — American Mathematical Society, 1975.

[2] K.-J. Engel, R. Nagel. One-Parameter Semigroups for Linear Evolution Equations. — Springer, 2000.

[3] Ya. A. Butko. The Method of Chernoff Approximation.// In book: Semigroups of Operators - Theory and Applications — Springer International Publishing, 2020

[4] A. V. Vedenin, V. S. Voevodkin, V. D. Galkin, E. Yu. Karatetskaya, I. D. Remizov. Speed of Convergence of Chernoff Approximations to Solutions of Evolution Equations.// Math. Notes, 108:3 (2020), 451-456

[5] O. E. Galkin, I. D. Remizov. Upper and lower estimates for the speed of convergence of Chernoff approximations of operator semigroups// arXiv [Math.FA] 2021