Explicit formula for evolution semigroup for diffusion in Hilbert space

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Explicit formula for evolution semigroup for diffusion in Hilbert space

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A parabolic partial differential equation $u'_t(t, x) = Lu(t, x)$ is considered, where L is a linear second-order differential operator with time-independent (but dependent on x) coefficients. We assume that the spatial coordinate x belongs to a finite- or infinite-dimensional real separable Hilbert space H. The aim of the paper is to prove a formula that expresses the solution of the Cauchy problem for this equation in terms of initial condition and coefficients of the operator L. Assuming the existence of a strongly continuous resolving semigroup for this equation, we construct a representation of this semigroup using a Feynman formula (i.e. we write it in the form of a limit of a multiple integral over H as the multiplicity of the integral tends to infinity), which gives us a unique solution to the Cauchy problem in the uniform closure of the set of smooth cylindrical functions on H. This solution depends continuously on the initial condition. In the case where the coefficient of the first-derivative term in L is zero, we prove that the strongly continuous resolving semigroup indeed exists (which implies the existence of a unique solution to the Cauchy problem in the class mentioned above), and that the solution to the Cauchy problem depends continuously on the coefficients of the equation.

Keywords: Heat equation; diffusion equation; Hilbert space; Feynman formula; Cauchy problem solution.

AMS Subject Classification: 35C15, 47D06, 28C20

1. Introduction

Representation of a function in the form of a limit of a multiple integral as multiplicity tends to infinity is called a Feynman formula after Feynman, who was the first[34,35] to use them (on the physical level of rigor) to solve the Cauchy problem for PDEs, see Ref. 2 for contemporary mathematical interpretation. The term "Feynman formula" in this sense was introduced in 2002 by Smolyanov.[69] One can

find more about the research of Feynman formulas (up to 2009) in Ref. 63. Feynman formulas are closely related to Feynman–Kac formulas and can be used as approximations to Feynman path integrals; we do not touch this connection, but one can read about it in Ref. 67.

In this paper, the Cauchy problem for heat (diffusion) equation

$$\begin{cases} u_t'(t,x) = Lu(x,t); & t \geq 0, \ x \in H \\ u(0,x) = u_0(x); & x \in H \end{cases} \tag{1.1}$$

for the special space of functions on a real separable Hilbert space H, is considered. The operator L in (1.1) has the form

$$(L\varphi)(x) = g(x)\text{trace}[A\varphi''(x)] + \langle \varphi'(x), AB(x) \rangle + C(x)\varphi(x),$$

where A is a positive trace-class (nuclear) operator in H, B is a vector field, g and C are real-valued functions (one assumes that A, B, C and g satisfy some additional technical conditions). The aim of the paper is to express the solution of (1.1) in terms of A, g, B, C, u_0. The Feynman formula that we obtain is the representation of the solution of (1.1) by a limit of some multiple integrals over H with respect to specially constructed Gaussian measure on H as the multiplicity tends to infinity. We make use of the fact that if there exists a C_0-semigroup $(e^{tL})_{t \geq 0}$ with an infinitesimal generator L, the solution u can be represented in the form $u(t,x) = (e^{tL}u_0)(x)$. Then we find e^{tL} via the Chernoff theorem (Theorem 2.1). For the case $B = 0$ and $C \leq 0$, existence of the semigroup for Eq. (1.1) is proved and so is the Feynman formula; moreover, it is proved that in this case the solution depends continuously not only on u_0, but also on g and C. For the case $B \neq 0$, the Feynman formula is proved under the assumption of the existence of the semigroup.

Differential equations for functions of an infinite-dimensional argument[27,28] arise in quantum field theory,[62,1] string theory,[3] theory of stochastic processes[37] and financial mathematics.[36]

Evolutionary equations (i.e. PDEs in the form $u_t'(t,x) = \cdots$) in infinite-dimensional spaces have been studied since 1960s by Accardi, Albeverio, Hida, Khrennikov, Nelson, Shavgulidze, Smolyanov and others. We will only mention the most recent and relevant (for our study) publications.

There are many papers dedicated to study of diffusion equation in Hilbert space in different aspects, but without aim of finding exact formula for solution, see Refs. 4, 5, 24, 25, 26, 32, 43, 45, 50, 54, 71 and references therein.

There is a group of papers where representations of Cauchy problem solution were obtained with methods similar to what we use. Let us provide some comments on them. In Ref. 15, the Schrödinger equation in Hilbert space is studied. The equation includes the terms of the second, first and zero order, the coefficient of the second-order term being constant. The solution to the Cauchy problem is given by a Feynman–Kac–Ito formula. In Ref. 68, authors describe classes of polynomial potentials for which infinite-dimensional Schrödinger equations have solutions that can be represented in terms of sequential Feynman path integrals in

the configuration space. In Ref. 28, a solution to a heat equation in Hilbert space and without the terms of the first and zero order, and with the constant coefficient of the second-derivative term, is discussed. The solution is given in the form of a convolution with the Gaussian measure (analogous to the finite dimensional equation with constant coefficients), the existence of the resolving semigroup is proved. In Ref. 10, the solution to the same equation is given by a Feynman–Kac formula. In Ref. 18, the parabolic equation in finite-dimensional space is studied for the case of variable coefficients. Under the assumption that the resolving semigroup exists for the Cauchy problem, Feynman and Feynman–Kac formulas were proven in Ref. 18, giving the solution. In Refs. 14 and 12, nonlinear and semilinear heat-type equations are studied with the path integral approach, and in Refs. 11 and 13 the same approach is applied to the wave equation. In Ref. 58 an equation similar to (1.1) (but with less terms in L and constant coefficients) is studied, and solution to the Cauchy problem is constructed using finitely-additive measure on Hilbert space; after that the corresponding Schrödinger equation is solved using the ideas from Ref. 57. In spaces over the field of p-adic numbers, Feynman and Feynman–Kac formulas for solutions of the Cauchy problem for evolutionary equations are given in Refs. 66 and 65. In Refs. 52 and 53, Schrödinger and heat equations in $\mathbb{R}^n$ are studied in the case of time-dependent coefficients, and Chernoff-type theorem is proven for this case. In Refs. 16 and 20, Feynman formulas for perturbed semigroups are obtained. See also Refs. 21, 61, 49, 59, 57, 19, 39, 64, 46, 40, 9, 47, 38 and references therein.

Note also that usage of Feynman and Feynman–Kac formulas may include exact or numerical evaluation of integrals over Gaussian measures on spaces of high or infinite dimension; some useful approaches to this topic are developed in Refs. 30 and 44.

This paper extends the results of Ref. 55, where the Feynman formula is proved for the case where the last two terms vanish in the operator L. We also prove and use several statements that were announced without proofs in Ref. 56.

2. Preliminaries

2.1. *Notation and definitions*

The symbol H stands for the real separable Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$.

The self-adjoint, positive, non-degenerate (hence injective), linear operator $A : H \to H$ is assumed to be defined everywhere on H. The operator A is assumed to be of trace class, which means that for every orthonormal basis (e_k) in H the sum $\sum_{k=1}^{\infty} \langle Ae_k, e_k \rangle = \mathrm{tr}A$ is finite; this sum is called the trace of A (it is independent of the choice of the basis (e_k)).

The symbol $\mathcal{X}$ below stands for any complex Banach space. The symbol $L_b(\mathcal{X}, \mathcal{X})$ stands for space of all linear bounded operators in $\mathcal{X}$, endowed with the classical operator norm.

Symbol $C(M, N)$ will mean the set of all continuous functions from M to N, where M and N are topological spaces.

A function $f : H \to \mathbb{R}$ is called cylindrical,[27,67] if there exist vectors $e_1, \ldots, e_n$ from H and function $f^n : \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in H$ the equality $f(x) = f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$ holds. In other words, the function $f : H \to \mathbb{R}$ is cylindrical if there exists an n-dimensional subspace $H_n \subset H$ and orthogonal projector $P : H \to H_n$ such that $f(x) = f(Px)$ for every $x \in H$. The cylindrical function f can be imagined as a function, which is first defined on H_n and then continued to the entire space H in such a way that $f(x) = f(x_0)$ if $x_0 \in H_n$ and $x \in (x_0 + \ker P)$.

Symbol $D = C_{b,c}^\infty(H, \mathbb{R})$ stands for the space of all continuous bounded cylindrical functions $H \to \mathbb{R}$ such that they have Fréchet derivatives[22,29] of all positive integer orders at every point of H, and their Fréchet derivatives of any positive integer order are bounded and continuous.

If $f : H \to \mathbb{R}$ is twice Fréchet differentiable, then $f'(x)$ will stand for the first Fréchet derivative of f at the point x, and $f''(x)$ will denote the second derivative. Riesz–Fréchet representation theorem allows us to assume $f'(x) \in H$ and $f''(x) \in L_b(H, H)$ for every $x \in H$.

Symbol $C_b(H, \mathbb{R})$ stands for the Banach space of all bounded continuous functions $H \to \mathbb{R}$, endowed with a uniform norm $\|f\| = \sup_{x \in H} |f(x)|$. It is regarded as a closed subspace of a complex Banach space $C_b(H, \mathbb{C})$.

Let $X = \overline{C_{b,c}^\infty(H, \mathbb{R})}$ be the closure of the space D in $C_b(H, \mathbb{R})$. It is clear, that X with the norm $\|f\| = \sup_{x \in H} |f(x)|$ is a Banach space, as it is a closed linear subspace of the Banach space $C_b(H, \mathbb{R})$. Function f belongs to X if and only if there is a sequence of functions $(f_j) \subset D$ such that $\lim_{j \to \infty} f_j = f$, i.e. $\lim_{j \to \infty} \sup_{x \in H} |f(x) - f_j(x)| = 0$.

Symbol $C_b(H, H)$ stands for a Banach space of all bounded continuous functions $B : H \to H$, endowed with the uniform norm $\|B\| = \sup_{x \in H} \|B(x)\|$.

Denote $D_H = \{B : H \to H \,|\, \exists N \in \mathbb{N}, b_k \in H, B_k \in D : B(x) = B_1(x)b_1 + \cdots + B_N(x)b_N\}$.

Let X_H be the closure of D_H in $C_b(H, H)$.

If $x \in H$, and $R : H \to H$ is linear, trace class, positive, non-degenerate operator, then symbol μ_R^x stands for the Gaussian probabilistic measure[6,27,42] on H with expectation x and correlation operator R, i.e. the unique sigma-additive measure on Borel sigma-algebra in H such that the equality $\int_H e^{i\langle z, y \rangle} \mu_R^x(dy) = \exp(i\langle z, x \rangle - \frac{1}{2}\langle Rz, z \rangle)$ holds for every $z \in H$. To make it shorter, we will write μ_R instead of μ_R^0.

If $B : H \to H$ is a vector field, and $g : H \to \mathbb{R}$ and $C : H \to \mathbb{R}$ are real-valued functions, then symbol L defines a differential operator on the space of functions $\varphi : H \to \mathbb{R}$

$$(L\varphi)(x) := g(x)\mathrm{tr}A\varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x)\varphi(x), \quad x \in H.$$

The pair $(\mathcal{L}, M)$ defines a linear operator $\mathcal{L}$ with the domain M. It will be shown in Theorem 3.2 that $L(D) \subset X$ when A, B, g and C have certain properties. So

(L, D) is a densely defined (on D) operator $L : X \supset D \to X$. Here the earlier defined spaces D and X are endowed with the uniform norm, induced from $C_b(H, \mathbb{R})$. Let $(\overline{L}, D_1)$ be the closure of (L, D) in X. This means that

$$D_1 = \left\{ f \in X \, | \, \exists (f_j) \subset D : \lim_{j \to \infty} f_j = f, \exists \lim_{j \to \infty} L f_j \right\},$$

and, if $f \in D_1$, then, by definition, $\overline{L} f = \lim_{j \to \infty} L f_j$.

If for every fixed first argument $t > 0$ of the function $u : [0, +\infty) \times H \to \mathbb{R}$ we have $[x \mapsto u(t, x)] \in D_1$, then the expression $\overline{L} u(t, x)$ means the result of applying the operator $\overline{L}$ to the function $x \mapsto u(t, x)$ with the fixed $t > 0$.

Expression $(S_t)_{t \geq 0}$ defines the one-parameter family of linear operators in the space of functions $\varphi : H \to \mathbb{R}$, where $S_0 f = f$ and S_t for $t > 0$ is defined as follows:

$$(S_t \varphi)(x) := e^{tC(x) - t \frac{\langle AB(x), B(x) \rangle}{g(x)}} \int_H \varphi(x + y) e^{\langle \frac{1}{g(x)} B(x), y \rangle} \mu_{2tg(x)A}(dy).$$

Remark 2.1. Further, in Theorem 3.1, we will prove that for every $t \geq 0$ and for A, B, g and C having certain properties the following holds (i) $S_t(X) \subset X$, (ii) operator S_t is bounded, and (iii) $\frac{d}{dt} S_t \varphi \big|_{t=0} = L \varphi$ for all $\varphi \in D$. This will allow us to use the Chernoff approximation (Theorems 2.1, 2.2) and prove the main result of this paper, Theorem 3.4.

