Convergence rate of Chernoff approximations

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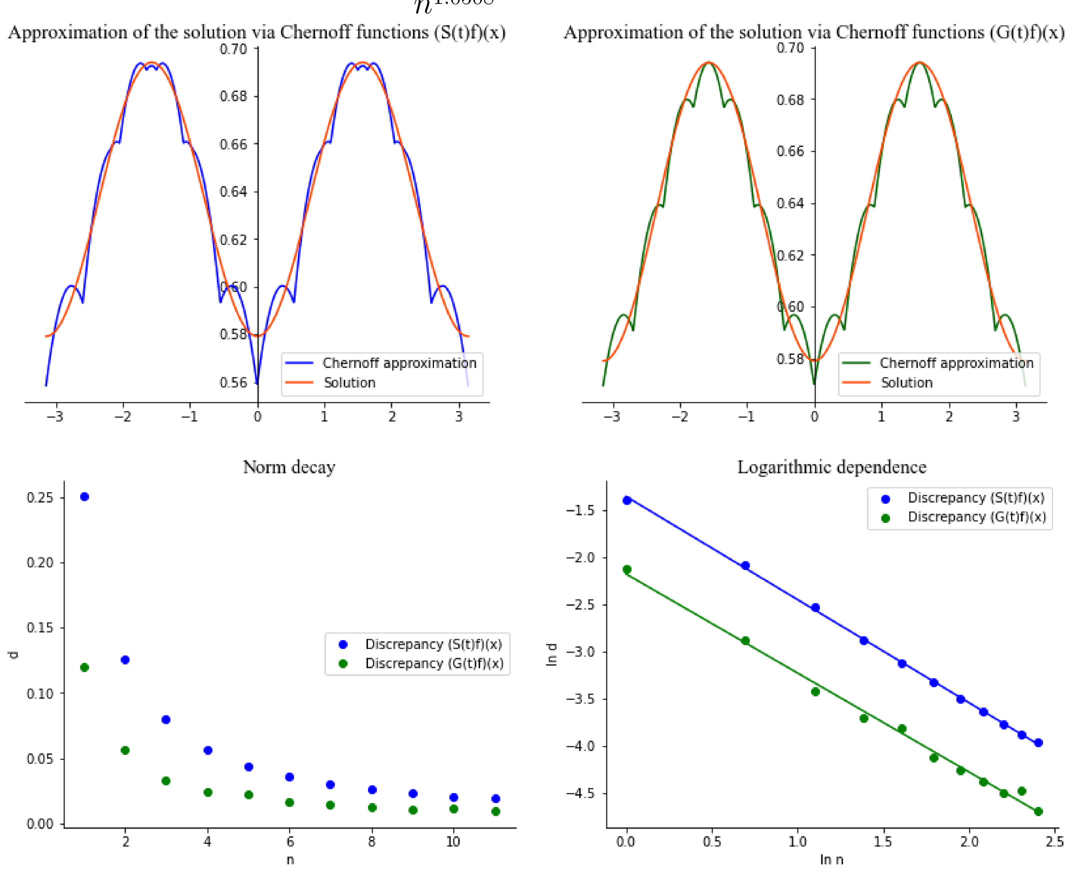
Introduction

Chernoff approximations are flexible and powerful tool of functional analysis [1, 2], which can be used, in particular, to find numerically approximate solutions of some partial differential equations with variable coefficients. In the paper by I.D. Remizov [3], conjectures about the rate of convergence were explicitly formulated, and they were recently proved in paper by I.D. Remizov and O.E. Galkin [4]. Our talk is devoted to constructing examples to the Galkin-Remizov theorem illustrating the speed of convergence of Chernoff approximations to the solution of the Cauchy problem for the heat equation on the real line.

Solutions of the heat equation are also known without Chernoff's theorem. We use the heat equation only as a model to study the general properties of Chernoff approximations. Thus, this paper is primarily devoted to Chernoff approximations as a new method, the scope of which is much wider than the thermal conductivity equation.

Numerical solution of the heat conduction equation with the initial condition $u_0(x) = |\sin x|$

Figure shows two graphs of the approximate solution for the functions we are studying, at $n = 10$, and the exact solution under the initial condition $u_0(x) = |\sin x|$. Rounding off the coefficients, we see that for the blue line (see Figure 2.2) the equation is as follows: $\ln(d) = -1.0948\ln(n) - 1.355$, i.e. $d = n^{-1.0948}e^{-1.355} = \frac{0.2579}{n^{1.0948}}$.
Similarly, for the green line the equation $\ln(d) = -1.0508\ln(n) - 2.1782$, i.e. $d = n^{-1.0508}e^{-2.1782} = \frac{0.1132}{n^{1.0508}}$.