2.2. *Integration in Hilbert space*

Lemma 2.1 (Ref. 27, Chap. II, §2, 3°). *If a function $\varphi : H \to \mathbb{R}$ is cylindrical and measurable, i.e. $\varphi(x) = \varphi^n(\langle x, e_1 \rangle, \dots, \langle x, e_n \rangle)$ for some $n \in \mathbb{N}$, some measurable function $\varphi^n : \mathbb{R}^n \to \mathbb{R}$, and some finite orthonormal family of vectors $e_1, \dots, e_n$ from space H, then*

$$\int_H \varphi(y) \mu_A(dy) = \left(\frac{1}{\sqrt{2\pi}} \right)^n \frac{1}{\sqrt{\det M_Q}} \int_{\mathbb{R}^n} \varphi^n(z) \exp\left(-\frac{1}{2} \langle M_Q^{-1} z, z \rangle_{\mathbb{R}^n} \right) dz, \tag{2.1}$$

where $H_n = \text{span}(e_1, \dots, e_n)$, and $P : H \ni h \mapsto \langle h, e_1 \rangle e_1 + \cdots + \langle h, e_n \rangle e_n \in H_n$, $Q = PA$, $Q : H_n \to H_n$, and M_Q is the matrix of the operator Q in basis $e_1, \dots, e_n$ of the space H_n. If $e_1, \dots, e_n$ is a full set of eigenvectors of the operator Q, and $q_1, \dots, q_n$ is the corresponding set of eigenvalues, then

$$\int_H \varphi(y) \mu_A(dy) = \left(\frac{1}{\sqrt{2\pi}} \right)^n \frac{1}{\sqrt{\prod_{i=1}^n q_i}} \int_{\mathbb{R}^n} \varphi^n(z_1, \dots, z_n)$$

$$\cdot \exp\left(-\sum_{i=1}^n \frac{z_i^2}{2q_i} \right) dz_1 \dots dz_n. \tag{2.2}$$

Lemma 2.2 (Explicit form of some integrals over Gaussian measure). *Let H be a real separable Hilbert space of finite or infinite dimension, $\widetilde{A} : H \to H$ be a linear, trace class, symmetric, positive, non-degenerate operator, $\mu_{\widetilde{A}}$ be the centered*

Gaussian measure on H with the correlation operator $\widetilde{A}$, and $G : H \to H$ be a bounded linear operator. Let w and z be nonzero vectors from H.

Then the following equalities hold:

$$\int_H \langle Gy, y \rangle \mu_{\widetilde{A}}(dy) = \text{tr}(\widetilde{A}G), \tag{2.3}$$

$$\int_H e^{\langle z,y \rangle} \mu_{\widetilde{A}}(dy) = e^{\frac{1}{2}\langle \widetilde{A}z, z \rangle}, \tag{2.4}$$

$$\int_H \langle w, y \rangle e^{\langle z,y \rangle} \mu_{\widetilde{A}}(dy) = \langle \widetilde{A}w, z \rangle e^{\frac{1}{2}\langle \widetilde{A}z, z \rangle}, \tag{2.5}$$

$$\int_H \langle Gy, y \rangle e^{\langle z,y \rangle} \mu_{\widetilde{A}}(dy) = (\text{tr}\widetilde{A}G + \langle G\widetilde{A}z, \widetilde{A}z \rangle)e^{\frac{1}{2}\langle \widetilde{A}z, z \rangle}. \tag{2.6}$$

Proof. Formulas (2.3) and (2.4) can be found in Ref. 27, Chap. II, §2, 1°. Formula (2.5) can be derived from the fact that the function under the integral is cylindrical, so Lemma 2.1 can be employed. For a proof of (2.6), one can make the change of variable in the integral, $h = y - Aw$, then (Ref. 27, Chap. II, §4, 2°, Theorem 4.2) we have $\mu_{\widetilde{A}}(dy) = e^{-\frac{1}{2}\langle \widetilde{A}w, w \rangle - \langle h, w \rangle} \mu_{\widetilde{A}}(dh)$, and the integral reduces to (2.3). □

Lemma 2.3 (On a linear change of variable in the integral over Gaussian measure). *Let H be a real separable Hilbert space. Suppose a linear operator $A : H \to H$ is positive, non-degenerate, trace class, and self-adjoint. We will identify with the symbol μ_A the centered Gaussian measure on H with the correlational operator A. Let $t > 0$; the symbol tA denotes operator, that takes $x \in H$ to $tAx \in H$. Let $f : H \to \mathbb{R}$ be a continuous integrable function.*

Then

$$\int_H f(x)\mu_{tA}(dx) = \int_H f(\sqrt{t}x)\mu_A(dx). \tag{2.7}$$

Proof. It uses the uniqueness of the Gaussian measure with a given Fourier transform, and the standard theorem of changing variable in the Lebesgue integral.

□

Lemma 2.4 (On integrability of a polynomial multiplied by an exponent). Let H, A, μ_A be as above, $P : \mathbb{R} \to \mathbb{R}$ be a polynomial, and $\beta \in \mathbb{R}$. *Then function $H \ni x \mapsto P(\|x\|)e^{\beta\|x\|} \in \mathbb{R}$ is integrable over μ_A.*

Proof. It is easy to construct by relying on Fernique's theorem,[33] which (applied to this case) says that there exists such $\alpha > 0$ that $\int_H e^{\alpha\|y\|^2} \mu_A(dy) < +\infty$. □

2.3. *Differentiation in Hilbert space*

Proposition 2.1. *Let f be a cylindrical real-valued function on H, i.e. there is a number $n \in \mathbb{N}$ and a function $f^n : \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in H$ the equality*

$f(x) = f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$ *holds. A set of vectors* $e_1, \ldots, e_n$ *can be considered orthonormal without loss of generality. Let us complete this set to an orthonormal basis* $(e_k)_{k \in \mathbb{N}}$ *in* H.

Then:

(1) *Function* f *is differentiable in the direction* h *if and only if the function* f^n *is differentiable in the direction* $(\langle h, e_1 \rangle, \ldots, \langle h, e_n \rangle) \in \mathbb{R}^n$, *and*

$$f'(x)h = \langle h, (\partial_1 f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), \ldots, \partial_n f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), 0, 0, \ldots) \rangle,$$

where the symbol $\partial_j f^n$ *defines the partial derivative with respect to the* j*th argument of the function* f^n, *and* $(\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) = \alpha_1 e_1 + \cdots + \alpha_n e_n$. *If the function* f *has a Fréchet derivative at the point* x, *then* $f'(x)$ *is a vector whose first* n *coordinates yield the gradient of the function* f^n, *and the other coordinates are zero*:

$$f'(x) = (\partial_1 f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), \ldots, \partial_n f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), 0, 0, \ldots). \tag{2.8}$$

(2) *Function* f *has a Fréchet derivative in* H *if and only if the function* f^n *has a Fréchet derivative in* $\mathbb{R}^n$.

(3) *Let* $A : H \to H$ *be a trace-class operator (i.e. let* $\mathrm{tr}A < \infty$). *Then*

$$\mathrm{tr}Af''(x) = \sum_{s=1}^{n} \sum_{k=1}^{n} \langle Ae_s, e_k \rangle (\partial_k \partial_s f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle))$$

$$= \mathrm{tr}(A_n(f^n)''(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)), \tag{2.9}$$

where A_n *is the matrix of the operator* PA *in the basis* $e_1, \ldots, e_n$, *where* P *is the projector to the linear span of the vectors* $e_1, \ldots, e_n$.

Proof. It is a straight-forward application of the derivative's definition. $\qquad \square$

Proposition 2.2 (See Ref. 7 or other book on infinite-dimensional analysis). *For* $(n + 1)$-*times Fréchet differentiable function* $f : H \to \mathbb{R}$ *there is*[8] *a Taylor decomposition*

$$f(x + h) = f(x) + \frac{1}{1!} f'(x)h + \frac{1}{2!} f''(x)(h, h) + \cdots$$

$$+ \frac{1}{n!} f^{(n)}(x)(h, \ldots, h) + R_n(x, h), \tag{2.10}$$

where the following estimate holds:

$$|R_n(x, h)| \leq \frac{\|h\|^{n+1}}{(n + 1)!} \sup_{z \in [x, x+h]} \|f^{(n+1)}(z)\|. \tag{2.11}$$

2.4. *Differential operator on a finite-dimensional space*

Lemma 2.5 (Ref. 41, Theorems 4.3.1, 4.3.2. and Corollary 4.3.4). *Suppose for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$ functions $a^{ij} : \mathbb{R}^n \to \mathbb{R}$, $b^i : \mathbb{R}^n \to \mathbb{R}$, $c : \mathbb{R}^n \to \mathbb{R}$ from $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ are given, where $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ is the class of all bounded real-valued functions on $\mathbb{R}^n$, which have bounded partial derivatives of all orders. Suppose also that $c(x) \leq 0$ for all $x \in \mathbb{R}^n$.*

For $u \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ we define a differential operator T by the formula

$$(Tu)(x) = \sum_{i=1}^n \sum_{j=1}^n a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + \sum_{i=1}^n b^i(x) \frac{\partial}{\partial x_i} u(x) + c(x)u(x).$$

Suppose that there exists a constant $\varkappa > 0$ such that for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$ the ellipticity condition is fulfilled: $\sum_{i=1}^n \sum_{j=1}^n a^{ij}(x)\xi_i\xi_j \geq \varkappa \|\xi\|^2$. Take an arbitrary constant $\lambda > 0$ and function $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$.

Then:

(1) *There is a unique function $u \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$, which is a solution of the equation*

$$(Tu)(x) - \lambda u(x) = f(x). \tag{2.12}$$

(2) *For every function $v \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ the following estimate is true*

$$\sup_{x \in \mathbb{R}^n} |(Tv)(x) - \lambda v(x)| \geq \lambda \sup_{x \in \mathbb{R}^n} |v(x)|. \tag{2.13}$$

Note that Eq. (2.12) can have unbounded solutions; this does not contradict the lemma.

2.5. *C_0-semigroups and evolution equations*

Let $\mathcal{X}$ be a complex Banach space.

Definition 2.1. By a C_0-semigroup, or a strongly continuous one-parameter semigroup $(T_s)_{s \geq 0}$ of linear bounded operators in $\mathcal{X}$ we (following Refs. 31 and 51) mean the mapping

$$T : [0, +\infty) \to L_b(\mathcal{X}, \mathcal{X})$$

of the non-negative half-line into the space of all bounded linear operators on $\mathcal{X}$, which satisfies the following conditions:

(1) $\forall \varphi \in \mathcal{X} : T_0 \varphi = \varphi$.
(2) $\forall t \geq 0, \forall s \geq 0 : T_{t+s} = T_t \circ T_s$.
(3) $\forall \varphi \in \mathcal{X}$ function $s \mapsto T_s \varphi$ is continuous as a mapping $[0, +\infty) \to \mathcal{X}$.

Definition 2.2. By the generator of a strongly continuous one-parameter semigroup $(T_s)_{s \geq 0}$ of linear bounded operators on $\mathcal{X}$ we mean a linear operator

$\overline{\mathcal{L}} : \mathcal{X} \supset \mathrm{Dom}(\overline{\mathcal{L}}) \to \mathcal{X}$ given by the formula

$$\overline{\mathcal{L}}\varphi = \lim_{s \to +0} \frac{T_s\varphi - \varphi}{s}$$

on its domain

$$\mathrm{Dom}(\overline{\mathcal{L}}) = \left\{ \varphi \in \mathcal{X} : \exists \lim_{s \to +0} \frac{T_s\varphi - \varphi}{s} \right\},$$

where the limit is understood in the strong sense, i.e. it is defined in terms of the norm in the space $\mathcal{X}$.

The use of the symbol $\overline{\mathcal{L}}$ for the generator is related to the fact that the generator is always a closed operator:

Proposition 2.3 (Theorem 1.4 in Ref. 31, p. 51). *The generator of a strongly continuous semigroup is a closed linear operator with a dense domain. The generator defines its semigroup uniquely.*

Proposition 2.4 (Lemma 1.1 and Definition 1.2. in Ref. 31, pp. 48–49). *The set $\mathrm{Dom}(\overline{\mathcal{L}})$ coincides with the set of those $\varphi \in \mathcal{X}$, for which the mapping $s \mapsto T_s\varphi$ is differentiable with respect to s at every point $s \in [0, +\infty)$.*

Definition 2.3. (1) The problem of finding a function $U : [0, +\infty) \to \mathcal{X}$ such that

$$\begin{cases} \dfrac{d}{dt}U(t) = \overline{\mathcal{L}}U(t); & t \geq 0, \\ U(0) = U_0, \end{cases} \tag{2.14}$$

is called the abstract Cauchy problem, associated with the closed linear operator $\overline{\mathcal{L}} : \mathcal{X} \supset \mathrm{Dom}(\overline{\mathcal{L}}) \to \mathcal{X}$ and a vector $U_0 \in \mathcal{X}$.

(2) A function $U : [0, +\infty) \to \mathcal{X}$ is called a classic solution to abstract Cauchy problem (2.14) if, for every $t \geq 0$, the function U has a continuous derivative $U' : [0, +\infty) \to \mathcal{X}$, $U(t) \in \mathrm{Dom}(\overline{\mathcal{L}})$, and (2.14) holds.

(3) A continuous function $U : [0, +\infty) \to \mathcal{X}$ is called a mild solution to abstract Cauchy problem (2.14) if for every $t \geq 0$ we have $\int_0^t U(s)ds \in \mathrm{Dom}(\overline{\mathcal{L}})$ and $U(t) = \overline{\mathcal{L}} \int_0^t U(s)ds + U_0$.

Proposition 2.5 (Proposition 6.2 in Ref. 31, p. 145). *If the operator $(\overline{\mathcal{L}}, \mathrm{Dom}(\overline{\mathcal{L}}))$ is a generator of a strongly continuous semigroup $(T_s)_{s \geq 0}$, then:*

(1) *For every $U_0 \in \mathrm{Dom}(\overline{\mathcal{L}})$ there is a unique classic solution to abstract Cauchy problem (2.14), which is given by the formula $U(t) = T(t)U_0$.*

(2) *For every $U_0 \in \mathcal{X}$ there is a unique mild solution to abstract Cauchy problem (2.14), which is given by the formula $U(t) = T(t)U_0$.*

Definition 2.4. Linear operator $\mathcal{L} : \mathcal{X} \supset \mathrm{Dom}(\mathcal{L}) \to \mathcal{X}$ in Banach space $\mathcal{X}$ is called dissipative if for every $\lambda > 0$ and every $x \in \mathrm{Dom}(\mathcal{L})$ the estimate $\|\mathcal{L}x - \lambda x\| \geq \lambda\|x\|$ holds.

Proposition 2.6 (On the closability of a densely defined dissipative operator, Proposition 3.14 in Ref. 31). *A linear dissipative operator $\mathcal{L} : \mathcal{X} \supset \mathrm{Dom}(\mathcal{L}) \to \mathcal{X}$ in the Banach space $\mathcal{X}$ with the domain $\mathrm{Dom}(\mathcal{L})$ dense in $\mathcal{X}$ is closable. The closure $\overline{\mathcal{L}} : \mathcal{X} \supset \mathrm{Dom}(\overline{\mathcal{L}}) \to X$ is also a dissipative operator.*

The main tool for the construction of Feynman formulas for the solutions of the Cauchy problem is Chernoff's theorem. For convenience we decompose its conditions into several blocks and give them separate names, as follows.

Theorem 2.1 (Chernoff, 1968; see Ref. 23 and Theorem 10.7.21 in Ref. 7).
Let $\mathcal{X}$ be Banach space, and $L_b(\mathcal{X}, \mathcal{X})$ be the space of all linear bounded operators in $\mathcal{X}$ endowed with the operator norm. Let $\overline{\mathcal{L}} : \mathcal{X} \supset \mathrm{Dom}(\overline{\mathcal{L}}) \to \mathcal{X}$ be a linear operator.
Suppose there is a function F such that:

(E) *There exists a strongly continuous semigroup $(e^{t\overline{\mathcal{L}}})_{t \geq 0}$, and its generator is $(\overline{\mathcal{L}}, \mathrm{Dom}(\overline{\mathcal{L}}))$.*

(CT1) *F is defined on $[0, +\infty)$, takes values in $L_b(\mathcal{X}, \mathcal{X})$ and $t \mapsto F(t)f$ is continuous for every vector $f \in \mathcal{X}$.*

(CT2) *$F(0) = I$.*

(CT3) *There exists a dense subspace $\mathcal{D} \subset \mathcal{X}$ such that for every $f \in \mathcal{D}$ there exists a limit $F'(0)f = \lim_{t \to 0}(F(t)f - f)/t = \mathcal{L}f$.*

(CT4) *The operator $(\mathcal{L}, \mathcal{D})$ has a closure $(\overline{\mathcal{L}}, \mathrm{Dom}(\overline{\mathcal{L}}))$.*

(N) *There exists $\omega \in \mathbb{R}$ such that $\|F(t)\| \leq e^{\omega t}$ for all $t \geq 0$.*

Then for every $f \in \mathcal{X}$ we have $(F(t/n))^n f \to e^{t\overline{\mathcal{L}}}f$ as $n \to \infty$, and the limit is uniform with respect to t from every segment $[0, t_0]$ for every fixed $t_0 > 0$.

Definition 2.5. In this paper, two mappings F_1 and F_2 are called Chernoff-equivalent if there exists a C_0-semigroup $(e^{t\overline{\mathcal{L}}})_{t \geq 0}$ such that $(F_1(t/n))^n f \to e^{t\overline{\mathcal{L}}}f$, $(F_2(t/n))^n f \to e^{t\overline{\mathcal{L}}}f$ for every $f \in \mathcal{X}$ as $n \to \infty$, and the limit is uniform with respect to t from every segment $[0, t_0]$ for every fixed $t_0 > 0$.

Remark 2.2. There are several slightly different definitions of the Chernoff equivalence, see e.g., Refs. 70, 48 and 17. We will just use this one not going into details. The only thing we need from this definition is that if F satisfies all the conditions of Chernoff's theorem, then by Chernoff's theorem the mapping F is Chernoff-equivalent to the mapping $F_1(t) = e^{t\overline{\mathcal{L}}}$, i.e. the limit of $(F(t/n))^n$ as n tends to infinity yields the C_0-semigroup $(e^{t\overline{\mathcal{L}}})_{t \geq 0}$.

Definition 2.6. Let us follow Ref. 57 and call a mapping F Chernoff-tangent to the operator $\mathcal{L}$ if it satisfies the conditions (CT1)-(CT4) of Chernoff's theorem.

Remark 2.3. With these definitions the Chernoff-equivalence of F to $(e^{t\overline{\mathcal{L}}})_{t \geq 0}$ follows from: existence (E) of the C_0-semigroup + Chernoff-tangency (CT) + growth of the norm bound (N).

Theorem 2.2 (Chernoff-type theorem, Corollary 5.3 from Theorem 5.2 in Ref. 31). *Let $\mathcal{X}$ be a Banach space, and $L_b(\mathcal{X}, \mathcal{X})$ be the space of all linear bounded operators on $\mathcal{X}$ endowed with the operator norm. Suppose there is a function*

$$V : [0, +\infty) \to L_b(\mathcal{X}, \mathcal{X}),$$

meeting the condition $V_0 = I$, where I is the identity operator. Suppose there are numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|(V_t)^k\| \leq Me^{k\omega t}$ for every $t \geq 0$ and every $k \in \mathbb{N}$. Suppose the limit

$$\lim_{t \downarrow 0} \frac{V_t \varphi - \varphi}{t} =: \mathcal{L}\varphi$$

exists for every $\varphi \in \mathcal{D} \subset \mathcal{X}$, where $\mathcal{D}$ is a dense subspace of $\mathcal{X}$. Suppose there is a number $\lambda_0 > \omega$ such that $(\lambda_0 I - \mathcal{L})(\mathcal{D})$ is a dense subspace of $\mathcal{X}$.

Then the closure $\overline{\mathcal{L}}$ of the operator $\mathcal{L}$ is a generator of a strongly continuous semigroup of operators $(T_t)_{t \geq 0}$ given by the formula

$$T_t \varphi = \lim_{n \to \infty} (V_{\frac{t}{n}})^n \varphi$$

where the limit exists for every $\varphi \in \mathcal{X}$ and is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$. Moreover $(T_t)_{t \geq 0}$ satisfies the estimate $\|T_t\| \leq Me^{\omega t}$ for every $t \geq 0$.

Theorem 2.3 (Approximation of generator implies approximation of semigroup, Theorem 4.9 in Ref. 31). *Let $(e^{\overline{\mathcal{L}_j}t})_{t \geq 0}$ be a sequence of strongly continuous semigroups of operators in a Banach space $\mathcal{X}$ with the generators $(\overline{\mathcal{L}_j}, \mathrm{Dom}(\overline{\mathcal{L}_j}))$, which satisfies, for some fixed constants $M \geq 1, w \in \mathbb{R}$, the condition $\|e^{\overline{\mathcal{L}_j}t}\| \leq Me^{wt}$ for all $t \geq 0$ and every $j \in \mathbb{N}$. Suppose there is a closed linear operator $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$ on $\mathcal{X}$ with a dense domain $\mathrm{Dom}(\mathcal{L})$, such that $\overline{\mathcal{L}_j}x \to Lx$ for every $x \in \mathrm{Dom}(\mathcal{L})$. Suppose the image of the operator $(\lambda_0 I - \mathcal{L})$ is dense in $\mathcal{X}$ for some $\lambda_0 > 0$.*

Then the semigroups $(e^{\overline{\mathcal{L}_j}t})_{t \geq 0}, j \in \mathbb{N}$ converge strongly (and uniformly in $t \in [0, t_0]$ for every fixed $t_0 > 0$) to a strongly continuous semigroup $(e^{\overline{\mathcal{L}}t})_{t \geq 0}$ with the generator $\overline{\mathcal{L}}$. In other words, for every $x \in \mathcal{X}$ there exists $\lim_{j \to \infty} e^{\overline{\mathcal{L}_j}t}x = e^{\overline{\mathcal{L}}t}x$ uniformly in $t \in [0, t_0]$ for every fixed $t_0 > 0$.

Remark 2.4. Below, the role of $\mathcal{X}$ will be played by space X, a closed real subspace of the complex Banach space $C_b(H, \mathbb{C})$. Because all the operators used in this paper below are real, and (as it will be proven further in Theorems 3.1 and 3.2) X is invariant with respect to them, the above theorems about $\mathcal{X}$ are applicable to X.

3. Main Results

3.1. *Properties of spaces D, X, D_1*

Remark 3.1. It directly follows from the definitions of these spaces that

(i) $D \subset D_1 \subset X \subset C_b(H, \mathbb{R}) \subset C_b(H, \mathbb{C})$;
(ii) D and D_1 are dense in X;
(iii) X is a Banach space.

Proposition 3.1. *If $f \in D$, then f is uniformly continuous.*

Proof. It follows from the definition of the space D that the function $D \ni f :$ $H \to \mathbb{R}$ is bounded and its Fréchet derivatives of all orders exist and are bounded. In particular, there exists

$$\sup_{x \in H} \|f'(x)\| = M < \infty. \tag{3.1}$$

One can set $n = 0$ in Taylor's formula (2.10) to ensure that for every $x \in H$ and every $y \in H$ there exists a real number $R_1(x, y)$ such that

$$f(x) - f(y) = R_1(x, y), \tag{3.2}$$

and the estimate holds

$$|R_1(x,y)| \overset{(2.11)}{\leq} \frac{\|y - x\|^1}{1!} \sup_{z \in [x,y]} \|f'(z)\| \overset{(3.1)}{\leq} M\|x - y\|. \tag{3.3}$$

Hence, for each $x \in H$ and each $y \in H$ we have

$$|f(x) - f(y)| \overset{(3.2)}{=} \|R_1(x, y)\| \overset{(3.3)}{\leq} M\|x - y\|, \tag{3.4}$$

which implies the uniform continuity of f. $\qquad\square$

Proposition 3.2. *If $\varphi \in X$, then φ is uniformly continuous.*

Proof. Take any given $\varepsilon > 0$. Let us find $\delta > 0$ such that $\|x - y\| < \delta$ implies $|\varphi(x) - \varphi(y)| < \varepsilon$.

As $\varphi \in X$, there exists a sequence of functions $(f_j) \subset D$ converging to φ uniformly. Hence, there exists a number j_0 such that (introducing the notation $f_{j_0} = f$) we have

$$\|\varphi - f_{j_0}\| = \|\varphi - f\| = \sup_{x \in H} |\varphi(x) - f(x)| < \frac{\varepsilon}{3}. \tag{3.5}$$

Moreover, as $f \in D$, Proposition 3.1 implies estimate (3.4) with some $M > 0$. Let us set $\delta = \frac{\varepsilon}{3M}$ and note that $\|x - y\| < \delta$. Then

$$|\varphi(x) - \varphi(y)| \leq |\varphi(x) - f(x)| + |f(x) - f(y)| + |f(y) - \varphi(y)|$$

$$\overset{(3.4),\,(3.5)}{<} \frac{\varepsilon}{3} + M\frac{\varepsilon}{3M} + \frac{\varepsilon}{3} = \varepsilon. \qquad\square$$

Proposition 3.3. *Suppose that a sequence of functions $(f_j)_{j=1}^{\infty} \subset X$ converges uniformly to a function $f_0 \in X$. Then the family $(f_j)_{j=0}^{\infty}$ is equicontinuous.*

Proof. Suppose $\varepsilon > 0$ is given. Let us find $\delta > 0$ such that $\|x - y\| < \delta$ implies that $|\varphi(x) - \varphi(y)| < \varepsilon$.

By Proposition 3.2, function f_j is uniformly continuous for each $j = 0, 1, 2, \ldots$. Thus, for each $j = 0, 1, 2, \ldots$ there exists $\delta_j > 0$ such that $\|x - y\| < \delta_j$ implies

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{3}. \tag{3.6}$$

As $f_j \to f_0$ uniformly, there exists j_0 such that for all $j > j_0$

$$\sup_{x \in H} |f_0(x) - f_j(x)| < \frac{\varepsilon}{3}. \tag{3.7}$$

Let us set $\delta = \min(\delta_0, \delta_1, \ldots, \delta_{j_0})$. Then for $j > j_0$ we have that $\|x - y\| < \delta$ implies

$$|f_j(x) - f_j(y)| \leq |f_j(x) - f_0(x)| + |f_0(x) - f_0(y)|$$

$$+ |f_0(y) - f_j(y)| \stackrel{(3.6), (3.7)}{<} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Now, if $0 \leq j \leq j_0$, then $\|x - y\| < \delta$ implies estimate (3.6), which is even stronger. $\qquad \square$

Remark 3.2. A number $a \in \mathbb{R}$ is called a limit at infinity of a function $f : H \to \mathbb{R}$ if $\lim_{R \to +\infty} \sup_{\|x\| \geq R} |f(x) - a| = 0$. It is shown in Ref. 55 that if H is infinite-dimensional, then a non-constant function that belongs to X cannot have a limit at infinity. E.g., the function $x \mapsto \exp(-\|x\|^2)$ belongs to $C_b(H, \mathbb{R})$ but not to X.