Problem setting

Consider Cauchy problem for the heat equation:

$$\begin{cases} u_t'(t,x) = u_{xx}''(t,x), t \geq 0, x \in \mathbb{R} \\ u(0,x) = u_0(x), x \in \mathbb{R} \end{cases} \quad (1)$$

Function u_0 is given and belongs to the space $UC_b(\mathbb{R}$ of all uniformly continuous bounded real-valued functions on the real line. It is known that solutions of (1) that satisfy $u(t,\cdot) \in UC_b(\mathbb{R}$ for all $t \geq 0$, are given by the heat C_0-semigroup [2]. We are interested in Chernoff approximations to this semigroup. For all $t \geq 0$, $x \in \mathbb{R}$, $f \in UC_b(\mathbb{R})$ define:

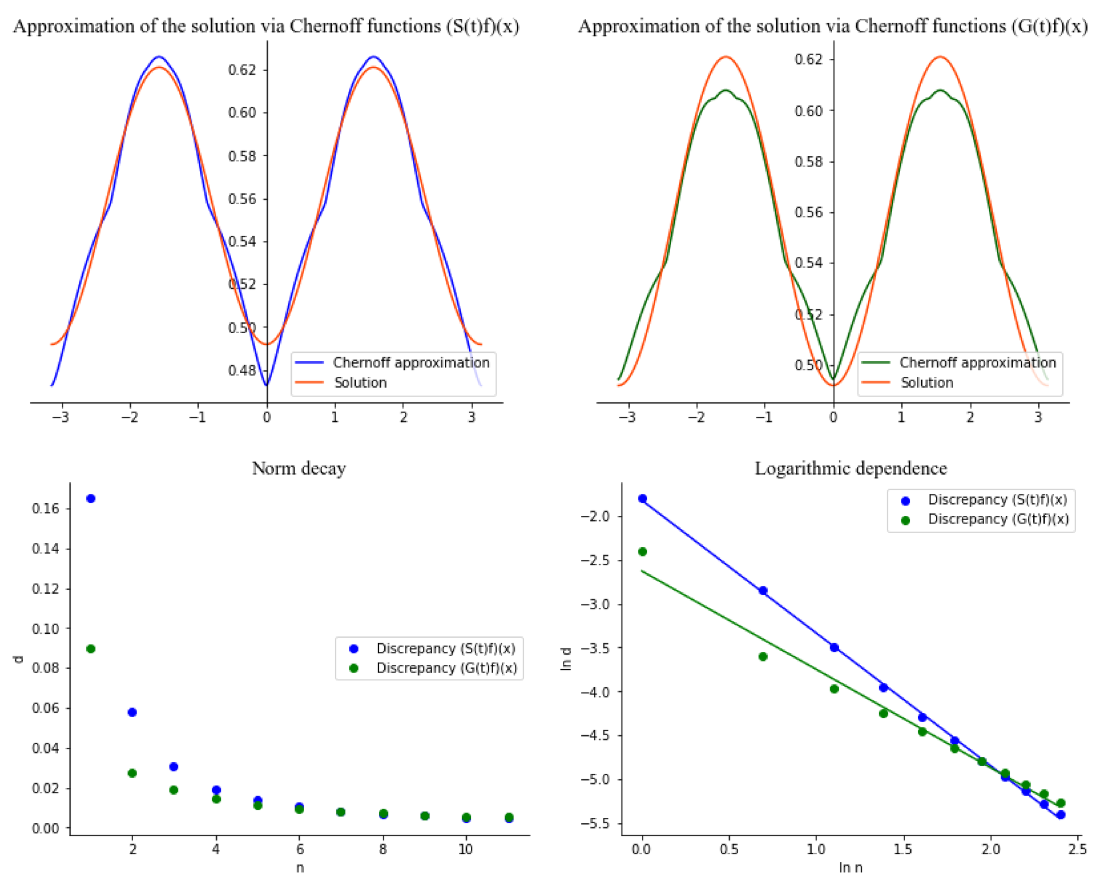
$$(G(t)f)(x) = \frac{1}{4}f(x + 2\sqrt{t}) + \frac{1}{4}f(x - 2\sqrt{t}) + \frac{1}{2}f(x),$$

$$(S(t)f)(x) = \frac{2}{3}f(x) + \frac{1}{6}f(x + \sqrt{6t}) + \frac{1}{6}f(x - \sqrt{6t}).$$

It follows from [3, 4] that if u_0 is smooth enough then Chernoff approximations $S(t/n)^n u_0 \to u(t,x), n \to \infty$ converge faster than $G(t/n)^n u_0 \to u(t,x), n \to \infty$. In this study test whether this is true for the case of a non-smooth u_0 which is not covered by theory from [3, 4].