Remark 3.3. Suppose that $\alpha_k : \mathbb{R} \to \mathbb{R}$ is a family of infinitely-smooth functions, uniformly bounded with their first and second derivatives:

$$\sup_{p \in \{0,1,2\}} \sup_{k \in \mathbb{N}} \sup_{t \in \mathbb{R}} \left| \frac{d^p \alpha_k(t)}{dt^p} \right| \leq M \equiv \text{const.}$$

For example: $\alpha_k(t) = \sin(d_k(t - t_k))$, where d_k and t_k are constants and $0 < d_k \leq 1$. Suppose numerical series $\sum_{k=1}^{\infty} b_k$ converges absolutely. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis in H. Then straightforward reasoning shows that function

$$f(x) = \sum_{k=1}^{\infty} b_k \alpha_k(\langle x, e_k \rangle)$$

belongs to the class D_1. This statement can be extended to the case $\alpha_k : \mathbb{R}^{n_k} \to \mathbb{R}$.

Remark 3.4. Space D is not separable (it does not have a countable dense subset). In the case of one-dimensional H it can be shown similar to the standard proof of the non-separability of $C_b(\mathbb{R}, \mathbb{R})$. If $\dim H > 1$, then $\mathbb{R}^1$ can be embedded into H as a linear span of a nonzero vector $e \in H$. Using this, one can embed the set of cylindrical functions contributing to the non-separability of D in the case of one-dimensional H, into the space D in the general case.

Remark 3.5. By Remark 3.4 and the inclusion $D \subset D_1 \subset X$, one can see that D_1 and X are not separable too.

3.2. *Family S_t provides a semigroup with generator $\overline{L}$*

Theorem 3.1 (On the properties of family $(S_t)_{t \geq 0}$ and its connection to the operator L). *Suppose that $g \in X$, and for every $x \in H$ we have $g(x) \geq g_o \equiv$ const > 0. Suppose that $B \in X_H$ and $C \in X$. Suppose that $t > 0$, and $\mu_{2tg(x)A}$ is the centered Gaussian measure on H with the correlation operator $2tg(x)A$.*

For $t \geq 0$ and $\varphi \in C_b(H, \mathbb{R})$ let us define

$$(S_t \varphi)(x) := e^{tC(x) - t\frac{\langle AB(x), B(x) \rangle}{g(x)}}$$

$$\cdot \int_H \varphi(x + y) e^{\langle \frac{1}{g(x)} B(x), y \rangle} \mu_{2tg(x)A}(dy) \quad for \ t > 0, \quad and \quad S_0 \varphi := \varphi. \tag{3.8}$$

Then:

(1) *If $t \geq 0$ and $\varphi \in C_b(H, \mathbb{R})$, then $S_t \varphi \in C_b(H, \mathbb{R})$. For every $t \geq 0$ the operator $S_t : C_b(H, \mathbb{R}) \to C_b(H, \mathbb{R})$ is linear and bounded; its norm does not exceed $e^{(\frac{2\|A\|\|B\|^2}{g_0} + \|C\|)t}$.*

(2) *If $g \in D$, $C \in D$, $B \in D_H$, then the space D for every $t \geq 0$ is invariant with respect to the operator S_t.*

(3) *If $g \in X$, $C \in X$, $B \in X_H$, then the space X for every $t \geq 0$ is invariant with respect to the operator S_t.*

(4) *For every function $\varphi \in D$, for $g \in X$, $C \in X$, $B \in X_H$ there exists (uniformly with respect to $x \in H$) a limit*

$$\lim_{t \to 0} \frac{(S_t \varphi)(x) - \varphi(x)}{t} = g(x) \mathrm{tr} A \varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x) \varphi(x)$$

$$= (L\varphi)(x).$$

(5) *If $\varphi \in X$, $g \in X$, $C \in X$, $B \in X_H$, then the function $[0, +\infty) \ni t \mapsto S_t \varphi \in X$ is continuous, i.e. if $t_0 \geq 0, t_n \geq 0$ and $t_n \to t_0$, then $\sup_{x \in H} |(S_{t_n} \varphi)(x) - (S_{t_0} \varphi)(x)| \to 0$.*

Proof. (1) Function φ is bounded, so integral (3.8) exists by Lemma 2.4. Suppose a number $t > 0$ and a function $\varphi \in C_b(H, \mathbb{R})$ are fixed. Recalling Lemma 2.3, one can see that $\int_H \varphi(x + y) e^{\langle \frac{1}{g(x)} B(x), y \rangle} \mu_{2tg(x)A}(dy) = \int_H \varphi(x + \sqrt{2tg(x)}y) e^{\langle \frac{1}{g(x)} B(x), \sqrt{2tg(x)}y \rangle} \mu_A(dy)$.

Introducing the notation $\|B\| = \sup_{x \in H} \|B(x)\|$, we obtain the estimate

$$\|S_t \varphi\| = \sup_{x \in H} \left| e^{tC(x) - \frac{\langle AB(x), B(x) \rangle}{g(x)} t} \int_H \varphi(x + \sqrt{2tg(x)}y) \right.$$

$$\left. \cdot e^{\langle \frac{1}{g(x)} B(x), \sqrt{2tg(x)}y \rangle} \mu_A(dy) \right|$$

$$\leq \sup_{x \in H} \left| e^{tC(x) - \frac{\langle AB(x), B(x) \rangle}{g(x)} t} \right| \sup_{x \in H} |\varphi(x)|$$

$$\cdot \sup_{x \in H} \int_H e^{\left\langle \sqrt{\frac{2t}{g(x)}} B(x), y \right\rangle} \mu_A(dy)$$

$$\overset{(2.4)}{=} e^{t\left(\|C\| + \left\| \frac{\langle AB(x), B(x) \rangle}{g(x)} \right\| \right)} \|\varphi\| \sup_{x \in H} e^{\frac{1}{2} \frac{2t}{g(x)} \langle AB(x), B(x) \rangle}$$

$$\overset{\langle x_1, x_2 \rangle \leq \|x_1\| \cdot \|x_2\|}{\leq} e^{\left(\frac{2\|A\|\|B\|^2}{g_0} + \|C\| \right) t} \|\varphi\|, \tag{3.9}$$

which implies that the function $x \mapsto (S_t \varphi)(x)$ is bounded. Let us prove that this function is continuous. Suppose $x_j \to x$, then for every $y \in H$

$$\varphi\left(x_j + \sqrt{2tg(x_j)}y\right) e^{\left\langle \sqrt{\frac{2t}{g(x_j)}} B(x_j), y \right\rangle} \to \varphi(x + \sqrt{2tg(x)}y) e^{\left\langle \sqrt{\frac{2t}{g(x)}} B(x), y \right\rangle}.$$

Moreover,

$$\left| \varphi\left(x_j + \sqrt{2tg(x_j)}y\right) e^{\left\langle \sqrt{\frac{2t}{g(x_j)}} B(x_j), y \right\rangle} \right| \leq \|\varphi\| e^{\sqrt{\frac{2t}{g_0}} \|B\|\|y\|}$$

and, in a similar way,

$$\left| \varphi(x_j + \sqrt{2tg(x)}y) e^{\left\langle \sqrt{\frac{2t}{g(x)}} B(x), y \right\rangle} \right| \leq \|\varphi\| e^{\sqrt{\frac{2t}{g_0}} \|B\|\|y\|}.$$

Lemma 2.4 implies that the function $y \mapsto e^{\sqrt{\frac{2t}{g_0}} \|B\|\|y\|}$ is integrable over the measure μ_A. Therefore by Lebesgue dominated convergence theorem we have $\lim_{j \to \infty} \int_H \varphi(x_j + \sqrt{2tg(x_j)}y) e^{\left\langle \sqrt{\frac{2t}{g(x_j)}} B(x_j), y \right\rangle} \mu_A = \int_H \varphi(x + \sqrt{2tg(x)}y) e^{\left\langle \sqrt{\frac{2t}{g(x)}} B(x), y \right\rangle} \mu_A$.

Because the functions C, B, g are continuous and $g(x) \geq g_0 > 0$, we have $e^{tC(x_j) - \frac{\langle AB(x_j), B(x_j) \rangle}{g(x_j)} t} \to e^{tC(x) - \frac{\langle AB(x), B(x) \rangle}{g(x)} t}$. Therefore, $(S_t \varphi)(x_j) \to (S_t \varphi)(x)$. So we have proved that $S_t \varphi \in C_b(H, \mathbb{R})$. Estimate (3.9) shows that $\|S_t\| \leq e^{\left(\frac{2\|A\|\|B\|^2}{g_0} + \|C\| \right) t}$.

(2) We fix $t > 0$ and prove that $S_t \varphi \in D$.

(i) First of all, if $g \in D$, $C \in D$, $B \in D_H$, then the operator S_t maps the cylindrical function φ into a cylindrical function $S_t \varphi$. This follows from the fact that (3.8) is a cylindrical function of cylindrical functions, which are functions of finite number of linear functionals of x. Therefore, the number $(S_t \varphi)(x)$ depends on x only via a finite number of linear functionals, hence $x \mapsto (S_t \varphi)(x)$ is a cylindrical function, see (3.11) for the exact formula.

(ii) Let us introduce some notation. As φ is cylindrical, then for every $x \in H$ the equality $\varphi(x) = \varphi^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$ holds for some $n \in \mathbb{N}$, some function $\varphi^n : \mathbb{R}^n \to \mathbb{R}$ and some set of vectors $e_1, \ldots, e_n$ of the space H. Functions g, C, B are also cylindrical, and without loss of generality we can accept that the set of vectors $e_1, \ldots, e_n$ is so large that the following holds: $g(x) = g^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$, $C(x) = C^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$, $B(x) = B_1(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)e_1 + \cdots + B_n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)e_n$.

At this moment vectors $e_1, \ldots, e_n$ can be in arbitrary position with respect to the eigenvectors of the operator A. Without loss of generality the set $e_1, \ldots, e_n$ can be considered orthonormalized.

Let us introduce the definitions: $\Psi_n : H \ni h \mapsto (\langle h, e_1 \rangle, \ldots, \langle h, e_n \rangle) \in \mathbb{R}^n$ — a projector, $H_n = \text{span}(e_1, \ldots, e_n)$ — a subspace in H, $I_n : H_n \ni h \mapsto (\langle h, e_1 \rangle, \ldots, \langle h, e_n \rangle) \in \mathbb{R}^n$ — an isomorphism, $P_n : H \ni h \mapsto \langle h, e_1 \rangle e_1 + \cdots + \langle h, e_n \rangle e_n \in H_n$ — a projector. Next, denote $\overrightarrow{x}_1^n = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\overrightarrow{B}_1^n (\overrightarrow{x}_1^n) = (B_1(x_1, \ldots, x_n), \ldots, B_n(x_1, \ldots, x_n)) \in \mathbb{R}^n$. With these definitions we have $\Psi_n = I_n P_n$ and $\varphi(x) = \varphi^n(\Psi_n x)$, $g(x) = g^n(\Psi_n x)$, $C(x) = C^n(\Psi_n x)$, $B(x) = \overrightarrow{B}_1^n (\Psi_n x)$.

Let us introduce the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ in the following way:

$$
\Phi\left(\overrightarrow{x}_1^n \right) = \int_H \varphi^n\left(\overrightarrow{x}_1^n + \sqrt{2tg^n(\overrightarrow{x}_1^n)}\Psi_n(y) \right) \cdot e^{\sqrt{\frac{2t}{g^n(\overrightarrow{x}_1^n)}}\left\langle \overrightarrow{B}_1^n (\overrightarrow{x}_1^n), \Psi_n(y) \right\rangle} \mu_A(dy).
$$
(3.10)

Then for every $x \in H$ we have

$$
(S_t\varphi)(x) = \Phi(\Psi_n x) \exp\left(tC^n(\Psi_n x) - \frac{\left\langle A \overrightarrow{B}_1^n (\Psi_n x), \overrightarrow{B}_1^n (\Psi_n x) \right\rangle}{g^n(\Psi_n x)} t \right).
$$
(3.11)

(iii) Now let us prove that $S_t\varphi$ has bounded Fréchet derivatives of all orders employing Proposition 2.1. To do this we need to prove that the functions $\mathbb{R}^n \to \mathbb{R}$ have bounded Fréchet derivatives of all orders. The exponent in (3.11) has this property because the exponent is obtained by composition and arithmetical operations from the functions with this property.

Let us show that Φ has Fréchet derivatives of all orders. The product of differentiable functions under the sign of integral in (3.10) is differentiable, so the problem is reduced to the verification of the differentiability of the integral. To do this, we apply Lemma 2.1 and arrive from an integral over H to an integral over $\mathbb{R}^n$ in the expression for Φ (this is possible because the integrand is cylindrical).

Operator A is non-degenerate and symmetric on H, therefore the operator $P_n A$ is non-degenerate and symmetric on H_n, and therefore it can be diagonalized in some orthonormal basis $b_1, \ldots, b_n$. Without loss of generality we can assume that the vectors $e_1, \ldots, e_n$ form such a basis. Indeed, changing the basis in the space H_n will just produce linear non-degenerate change of variables in the functions $\mathbb{R}^n \to \mathbb{R}$ used to define cylindrical functions $H \to \mathbb{R}$. This will give us new functions $\mathbb{R}^n \to \mathbb{R}$, but all their properties that we need will be preserved.

The matrix of the operator $P_n A$ in H_n coincides with the matrix of the operator $Q_n = I_n P_n A I_n^{-1}$ in $\mathbb{R}^n$. Next, let $q_1, \ldots, q_n$ be the eigenvalues of the operator Q_n, corresponding to the eigenvectors $\Psi_n e_1, \ldots, \Psi_n e_n$. Note, that $q_i > 0$ and

$g^n(\vec{x}_1^{\,n}) \geq g_0 \equiv \text{const} > 0$ for every $\vec{x}_1^{\,n} \in \mathbb{R}^n$. Then by (2.2) we have

$$\Phi\left(\vec{x}_1^{\,n}\right) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\prod_{i=1}^n q_i}} \int_{\mathbb{R}^n} \varphi^n\left(\vec{x}_1^{\,n} + \sqrt{2tg^n(\vec{x}_1^{\,n})}z\right)$$

$$\cdot\, e^{\sqrt{\frac{2t}{g^n(\vec{x}_1^{\,n})}}\left\langle \vec{B}_1^n(\vec{x}_1^{\,n}), z\right\rangle} \exp\left(-\sum_{i=1}^n \frac{z_i^2}{2q_i}\right) dz.$$

Now we introduce a measure ν on $\mathbb{R}^n$ given by its density with respect to the Lebesgue measure: for every measurable set $\mathcal{A} \subset \mathbb{R}^n$ we set

$$\nu(\mathcal{A}) := \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\prod_{i=1}^n q_i}} \int_{\mathcal{A}} \exp\left(-\sum_{i=1}^n \frac{z_i^2}{2q_i}\right) dz.$$

It follows from the definitions given above that

$$\Phi\left(\vec{x}_1^{\,n}\right) = \int_{\mathbb{R}^n} \varphi^n\left(\vec{x}_1^{\,n} + \sqrt{2tg^n\left(\vec{x}_1^{\,n}\right)}z\right) e^{\sqrt{\frac{2t}{g^n(\vec{x}_1^{\,n})}}\left\langle \vec{B}_1^n(\vec{x}_1^{\,n}), z\right\rangle} \nu(dz). \qquad (3.12)$$

The integrand in (3.12) is a composition of mappings with the continuous bounded Fréchet derivative. Thus, it has a continuous bounded Fréchet derivative. The Fréchet derivative of the integrand is uniformly bounded (the estimate is obtained from the chain rule formula), and $(\mathbb{R}^n, \nu)$ is locally compact, countable at infinity, linear normed space with the non-negative Radon measure. Therefore we can apply Theorem 115 from Ref. 60 on the Fréchet differentiation under the Lebesgue integral. Repeating this reasoning for every $k \in \mathbb{N}$, we conclude that as the integrand has, everywhere in $\mathbb{R}^n$, continuous Fréchet derivatives of kth order, then the function Φ has, everywhere in $\mathbb{R}^n$, continuous bounded Fréchet's derivatives of kth order. So the functions $\mathbb{R}^n \to \mathbb{R}$ in the right-hand side of (3.11) all have continuous bounded Fréchet's derivatives of kth order.

Therefore, according to point (2) of Proposition 2.1, the function $x \mapsto (S_t\varphi)(x)$ also has, for every $k \in \mathbb{N}$, Fréchet derivatives of order k, continuous and bounded everywhere on H. Therefore $S_t\varphi \in D$.

(3)(i) Now suppose $\varphi \in X$, which means that $\varphi \in C_b(H, \mathbb{R})$ and there exists a sequence $(\varphi_j) \subset D$ such that $\varphi_j(x) \to \varphi(x)$ uniformly with respect to $x \in H$. Suppose also that $g \in X$, so $g \in C_b(H, \mathbb{R})$ and there exists a sequence $(g_j) \subset D$ such that $g_j \to g$ uniformly. It follows from $g(x) \geq g_0 \equiv \text{const} > 0$ for all $x \in H$ that there exists a number $j_0 \in \mathbb{N}$ such that for all $j > j_0$ and for all $x \in H$ the inequality $g_j(x) \geq \frac{g_0}{2}$ holds. Therefore, we will not restrict the generality when assuming that for the sequence (g_j) the inequality $g_j(x) \geq \frac{g_0}{2}$ already holds for all $j \in \mathbb{N}$ and for all $x \in H$.

Also suppose $C \in X$, so $C \in C_b(H, \mathbb{R})$ and there exists a sequence $(C_j) \subset D$ such that $C_j \to C$ uniformly. Finally, suppose $B \in X_H$, so $B \in C_b(H, H)$ and there exists a sequence $(B_j) \subset D_H$ such that $B_j \to B$ uniformly. Let $t > 0$ be fixed as before.

Let us denote the operator S_t constructed with functions g_j, B_j, C_j by $(S_j)_t$. According to (just proven above) item (2) of the theorem, $(S_j)_t \varphi_j \in D$ for all $j \in \mathbb{N}$. Next, in (ii) and (iii) we will prove that $((S_j)_t \varphi_j)(x) \to (S_t \varphi)(x)$ uniformly with respect to $x \in H$; by the definition of the space X this will mean that $S_t \varphi \in X$.

(ii) First of all let us prove that for every fixed $y \in H$ the sequence of functions $x \mapsto \varphi_j(x + \sqrt{2tg_j(x)}y)$ converges to the function $x \mapsto \varphi(x + \sqrt{2tg(x)}y)$ uniformly with respect to $x \in H$. Indeed, suppose $\varepsilon > 0$ is given. Let us find $j^* \in \mathbb{N}$ such that for all $j > j^*$ the following estimate holds

$$\sup_{x \in H} \left| \varphi_j \left(x + \sqrt{2tg_j(x)}y \right) - \varphi(x + \sqrt{2tg(x)}y) \right| \leq \varepsilon. \tag{3.13}$$

Notice that $\varphi, \varphi_j, g, g_j$ are elements of X by the hypothesis made above, and all the functions in X are uniformly continuous, according to Proposition 3.2. According to Proposition 3.3, the family of functions $\{\varphi_j : j \in \mathbb{N}\}$ is equicontinuous. Thus, there exists $\delta > 0$ such that, for all $j \in \mathbb{N}$, if $\|x_1 - x_2\| < \delta$, then

$$|\varphi_j(x_1) - \varphi_j(x_2)| < \frac{\varepsilon}{2}. \tag{3.14}$$

Function $[0, +\infty) \ni a \mapsto \sqrt{2ta} \in \mathbb{R}$ is uniformly continuous, so the uniform (with respect to $x \in H$) convergence $g_j(x) \to g(x)$ implies the uniform (with respect to $x \in H$) convergence $[H \ni x \mapsto \sqrt{2tg_j(x)} \in \mathbb{R}] \to [H \ni x \mapsto \sqrt{2tg(x)} \in \mathbb{R}]$. This shows that for every fixed $y \in H$ there exists $j_1 \in \mathbb{N}$ such that for all $j > j_1$ and all $x \in H$ we have

$$\left\| \left(x + \sqrt{2tg_j(x)}y \right) - \left(x + \sqrt{2tg(x)}y \right) \right\| < \delta. \tag{3.15}$$

Besides, because $\varphi_j(z) \to \varphi(z)$ uniformly with respect to $z \in H$, there exists a number j_2 such that for all $j > j_2$ and all $z \in H$ we have

$$|\varphi_j(z) - \varphi(z)| < \frac{\varepsilon}{2}. \tag{3.16}$$

For each fixed $y \in H$, for all $j \in \mathbb{N}$ and for all $x \in H$, we have

$$\left| \varphi_j \left(x + \sqrt{2tg_j(x)}y \right) - \varphi(x + \sqrt{2tg(x)}y) \right| \leq \left| \varphi_j \left(x + \sqrt{2tg_j(x)}y \right) \right.$$
$$\left. - \varphi_j \left(x + \sqrt{2tg(x)}y \right) \right| + \left| \varphi_j \left(x + \sqrt{2tg(x)}y \right) - \varphi(x + \sqrt{2tg(x)}y) \right|.$$

Now, we define $j^* = \max(j_1, j_2)$. For all $j > j^*$ the first summand is less than $\frac{\varepsilon}{2}$ due to (3.14) and (3.15); the easiest way to see that is to set $x_1 = x + \sqrt{2tg_j(x)}y$ and $x_2 = x + \sqrt{2tg(x)}y$. As for the second summand, it is less than $\frac{\varepsilon}{2}$ on account of (3.16); the easiest way to see that is to set $z = x + \sqrt{2tg(x)}y$.

So, for each fixed $y \in H$ we have found a number $j^* \in \mathbb{N}$ such that for all $j > j^*$ and for all $x \in H$ the following inequality holds:

$$\left| \varphi_j \left(x + \sqrt{2tg_j(x)}y \right) - \varphi(x + \sqrt{2tg(x)}y) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By taking $\sup_{x \in H}$ we obtain the needed estimate (3.13), as the right-hand side of the inequality above does not depend on x.

(iii) A reasoning similar to (ii) shows that for fixed y we have $e^{\left\langle \sqrt{\frac{2t}{g_j(x)}}B_j(x),y\right\rangle} \to$ $e^{\left\langle \sqrt{\frac{2t}{g(x)}}B(x),y\right\rangle}$ uniformly with respect to $x \in H$; let us omit the detailed proof. Uniting this with the results of (ii) and keeping in mind that for fixed y all the sequences of functions are bounded collectively, we obtain that for each fixed y we have

$$\varphi_j\left(x + \sqrt{2tg_j(x)}y\right)e^{\left\langle \sqrt{\frac{2t}{g_j(x)}}B_j(x),y\right\rangle} \to \varphi\left(x + \sqrt{2tg(x)}y\right)e^{\left\langle \sqrt{\frac{2t}{g(x)}}B(x),y\right\rangle}$$

(3.17)

uniformly with respect to $x \in H$.

(iv) For fixed y, functions in (3.17) are bounded, therefore the sequence of functions $Y_j : H \to \mathbb{R}$

$$Y_j = \left[y \mapsto \sup_{x \in H}\left|\varphi_j\left(x + \sqrt{2tg_j(x)}y\right)e^{\left\langle \sqrt{\frac{2t}{g_j(x)}}B_j(x),y\right\rangle}\right.\right.$$
$$\left.\left. - \varphi\left(x + \sqrt{2tg(x)}y\right)e^{\left\langle \sqrt{\frac{2t}{g(x)}}B(x),y\right\rangle}\right|\right]$$

is well defined. It follows from (iii), that $Y_j(y)$ converges to zero pointwise, in other words for each y. Functions Y_j are non-negative and bounded collectively by an integrable (due to Lemma 2.4) function $y \mapsto \alpha e^{\beta\|y\|} + \gamma$ with appropriate constants α, β and γ. The Lebesgue dominated convergence theorem gives us that $\int_H Y_j(y)\mu_A(dy) \to 0$. As the number sequence $\int_H Y_j(y)\mu_A(dy)$ converges, it is bounded. For brevity, let us denote

$$\Psi(x,y) = \varphi\left(x + \sqrt{2tg(x)}y\right)e^{\left\langle \sqrt{\frac{2t}{g(x)}}B(x),y\right\rangle},$$

$$\Psi_j(x,y) = \varphi_j\left(x + \sqrt{2tg_j(x)}y\right)e^{\left\langle \sqrt{\frac{2t}{g_j(x)}}B_j(x),y\right\rangle},$$

$$E(x) = \exp\left(tC(x) - t\frac{\langle AB(x), B(x)\rangle}{g(x)}\right),$$

$$E_j(x) = \exp\left(tC_j(x) - t\frac{\langle AB_j(x), B_j(x)\rangle}{g_j(x)}\right).$$

A reasoning similar to (ii) shows that $E_j(x) \to E(x)$ uniformly with respect to $x \in H$. We obtain the estimate

$$\|(S_j)_t\varphi_j - S_t\varphi\| = \sup_{x \in H}\left|E_j(x)\int_H \Psi_j(x,y)\mu_A(dy) - E(x)\int_H \Psi(x,y)\mu_A(dy)\right|$$

$$\leq \sup_{x \in H}\left|E_j(x)\int_H \Psi_j(x,y)\mu_A(dy) - E_j(x)\int_H \Psi(x,y)\mu_A(dy)\right|$$

$$+ \sup_{x \in H}\left|E_j(x)\int_H \Psi(x,y)\mu_A(dy) - E(x)\int_H \Psi(x,y)\mu_A(dy)\right|$$

$$\leq \sup_{x\in H} |E_j(x)| \int_H Y_j(y)\mu_A(dy) + \sup_{x\in H} |E_j(x) - E(x)|$$

$$\cdot \int_H \Psi(x,y)\mu_A(dy).$$

Finally let us note that $\sup_{x\in H} |E_j(x) - E(x)| \to 0$ and $\int_H Y_j(y)\mu_A(dy) \to 0$, and the multipliers of these terms in the above estimate are bounded, which implies $\|(S_j)_t\varphi_j - S_t\varphi\| \to 0$.