Numerical solution of the heat conduction equation with the initial condition $u_0(x) = \sqrt[2]{|\sin x|^3}$

The line (green) corresponding to the decreasing error of the function $G(t)$ in the logarithmic scale was constructed without taking into account $n = 1$.
For the green line equation
$\ln(d) = -0.9785\ln(n) - 2.8973$, i.e. $d = n^{-0.9785}e^{-2.8973} = \frac{0.0552}{n^{0.9785}}$.
Similarly, for the blue line, the equation is $\ln(d) = -1.5109\ln(n) - 1.8234$, i.e. $d = n^{-1.5109}e^{-1.8234} = \frac{0.1615}{n^{1.5109}}$.
As can be seen from the Figure, the difference between the error decay rates using Chernoff functions $S(t)$ and $G(t)$ for $u_0(x) = \sqrt[2]{|\sin x|^3}$ is larger than for $u_0(x) = |\sin x|$. This is explained by the greater smoothness of $u_0(x) = \sqrt[2]{|\sin x|^3}$.

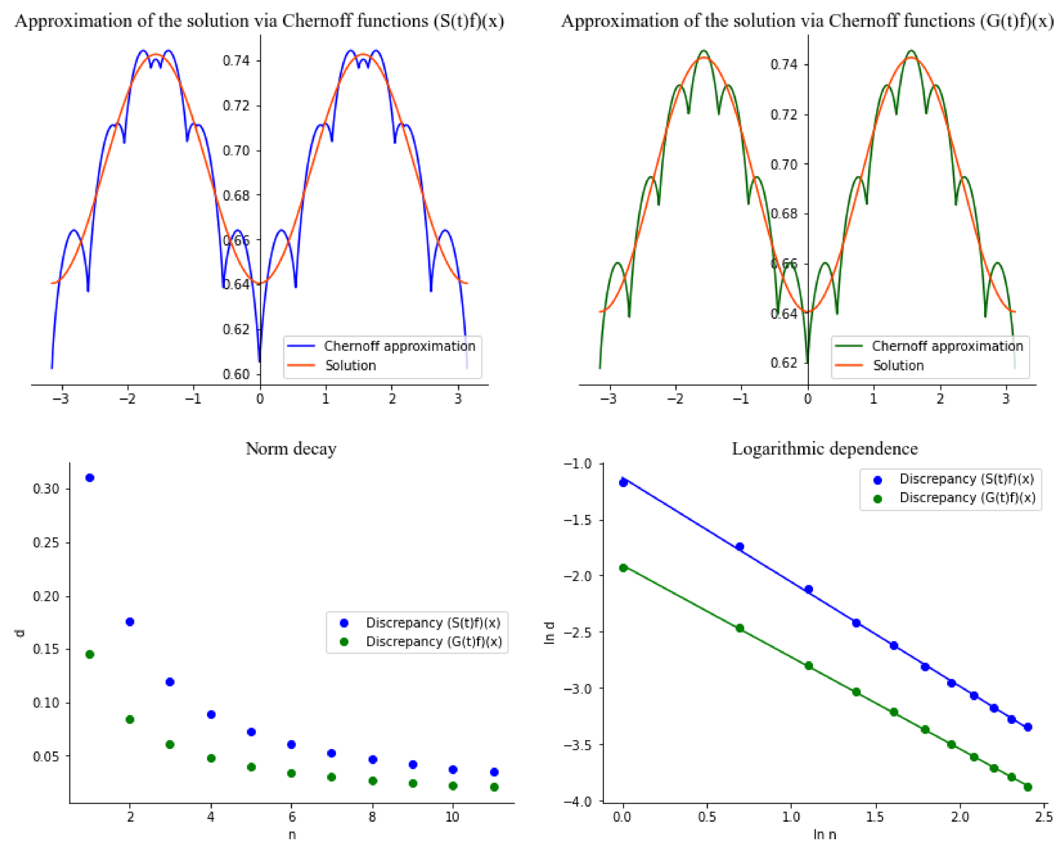


Description of the experiment

The calculations were performed in a Python environment using a program we wrote. In order to reduce the complexity and computation time, all measurements were performed at 11 iterations for $t = 1/2$. This t was chosen because it gives a good visual representation of the different rates of convergence.
The program code was written with the possibility to set any operator and any initial condition, i.e. without simplifying Chernoff functions and using binomial coefficients, in contrast to the works published earlier. The number of iterations n can also be changed.

Numerical solution of the heat conduction equation with the initial condition $u_0(x) = \sqrt[4]{|\sin x|^3}$

For the blue line the equation is as follows: $\ln(d) = -0.9262\ln(n) - 1.1314$, i.e. $d = n^{-0.9262}e^{-1.1314} = \frac{0.3226}{n^{0.9262}}$.
Similarly, for the green line, the equation $\ln(d) = -0.6905\ln(n) - 1.4709$, i.e. $d = n^{-0.6905}e^{-1.4709} = \frac{0.2297}{n^{0.6905}}$.



Conclusion

In this paper, a numerical simulation of a new method of approximate solution of differential equations based on Chernoff theorem was carried out. The heat conduction equation was chosen as the model equation. Two Chernoff functions constructed by I. Remizov [5] and A. Vedenin (2019) were considered. The main attention was paid to the error arising due to the fact that the method gives approximate solutions of Cauchy problem for heat equation (1).

References

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