(4) Suppose $\varphi \in D$, $t > 0$ and $x \in H$ are fixed. Let us consider the integral

$$\int_H \varphi(x+y)e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy).$$

We will work with the approximation of the function φ by its Taylor polynomial (2.10) of the second order with the center at the point x. Before we start, it is important to note that the remainder term $R(x,y)$ will not be small, as the vector y ranges over the whole space H, and vector x is fixed. However as φ is three times Fréchet differentiable on H, for each $x \in H$ and each $y \in H$ there exists a real number $R(x,y)$ such that

$$\int_H \varphi(x+y)e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy)$$

$$= \int_H \left\{ \varphi(x) + \langle\varphi'(x),y\rangle + \frac{1}{2!}\langle\varphi''(x)y,y\rangle + R(x,y) \right\} e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy),$$

and, according to (2.11), the estimate

$$|R(x,y)| \leq \frac{\|y\|^3}{(3)!} \sup_{z\in[x,x-y]} \|\varphi^{(3)}(z)\| \leq \frac{1}{3!}\|\varphi'''\|\|y\|^3 \tag{3.18}$$

holds, where we define $\|\varphi'''\| = \sup_{z\in H} \|\varphi^{(3)}(z)\|$. Also, let us denote $\|g\| = \sup_{z\in H} |g(z)|$.

The sum can be integrated termwise as every term can be bounded by a polynomial of $\|y\|$, multiplied by an exponent of $\|y\|$ and such functions are integrable due to Lemma 2.4. Let us calculate integrals in the sum and for fixed φ evaluate the decay rate as $t \to 0$ uniformly with respect to $x \in H$.

$$\int_H \varphi(x)e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy) = \varphi(x)\int_H e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy)$$

(set $z = \frac{B(x)}{g(x)}$ and $\widetilde{A} = 2tg(x)A$ in (2.4))

$$\varphi(x)\exp\left(\frac{1}{2}\left\langle 2tg(x)A\frac{B(x)}{g(x)}, \frac{B(x)}{g(x)}\right\rangle\right) = \varphi(x)\exp\left(\frac{\langle AB(x),B(x)\rangle}{g(x)}t\right)$$

$$= \varphi(x)\left(1 + \frac{\langle AB(x),B(x)\rangle}{g(x)}t + o(t)\right)$$

$$= \varphi(x) + \varphi(x)\frac{\langle AB(x),B(x)\rangle}{g(x)}t + o(t).$$

Next, let z and $\widetilde{A}$ be as before, and set $w = \varphi'(x)$ in (2.5).

$$\int_H \langle \varphi'(x), y \rangle \ e^{\left\langle \frac{1}{g(x)} B(x), y \right\rangle} \mu_{2tg(x)A}(dy)$$

$$\stackrel{(2.5)}{=} \left\langle 2tg(x)A\varphi'(x), \frac{1}{g(x)}B(x) \right\rangle$$

$$\cdot \exp\left(\frac{1}{2} \left\langle 2tg(x)A\frac{1}{g(x)}B(x), \frac{1}{g(x)}B(x) \right\rangle \right)$$

$$= 2t\langle A\varphi'(x), B(x) \rangle \exp\left(\frac{t}{g(x)} \|\sqrt{A}B(x)\|^2 \right)$$

$$= 2t\langle A\varphi'(x), B(x) \rangle + o(t).$$

For the term with φ'' we have

$$\int_H \langle \varphi''(x)y, y \rangle e^{\left\langle \frac{1}{g(x)} B(x), y \right\rangle} \mu_{2tg(x)A}(dy)$$

$$\stackrel{(2.6)}{=} (2tg(x)\text{tr}A\varphi''(x) + 4t^2\langle \varphi''(x)AB(x), AB(x) \rangle) \exp\left(\frac{\langle AB(x), B(x) \rangle}{g(x)}t \right)$$

$$= 2tg(x)\text{tr}A\varphi''(x) + o(t).$$

Finally,

$$\left| \int_H R(x, y)e^{\left\langle \frac{1}{g(x)} B(x), y \right\rangle} \mu_{2tg(x)A}(dy) \right|$$

$$\stackrel{(2.7)}{\leq} \int_H |R(x, \sqrt{2tg(x)}y)| e^{\left\langle \frac{1}{g(x)} B(x), \sqrt{2tg(x)}y \right\rangle} \mu_A(dy)$$

$$\stackrel{(3.18)}{\leq} \int_H \frac{1}{3!} \|\varphi'''\| \left(\sqrt{2tg(x)}\right)^3 \|y\|^3 e^{\left\langle \sqrt{\frac{2t}{g(x)}} B(x), y \right\rangle} \mu_A(dy)$$

$$\leq t^{\frac{3}{2}} \left(\frac{\sqrt{2}}{3} \|\varphi'''\| \|g\|^{\frac{3}{2}} \int_H \|y\|^3 e^{\left\langle \sqrt{\frac{2t}{g(x)}} B(x), y \right\rangle} \mu_A(dy) \right)$$

$$\leq \text{(due to the inequality } \langle z, y \rangle \leq \|z\|\|y\| \text{ and monotonicity}$$

$$\text{of the exponent function)}$$

$$t^{\frac{3}{2}} \left(\frac{\sqrt{2}}{3} \|\varphi'''\| \|g\|^{\frac{3}{2}} \int_H \|y\|^3 e^{\left\| \sqrt{\frac{2t}{g(x)}} B(x) \right\| \|y\|} \mu_A(dy) \right)$$

$$\leq \text{(due to } \|B(x)\| \leq B_0, \ g_0 \leq g(x) \text{ and monotonicity of the}$$

$$\text{exponent function)}$$

$$t^{\frac{3}{2}} \left(\frac{\sqrt{2}}{3} \|\varphi'''\| \|g\|^{\frac{3}{2}} \int_H \|y\|^3 e^{\sqrt{\frac{2t}{g_0}} B_0 \|y\|} \mu_A(dy) \right)$$

$$= o(t) \text{ uniformly with respect to } x \in H.$$

Summing everything up, one can see that

$$\int_H \varphi(x+y)e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy)$$

$$= \varphi(x) + t\langle\varphi'(x), AB(x)\rangle + tg(x)\mathrm{tr}A\varphi''(x)$$

$$- t\varphi(x)\frac{\langle AB(x), B(x)\rangle}{g(x)} + o(t),$$

uniformly with respect to $x \in H$.

Consider the term $\exp(tC(x) - \frac{\langle AB(x),B(x)\rangle}{g(x)}t)$. As $\|C\|, \|A\|, \|B\|$ are bounded from infinity and g is bounded from zero, we have $e^{tC(x) - \frac{\langle AB(x),B(x)\rangle}{g(x)}t} = 1 + tC(x) - \frac{\langle AB(x),B(x)\rangle}{g(x)}t + o(t)$ uniformly with respect to $x \in H$. Therefore, uniformly with respect to $x \in H$ for $t \to 0$ one obtains

$$(S_t\varphi)(x) = e^{tC(x) - \frac{\langle AB(x),B(x)\rangle}{g(x)}t}\int_H \varphi(x+y)e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy)$$

$$= \varphi(x) + t\langle\varphi'(x), AB(x)\rangle + tg(x)\mathrm{tr}A\varphi''(x) + tC(x)\varphi(x) + o(t).$$

This implies that uniformly with respect to $x \in H$

$$\lim_{t\to 0}\frac{(S_t\varphi)(x) - \varphi(x)}{t} = g(x)\mathrm{tr}A\varphi''(x) + \langle\varphi'(x), AB(x)\rangle + C(x)\varphi(x) = (L\varphi)(x).$$

(5)(i) First let us consider the case $t_0 \neq 0$. If $t_n \to t_0$, then $2t_ng(x) \to 2t_0g(x)$ uniformly with respect to $x \in H$. Because the function $z \mapsto \sqrt{z}$ is uniformly continuous, it follows that $\sqrt{2t_ng(x)} \to \sqrt{2t_0g(x)}$ uniformly with respect to $x \in H$.

According to Proposition 3.2, the function φ is uniformly continuous. Therefore, for every fixed $y \in H$ $\varphi(x+\sqrt{2t_ng(x)}y) \to \varphi(x+\sqrt{2t_0g(x)}y)$ uniformly with respect to $x \in H$.

Next, for every $y \in H$ the sequence $\langle\frac{1}{g(x)}B(x), \sqrt{2t_ng(x)}y\rangle$ converges to $\langle\frac{1}{g(x)}B(x), \sqrt{2t_0g(x)}y\rangle$ uniformly with respect to $x \in H$ because of the linearity of the scalar product. Since the number sequence t_n converges, it is bounded; besides, $g(x) \geq g_0 \equiv \mathrm{const} > 0$ and functions $x \mapsto g(x)$ and $x \mapsto \|B(x)\|$ are bounded. Therefore, there exists a constant $K > 0$ such that for fixed $y \in H$ for all $k = 0, 1, 2, 3, \dots$ and all $x \in H$ the inequality $|\langle\frac{1}{g(x)}B(x), \sqrt{2t_kg(x)}y\rangle| \leq K$ holds. Function $z \mapsto e^z$ is uniformly continuous with respect to $z \in [-K, K]$, therefore for every $y \in H$ we have the convergence $e^{\langle\frac{1}{g(x)}B(x),\sqrt{2t_ng(x)}y\rangle} \to e^{\langle\frac{1}{g(x)}B(x),\sqrt{2t_0g(x)}y\rangle}$, uniformly with respect to $x \in H$.

The product of two collectively bounded uniformly converging sequences is a sequence, uniformly converging to the product of the limits of these sequences. Therefore, from the two last paragraphs, it follows that for every $y \in H$ we have

$$\varphi(x + \sqrt{2t_ng(x)}y)e^{\langle\frac{1}{g(x)}B(x),\sqrt{2t_ng(x)}y\rangle} \to \varphi(x + \sqrt{2t_0g(x)}y)e^{\langle\frac{1}{g(x)}B(x),\sqrt{2t_0g(x)}y\rangle}$$

(3.19)

uniformly with respect to $x \in H$.

Since for fixed y the functions from (3.19) are bounded, the sequence of the number functions is well defined

$$Y_n = \left[y \mapsto \sup_{x \in H} \left| \varphi \left(x + \sqrt{2 t_n g(x)} y \right) e^{\left\langle \sqrt{\frac{2 t_n}{g(x)}} B(x), y \right\rangle} \right. \right.$$

$$\left. \left. - \varphi \left(x + \sqrt{2 t_0 g(x)} y \right) e^{\left\langle \sqrt{\frac{2 t_0}{g(x)}} B(x), y \right\rangle} \right| \right].$$

According to above results, $Y_n(y)$ converges to zero pointwise, i.e. for every y. Functions Y_n are bounded by an integrable function (see Lemma 2.4), therefore, by the Lebesgue dominated convergence theorem, we have $\int_H Y_n(y) \mu_A(dy) \to 0$.

Finally, $C \in X$, $B \in X_H$, and therefore $\sup_{x \in H} |C(x)| < \infty$, $\sup_{x \in H} \|B(x)\| < \infty$. United with $g(x) \geq g_0$ this implies $e^{t_n C(x) - \frac{\langle AB(x), B(x) \rangle}{g(x)} t_n} \to e^{t_0 C(x) - \frac{\langle AB(x), B(x) \rangle}{g(x)} t_0}$ uniformly with respect to $x \in H$.

So, the sequence $S_{t_n} \varphi(x)$ is a product of two collectively bounded uniformly converging sequences; thus it converges to the product of the limits of these sequences, i.e. to $S_{t_0} \varphi(x)$.

(ii) Now let us consider the case $t_0 = 0$. Recall that $S_0 = I$, so we need for each fixed φ prove that if $t_n \to 0$ then $S_{t_n} \varphi \to \varphi$, i.e. $\|S_{t_n} \varphi - \varphi\| \to 0$. Without loss of generality we can assume that $0 < t_n \leq 1$.

(iia) Let us first consider the case $\varphi \in D$. Then by item (4) of the present theorem $S_t \varphi = \varphi + t L \varphi + o(t)$, so $t_n \to 0$ implies $S_{t_n} \varphi \to \varphi$.

(iib) Now suppose that $\varphi \in X$, so there exists a sequence $(\varphi_k) \subset D$ such that $\|\varphi_k - \varphi\| \to 0$. Item (1) of the present theorem implies that there exists such a constant $\omega \geq 1$ such that for all $t \in [0, 1]$ we have $\|S_t\| \leq \omega$. Now for arbitrary $\varepsilon > 0$ we apply "$\varepsilon/3$-method" based on the estimate

$$\|S_{t_n} \varphi - \varphi\| \leq \|S_{t_n} \varphi - S_{t_n} \varphi_k\| + \|S_{t_n} \varphi_k - \varphi_k\| + \|\varphi_k - \varphi\|.$$

We can select such k that $\|\varphi_k - \varphi\| < \varepsilon/(3\omega)$. For fixed k we can thanks to (iia) find such a number n_0 that for all $n > n_0$ one has $\|S_{t_n} \varphi_k - \varphi_k\| < \varepsilon/3$. So for all $n > n_0$ we see that

$$\|S_{t_n} \varphi - \varphi\| \leq \|S_{t_n}\| \cdot \|\varphi - \varphi_k\| + \|S_{t_n} \varphi_k - \varphi_k\| + \|\varphi_k - \varphi\|$$

$$< \omega \frac{\varepsilon}{3\omega} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\omega} \leq \varepsilon. \qquad \square$$

Theorem 3.2 (On the properties of the operator L). *Suppose for each $x \in H$ the inequalities $g(x) \geq g_o \equiv \text{const} > 0$ and $C(x) \leq 0$ hold. As $C \in X$, there exists a sequence $(C_j) \subset D$ converging to C uniformly; let us additionally require that this sequence can be selected in such a way that $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$. The operator $L : D \to X$ is defined by the equation*

$$(L\varphi)(x) = g(x) \text{tr} A \varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x) \varphi(x).$$

Symbol I stands for the identity operator.

Then:

(1) *If $g \in D$, $C \in D$, $B \in D_H$ and $\varphi \in D$, then $L\varphi \in D$. If $g \in X$, $C \in X$, $B \in X_H$ and $\varphi \in D$, then $L\varphi \in X$.*
(2) *If $g \in D$, $C \in D$, $B \in D_H$, then for each $\lambda > 0$ the operator $\lambda I - L$ is surjective on D, therefore $(\lambda I - L)(D) = D$ is a dense subspace in X.*
(3) *If $g \in D$, $C \in D$, $B \in D_H$, then the operator (L, D) is dissipative and closable.*
(4) *If $g \in X$, $C \in X$, $B = 0$, then for each $\lambda > 0$ the space $(\lambda I - L)(D)$ is dense in X.*
(5) *If $g \in X$, $C \in X$, $B \in X_H$, then the operator (L, D) is dissipative and has the closure $(\overline{L}, D_1)$. This operator is also dissipative.*

Proof. (1) First part of the statement is obvious: the result of the applying a differential operator with smooth cylindrical coefficients to a smooth cylindrical function is a smooth cylindrical function. As it follows from the chain rule, all the derivatives are bounded. Let the coefficients of the operator L be uniform limits g, B, C of cylindrical functions g_j, B_j, C_j. We denote the operator L that corresponds to the functions g_j, B_j, C_j as L_j. Then, as $j \to \infty$ the sequence $L_j\varphi$ converges uniformly to $L\varphi$, therefore $L\varphi \in X$, because $L_j\varphi \in D$ as we proved above.

(2) Suppose $g \in D$, $B \in D_H$, $C \in D$. Recall that all these functions are cylindrical and thus they are closely related to functions on $\mathbb{R}^n$; this will be the core idea of the proof that follows.

Let us fix $\lambda > 0$, choose arbitrary function $\varphi \in D$ and then show, that there exists a function $f \in D$ satisfying the equation

$$\lambda f(x) - g(x)\mathrm{tr}Af''(x) - \langle f'(x), AB(x)\rangle - C(x)f(x) = \varphi(x). \tag{3.20}$$

Let the vectors $e_1, \ldots, e_n$ be such that for every $x \in H$ we have

$$C(x) = C^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle), \quad B(x) = \sum_{k=1}^{n} B_k^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)e_k, \tag{3.21}$$

$$\varphi(x) = \varphi^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle) \quad \text{and} \quad g(x) = g^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle), \tag{3.22}$$

where $\varphi^n : \mathbb{R}^n \to \mathbb{R}$, $g^n : \mathbb{R}^n \to \mathbb{R}$, $C^n : \mathbb{R}^n \to \mathbb{R}$ and $B_k^n : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions, bounded along with all the derivatives. Let us find a solution of Eq. (3.20) in the form

$$f(x) = f^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle), \tag{3.23}$$

where $f^n : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, bounded along with all its derivatives.

According to item (1) of Proposition 2.1

$$f'(x) = \sum_{s=1}^{n} \partial_s f^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)e_s.$$

Using (3.21) we obtain that

$$\langle f'(x), AB(x)\rangle$$

$$= \left\langle \sum_{s=1}^{n} \partial_s f^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)e_s, A\sum_{k=1}^{n} B_k^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)e_k \right\rangle$$

$$= \sum_{s=1}^{n} \left(\sum_{k=1}^{n} B_k^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)\langle Ae_s, e_k\rangle \right) \partial_s f^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle).$$
(3.24)

One can see, by taking into account (2.9), (3.22), (3.21), (3.23) and (3.24), that Eq. (3.20) for the unknown function f of form (3.23) is equivalent to the equation for an unknown function f^n:

$$g^n(x_1, \ldots, x_n) \sum_{s=1}^{n}\sum_{k=1}^{n}\langle Ae_s, e_k\rangle\partial_k\partial_s f^n(x_1, \ldots, x_n)$$

$$+ \left(\sum_{s=1}^{n} \left(\sum_{k=1}^{n} B_k^n(x_1, \ldots, x_n)\langle Ae_s, e_k\rangle \right) \partial_s f^n(x_1, \ldots, x_n) \right)$$

$$+ C^n(x_1, \ldots, x_n)f^n(x_1, \ldots, x_n) - \lambda f^n(x_1, \ldots, x_n)$$

$$= -\varphi^n(x_1, \ldots, x_n),$$
(3.25)

where we denote $x_j = \langle x, e_j\rangle$.

Equation (3.25) is a finite-dimensional PDE. Note that $g_n(x_1, \ldots, x_n) \geq g_0 \equiv$ const > 0. The quadratic form given by the $n \times n$-matrix $(\langle Ae_s, e_k\rangle)$ is positively defined because the operator A is positive and non-degenerate. So the differential operator in Eq. (3.25) is elliptic, and (3.25) has form (2.12), i.e.

$$L^n f^n(x_1, \ldots, x_n) - \lambda f^n(x_1, \ldots, x_n) = -\varphi(x_1, \ldots, x_n).$$

Functions g^n, C^n and B_k^n used to construct the coefficients of the operator L^n, and the function φ^n are bounded along with all the derivatives. (Note that L^n is not the nth power of the operator L.) Therefore, we can apply item (1) of Lemma 2.5 to Eq. (3.25), so (3.25) has the solution f^n, continuous and bounded along with all the derivatives. Function f defined by Eq. (3.23) belongs to D according to item (2) of Proposition 2.1.

Thus, for every $\lambda > 0$ the operator $\lambda I - L$ is surjective in D, because in the preimage of the function $\varphi \in D$ there is at least the function $f(x) = f^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)$, where $f^n : \mathbb{R}^n \to \mathbb{R}$ is the solution of Eq. (3.25).

(3) Suppose $g \in D, C \in D, B \in D_H$. Let us prove that the operator L is dissipative. Let $f \in D$ and $\lambda > 0$ be fixed.

As in the proof of item (2) of this theorem, the value of the function Lf at the point $x \in H$ is equal to the value of the function $L^n f^n$ at the point

$(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \in \mathbb{R}^n$. Again, the operator A is positive, and the function g satisfies the inequality $g(x) \geq g_0 \equiv \text{const} > 0$, so we can apply item (2) of Lemma 2.5 to the finite-dimensional operator. This gives us that the finite-dimensional operator is dissipative, which implies that the operator L is dissipative too. This idea can be expressed in a more formal way:

$$
\begin{aligned}
\|Lf - \lambda f\| &= \sup_{x \in H} |g(x)\text{tr}(Af''(x)) + \langle f'(x), AB(x) \rangle + C(x)f(x) - \lambda f(x)| \\
&= \sup_{(x_1, \ldots, x_n) \in \mathbb{R}^n} \bigg| g^n(x_1, \ldots, x_n) \sum_{s=1}^{n} \sum_{k=1}^{n} \langle Ae_s, e_k \rangle \partial_k \partial_s f^n(x_1, \ldots, x_n) \\
&\quad + \sum_{s=1}^{n} \left(\sum_{k=1}^{n} B_k^n(x_1, \ldots, x_n) \langle Ae_s, e_k \rangle \right) \partial_s f^n(x_1, \ldots, x_n) \\
&\quad + C^n(x_1, \ldots, x_n)f^n(x_1, \ldots, x_n) - \lambda f_n(x_1, \ldots, x_n) \bigg| \\
&\overset{(2.13)}{\geq} \lambda \sup_{(x_1, \ldots, x_n) \in \mathbb{R}^n} |f^n(x_1, \ldots, x_n)| = \lambda \sup_{x \in H} |f(x)| = \lambda \|f\|,
\end{aligned}
$$

where $x_j = \langle x, e_j \rangle$. The inequality

$$
\|Lf - \lambda f\| \geq \lambda \|f\| \tag{3.26}
$$

means that the operator L is dissipative.

Finally, according to Proposition 2.6, the closability of L follows from the fact L is dissipative and densely defined.

(4) (i) Suppose $g \in X$ and $C \in X$, i.e. there exists sequences $(g_j) \subset D$ and $(C_j) \subset D$ such that $\|g - g_j\| \to 0$ and $\|C - C_j\| \to 0$. Suppose that $B(x) = 0$ for each $x \in H$. The images of the operators $(\lambda I - L)$ and $(L - \lambda I)$ are equal. As D is dense in X, it is enough to show that the image of the operator $(L - \lambda I)$ is dense in D, then it will be dense in X. Let the number $\lambda > 0$ and function $\psi \in D$ be fixed. We will approximate f by the values of the operator $(L - \lambda I)$.

Since $g_j \to g$ and, for every $x \in H$, the estimate $g(x) \geq g_0$ holds, it follows that there is a number j_0' such that for every $x \in H$ and all $j > j_0'$ we have

$$
g_j(x) \geq \frac{g_0}{2}. \tag{3.27}
$$

We denote the operator L corresponding to the functions g_j and C_j by the symbol L_j. Note that $C_j(x) \leq 0$ and $g_j(x) \geq \frac{g_0}{2}$, so we can apply item (2) of this theorem to the operator L_j. Item (2) says that image of the operator $(L_j - \lambda I)$ is equal to D, so for every $j > j_0'$ there exists a function $f_j \in D$ such that

$$
L_j f_j - \lambda f_j = \psi. \tag{3.28}
$$

The goal is to prove that $Lf_j - \lambda f_j \to \psi$ as $j \to \infty$. This will imply that the image of the operator $(\lambda I - L)$ is dense in D.

(ii) Let us prepare several estimates. First, since $C_j \to C$, there is a number j_0 such that for every $j > j_0$

$$\|C_j\| \leq 2\|C\|. \qquad (3.29)$$

Second, it follows from (3.28) and (3.26) that for every $j > j_0'$ we have

$$\lambda\|f_j\| \overset{(3.26)}{\leq} \|L_j f_j - \lambda f_j\| \overset{(3.28)}{=} \|\psi\|,$$

i.e. for every $j > j_0'$

$$\|f_j\| \leq \frac{\|\psi\|}{\lambda}. \qquad (3.30)$$

Finally, expressing the term $\mathrm{tr}(Af_j'')$ by the use of the equation

$$\psi = L_j f_j - \lambda f_j = g_j \mathrm{tr}(Af_j'') + (C_j - \lambda)f_j,$$

we find that for every $j > \max(j_0, j_0')$

$$\|\mathrm{tr}(Af_j'')\| = \left\| \frac{\psi + (\lambda - C_j)f_j}{g_j} \right\| \overset{(3.30),\,(3.29),\,(3.27)}{\leq} \frac{\|\psi\| + (\lambda + 2\|C\|)\|\psi\|/\lambda}{g_0/2}. \qquad (3.31)$$

(iii) Now let us prove that if $j \to \infty$, then $Lf_j - \lambda f_j \to \psi$. Indeed, for every $j > \max(j_0, j_0')$ one obtains

$$\|Lf_j - \lambda f_j - \psi\| \overset{(3.28)}{=} \|Lf_j - \lambda f_j - (L_j f_j - \lambda f_j)\|$$

$$= \|(g - g_j)\mathrm{tr}(Af_j'') + (C - C_j)f_j\|$$

$$\overset{(3.31),\,(3.30)}{\leq} \|(g - g_j)\| \frac{\|\psi\| + (\lambda + 2\|C\|)\|\psi\|/\lambda}{g_0/2}$$

$$+ \|(C - C_j)\| \frac{\|\psi\|}{\lambda} \to 0,$$

because $\|(g - g_j)\| \to 0$ and $\|(C - C_j)\| \to 0$. Item (4) is proven.

(5) Let the coefficients of the operator L be uniform limits g, B, C of the continuously differentiable cylindrical functions g_j, B_j, C_j. As $g_j \to g$ and, for all $x \in H$, we have $g(x) \geq g_0$, it follows that there exists a number j_0 such that for all $x \in H$ and all $j > j_0$ we have $g_j(x) \geq \frac{g_0}{2}$. Also recall that $C_j(x) \leq 0$. This all allows us to use item (3) of this theorem.

According to (3.26), for every function $\varphi \in D$ and every $\lambda > 0$ we have

$$\|g_j \mathrm{tr}(A\varphi'') + \langle \varphi', AB_j \rangle + C_j \varphi - \lambda\varphi\| \geq \lambda\|\varphi\|.$$

Taking the limit as $j \to \infty$, we obtain the estimate $\|L\varphi - \lambda\varphi\| \geq \lambda\|\varphi\|$, which means that L is dissipative. According to Proposition 2.6, the dissipative operator (L, D) with the domain D dense in X is closable. Let us denote its closure with $(\overline{L}, D_1)$. Note that by Proposition 2.6 the closure also is a dissipative operator.

$\square$

The constructions above were built to prove the following result:

Theorem 3.3 (On the connection between the family $(S_t)_{t \geq 0}$ and the semigroup with the generator $\overline{L}$). *Suppose that $g \in X$, $B \in X_H$, $C \in X$, and for every $x \in H$ we have $g(x) \geq g_0 \equiv$ const > 0 and $C(x) \leq 0$. As $C \in X$, there exists a sequence $(C_j) \subset D$, converging to C uniformly; let us additionally claim that this sequence can be selected in such a way that $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$. Then the following holds:*

(1) *If the closure $(\overline{L}, D_1)$ of the operator (L, D) is a generator of a strongly continuous semigroup $(e^{t\overline{L}})_{t \geq 0}$ of linear continuous operators on the space X, then*

$$e^{t\overline{L}}\varphi = \lim_{n \to \infty} (S_{\frac{t}{n}})^n \varphi, \tag{3.32}$$

where limit exists for every $\varphi \in X$ and is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$.

(2) *If $B = 0$, then the operator $(\overline{L}, D_1)$ is a generator of a strongly continuous semigroup $(e^{\overline{L}t})_{t \geq 0}$ of linear continuous operators on the space X. Moreover for every $t \geq 0$ we have $\|e^{\overline{L}t}\| \leq 1$, i.e. the semigroup $(e^{\overline{L}t})_{t \geq 0}$ is contractive.*

(3) *Suppose $B = 0$, and for all $j \in \mathbb{N}$ the functions $g_j \in X$, $B_j \in X_H$ and $C_j \in X$ are given. Suppose $B_j = 0$ for all $j \in \mathbb{N}$. Suppose there exists a number $\varepsilon_0 > 0$ such that for all $j \in \mathbb{N}$ and all $x \in H$ we have $g_j(x) \geq \varepsilon_0$ and $C_j(x) \leq 0$. Let us denote by the symbol L_j the operator L, which corresponds to the functions g_j, B_j and C_j, and the operator L corresponding to the functions g, B and C will be denoted by L_0. Suppose also that $g_j(x) \to g(x)$ and $C_j(x) \to C(x)$, uniformly with respect to $x \in H$.*

   Then the (existing by item (2)) strongly continuous semigroups $(e^{\overline{L_j}t})_{t \geq 0}$ converge strongly (and uniformly with respect to $t \in [0, t_0]$ for every fixed $t_0 > 0$) to the (existing by item (2)) strongly continuous semigroup $(e^{\overline{L_0}t})_{t \geq 0}$ with the generator $\overline{L_0}$. In other words for every $t_0 > 0$ and every $\varphi \in X$ there exists a limit

$$\lim_{j \to \infty} (e^{\overline{L_j}t}\varphi)(x) = (e^{\overline{L_0}t}\varphi)(x), \tag{3.33}$$

   uniformly with respect to $x \in H$ and $t \in [0, t_0]$.

Proof. (1) Recall Theorem 2.1 and set $F(t) = S_t$, $\omega = \frac{2\|A\|\|B\|^2}{g_0} + \|C\|$, $\mathcal{X} = X$, $\mathcal{D} = D$, $F'(0) = L$, $G = \overline{L}$. One can see that according to items (1), (4) and (5) of Theorem 3.1 and item (5) of Theorem 3.2 all the conditions of Theorem 2.1 are fulfilled.

   (2) Note that $C(x) \leq 0$, so $\sup_{x \in H} e^{C(x)} \leq 1$ and for $B = 0$ one obtains the estimate $\|S_t\| \leq 1$. Conditions of Theorem 2.2 are fulfilled if one sets $\mathcal{X} = X$, $\mathcal{D} = D$, $\mathcal{L} = L$, $V_t = S_t$, $M = 1$, $\omega = 0$. Indeed, according to item (1) of Theorem 3.1, for all $t \geq 0$ the estimate $\|S_t\| \leq e^{\omega t} = 1$ holds true, therefore $\|(S_t)^k\| \leq 1 \cdots 1 = 1$.

Other conditions of Theorem 2.2 follow from item (4) of Theorem 3.1 and items (4) and (5) of Theorem 3.2.

(3) Recall Theorem 2.3, and set $\mathcal{X} = X$, $\mathcal{D} = D$, $\mathcal{L} = L_0$, $\mathcal{L}_n = L_j$. One can see that item (2) of this theorem and items (4) and (5) of Theorem 3.2 imply all the conditions of Theorem 2.3, except for the following one: if $\varphi \in D$, then $\lim_{j\to\infty} L_j\varphi = L_0\varphi$. A simple check shows that this condition is also fulfilled.

$\square$

3.3. *Feynman formula solves the Cauchy problem*

We want to find a function $u : [0, +\infty) \times H \to \mathbb{R}$ satisfying the following conditions (we call them Cauchy problem for the parabolic differential equation):

$$\begin{cases} u'_t(t, x) = Lu(x, t); & t \geq 0, \ x \in H, \\ u(0, x) = u_0(x); & x \in H. \end{cases} \tag{3.34}$$

To this Cauchy problem, we relate the so-called abstract Cauchy problem (see Definition 2.3), which we define as the following system of conditions upon the function $U : [0, +\infty) \to X$:

$$\begin{cases} \dfrac{d}{dt}U(t) = \overline{L}U(t); & t \geq 0, \\ U(0) = u_0, \end{cases} \tag{3.35}$$

Remark 3.6. Problem (3.34) can be considered as problem (3.35) in the following sense. Function $u : (t, x) \mapsto u(t, x)$ of two variables (t, x) can be considered as a function $u : t \mapsto [x \mapsto u(t, x)]$ of one variable t, with values in the space of functions of variable x. Then

$$u(t, x) = (U(t))(x), \quad t \geq 0, \ x \in H.$$

Using this correspondence, we start from Definition 2.3 and define the solution of problem (3.34).

Definition 3.1. We call a function $u : [0, +\infty) \times H \to \mathbb{R}$ a *strong solution* of problem (3.34) if it satisfies the following conditions:

$$\begin{cases} u(t, \cdot) \in D_1; & t \geq 0, \\ \text{function } t \mapsto u(t, \cdot) \text{ is continuous;} & t \geq 0, \\ \text{Uniformly for } x \in H \ \exists \lim_{\varepsilon \to 0} \dfrac{u(t + \varepsilon, x) - u(t, x)}{\varepsilon} = u'_t(t, x); & t \geq 0, \\ u'_t(t, \cdot) \in X; & t \geq 0, \\ \text{Function } t \mapsto u'_t(t, \cdot) \text{ is continuous;} & t \geq 0, \\ u'_t(t, x) = Lu(x, t); & t \geq 0, \ x \in H, \\ u(0, x) = u_0(x); & x \in H. \end{cases} \tag{3.36}$$

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Definition 3.2. We call a function $u : [0, +\infty) \times H \to \mathbb{R}$ a *mild solution* of problem (3.34) if it satisfies the following conditions:

$$
\begin{cases}
u(t, \cdot) \in X; & t \geq 0, \\
\text{Function } t \mapsto u(t, \cdot) \text{ is continuous}; & t \geq 0, \\
\int_0^t u(s, \cdot) ds \in D_1; & t \geq 0, \\
u(t, x) = L \int_0^t u(s, x) ds + u_0(x); & t \geq 0, \ x \in H, \\
u_0 \in X.
\end{cases}
\tag{3.37}
$$

Definition 3.3. Let us use the symbol $C([0, +\infty), X)$ for the class of all functions $u : [0, +\infty) \times H \to \mathbb{R}$ such that for every $t \geq 0$ the function $x \mapsto u(t, x)$ belongs to the class X, and the mapping $t \mapsto u(t, \cdot) \in X$ is continuous for every $t \geq 0$.

Finally, let us state and prove the main result of the paper. We use definitions and notation from Sec. 2.1.

Theorem 3.4 (On the solution of the Cauchy problem for a parabolic differential equation in Hilbert space). *Suppose $g \in X, C \in X, B \in X_H$. Suppose there is a number $g_0 > 0$ such that for all $x \in H$ we have $g(x) \geq g_0$ and $C(x) \leq 0$. As $C \in X$, there exists a sequence $(C_j) \subset D$, converging to C uniformly; let us additionally require that this sequence can be selected in such a way that $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$.*

Then the following holds:

(1) *If there exists a strongly continuous semigroup with the generator $\overline{L}$, then for every $u_0 \in D_1$ there exists a solution u of problem (3.36), unique in the class $C([0, +\infty), X)$. The solution depends continuously on u_0, and is given by the formula $u(t, x) = \lim_{n \to \infty} ((S_{\frac{t}{n}})^n u_0)(x)$, where the limit is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$.*

(2) *If there exists a strongly continuous semigroup with the generator $\overline{L}$, then for every $u_0 \in X$ there exists a solution u of problem (3.37), unique in the class $C([0, +\infty), X)$. It depends continuously on u_0, and is given by the formula $u(t, x) = \lim_{n \to \infty} ((S_{\frac{t}{n}})^n u_0)(x)$, where the limit is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$.*

(3) *If $B = 0$, then there exists a strongly continuous semigroup with the generator $\overline{L}$. The formula $u(t, x) = \lim_{n \to \infty} ((S_{\frac{t}{n}})^n u_0)(x)$ becomes simpler than in the case $B \neq 0$. Namely, for $B = 0$ we have*

$$
u(t, x) = \lim_{n \to \infty} \underbrace{\int_H \int_H \cdots \int_H \int_H}_{n} e^{\frac{t}{n}(C(x) + \sum_{k=1}^{n-1} C(y_k))} u_0(y_1) \mu_{\frac{2t}{n}g(y_2)A}^{y_2}(dy_1)
$$

$$
\cdot \mu_{\frac{2t}{n}g(y_3)A}^{y_3}(dy_2) \cdots \mu_{\frac{2t}{n}g(y_n)A}^{y_n}(dy_{n-1}) \mu_{\frac{2t}{n}g(x)A}^{x}(dy_n).
\tag{3.38}
$$

In this case the solution u for all $t > 0$ satisfies the estimate $\sup_{x \in H} |u(t,x)| \leq \sup_{x \in H} |u_0(x)|$.

(4) Let $B = 0$, and let the functions $g_j \in X$, $B_j \in X_H$ and $C_j \in X$ be given for all $j \in \mathbb{N}$. Let $B_j = 0$ for all $j \in \mathbb{N}$. Suppose there exists $\varepsilon_0 > 0$ such that $g_j(x) \geq \varepsilon_0$ and $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$. Let us use the symbol L_j for the operator L that corresponds to the functions g_j, B_j and C_j, and the symbol L_0 for the operator L that corresponds to the functions g, B and C. Suppose also that $g_j(x) \to g(x)$ and $C_j(x) \to C(x)$, uniformly with respect to $x \in H$. We denote as u_j the solution of problems (3.36) and (3.37) for the operator L_j. For solution of problems (3.36) and (3.37) with the operator L, we use the symbol u.

Then $u_j(t,x)$ converges to $u(t,x)$ as $j \to \infty$, uniformly with respect $x \in H$ and uniformly with respect to $t \in [0, t_0]$ for every fixed $t_0 > 0$.

Remark 3.7. Note that if $B = 0$, then solution depends continuously on the data of the Cauchy problem: the coefficients of the equation (item (4)) and the initial condition (items (1) and (2)).

Remark 3.8. Analogous theorems for $\mathbb{C}$- or $\mathbb{R}^n$-valued functions u can be formulated mutatis mutandis. The result will hold true due to the theorem above and the linearity of L and S_t. The only additional condition will be that the coefficients of the equation must be real-valued. The same remark is applicable to all the key theorems of this paper.

Proof of the Theorem 3.4.

(1) Suppose that there exists a strongly continuous semigroup with the generator $\overline{L}$. Then by item (1) of Proposition 2.5 we obtain the existence of a strong solution (Definition 2.3) to Cauchy problem (3.35), and the solution is unique in the class $C([0, +\infty), X)$. By item (1) of Theorem 3.3 the semigroup is given in the form described. Using the relation between problems (3.34) and (3.35) explained in Remark 3.6, we obtain the solution for problem (3.36). The solution is unique in the class $C([0, +\infty), X)$, as follows from Remark 3.6.

(2) The proof is similar to that in item (1). The only difference is that in Proposition 2.5 we use item (2) instead of item (1).

(3) The existence of the sought semigroup follows from item (2) of Theorem 3.3. The estimate for the supremum of the absolute value of the solution follows from the fact that the semigroup is contractive.

Let us explain how the equality $u(t,x) = \lim_{n \to \infty} ((S_{\frac{t}{n}})^n u_0)(x)$ implies formula (3.38). For a continuous bounded function $\psi : H \to \mathbb{R}$ and a point $x \in H$, the following change of variables rule in the integral is correct:

$$\int_H \psi(y) \mu_A(dy) = \int_H \psi(y - x) \mu_A^x(dy).$$

Applying this rule, and changing A to $2tg(x)A$, we come to the equality

$$(S_t\varphi)(x) = e^{tC(x)} \int_H \varphi(x+y)\mu_{2tg(x)A}(dy)$$

$$= e^{tC(x)} \int_H \varphi(x+(y-x))\mu_{2tg(x)A}^x(dy) = e^{tC(x)} \int_H \varphi(y)\mu_{2tg(x)A}^x(dy).$$

For $n=2$ in formula (3.38) we get the expression

$$((S_{\frac{t}{2}})^2\varphi)(x) = (S_{\frac{t}{2}}(S_{\frac{t}{2}}\varphi))(x)$$

$$= \int_H \left(\int_H e^{\frac{t}{2}(C(x)+C(y_1))}\varphi(y_1)\mu_{\frac{2t}{2}g(y_2)A}^{y_2}(dy_1) \right) \mu_{\frac{2t}{2}g(x)A}^x(dy_2).$$

In the same way expressions for $n>2$ are derived. Thus, the formula (3.38) is proven.

(4) The proof follows immediately from item (3) of Theorem 3.3. $\qquad\square$

Acknowledgments

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